# C*-Dynamical Systems from Number Theory

## Part 3: The Bost-Connes phase transition

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# The Hecke algebra of Bost and Connes

The inclusion of groups

$$P_{\mathbb{Z}}^{+} := \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & \mathbb{Q} \\ 0 & \mathbb{Q}_{+}^{*} \end{pmatrix} =: P_{\mathbb{Q}}^{+}$$

is a **Hecke pair** (each double coset contains finitely many right cosets), i.e. $R(\gamma) := |P_{\mathbb{Z}}^{+} \backslash (P_{\mathbb{Z}}^{+} \gamma P_{\mathbb{Z}}^{+})|$ is finite and hence
$L(\gamma) := |(P_{\mathbb{Z}}^{+} \gamma P_{\mathbb{Z}}^{+}) / P_{\mathbb{Z}}^{+}| = R(\gamma^{-1})$ is also finite .

## Definition

The *Hecke algebra* $\mathcal{H}_{\mathbb{Q}}$ is the $*$-algebra generated by the linear span of the characteristic functions of double cosets $[\gamma] \in P_{\mathbb{Z}}^{+} \backslash P_{\mathbb{Q}}^{+} / P_{\mathbb{Z}}^{+}$ with

- **convolution:** $(f * g)(\gamma) := \sum_{\gamma_1 \in P_{\mathbb{Z}}^{+} \backslash P_{\mathbb{Q}}^{+}} f(\gamma \gamma_1^{-1}) g(\gamma_1)$
  where the sum is over right-cosets $\gamma_1$;
- **involution:** $f^*(\gamma) := \overline{f(\gamma^{-1})}$;
- **identity:** $1 = [P_{\mathbb{Z}}^{+}]$.

Note: We denote by $[\gamma]$ the characteristic function of the double coset of $\gamma$.

## The regular representation of $\mathcal{H}_{\mathbb{Q}}$ on $\ell^2(P_{\mathbb{Z}}^+\backslash P_{\mathbb{Q}}^+)$.

### Definition

The **Hecke C*-algebra** $\mathcal{C}_{\mathbb{Q}}$ is the $C^*$-algebra generated by the left-convolution operators $L_f$ with $f \in \mathcal{H}_{\mathbb{Q}}$, defined via
$L_f(\xi)(\gamma) \ = \ (f * \xi)(\gamma) = \sum_{\gamma_1 \in P_{\mathbb{Z}}^+\backslash P_{\mathbb{Q}}^+} f(\gamma\gamma_1^{-1})\xi(\gamma_1), \quad$ for $\xi \in \ell^2(P_{\mathbb{Z}}^+\backslash P_{\mathbb{Q}}^+)$.

There is a strongly continuous one-parameter unitary group $t \mapsto U_t$ on $\ell^2(P_{\mathbb{Z}}^+\backslash P_{\mathbb{Q}}^+)$ given by $\quad U_t(\xi)(\gamma) = \big(R(\gamma)/L(\gamma)\big)^{it}\xi(\gamma)$.
Conjugation of double cosets by $U_t$ yields $U_t[\gamma]U_t^* = \big(R(\gamma)/L(\gamma)\big)^{it}[\gamma]$ and induces a natural time evolution $t \mapsto \sigma_t$ on $\mathcal{C}_{\mathbb{Q}}$.

### Definition

The **Bost-Connes C*-dynamical system** $(\mathcal{C}_{\mathbb{Q}}, \sigma)$ consists of the C*-algebra $\mathcal{C}_{\mathbb{Q}}$ with the dynamics $\sigma$, characterized on double cosets by

$$\sigma_t([\gamma]) = \left(\frac{R(\gamma)}{L(\gamma)}\right)^{it}[\gamma] \qquad \text{for } t \in \mathbb{R}.$$

# Double coset generators for $\mathcal{C}_{\mathbb{Q}}$

The elements $\mu_n := \frac{1}{n^{1/2}} \left[\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\right]$ with $n \in \mathbb{N}^\times$ and

$$e(r) := \left[\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\right] \quad \text{with } r \in \mathbb{Q}/\mathbb{Z}$$

generate $\mathcal{C}_{\mathbb{Q}}$ as a C*-algebra and satisfy the relations

1. $\mu_1 = 1, \quad \mu_n^* \mu_n = 1, \quad$ and $\quad \mu_m \mu_n = \mu_{mn}$;
2. $\mu_m^* \mu_n = \mu_n \mu_m^*$ if $(m, n) = 1$;
3. $e(0) = 1, \quad e(r)^* = e(-r), \quad$ and $\quad e(r)e(s) = e(r+s)$;
4. $\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{[s:ns=r]} e(s)$, *(since $s \in \mathbb{Q}/\mathbb{Z}$ there are $n$ summands.)*;
5. $e(r)\mu_n = \mu_n e(nr)$.

- the set $\{\mu_m e(r) \mu_n^* : r \in \mathbb{Q}/\mathbb{Z},\ m, n \in \mathbb{N}^\times,\ (m,n) = 1\}$ is linearly independent and has dense linear span in $\mathcal{C}_{\mathbb{Q}}$;
- analogous statements hold for $\mathcal{H}_{\mathbb{Q}}$, at the *-algebra level.

# The Bost Connes C*-algebra $\mathcal{C}_{\mathbb{Q}}$ by presentation

(Start here)
$\mathcal{C}_{\mathbb{Q}}$ = universal C*-algebra generated by elements $\mu_n$ for $n \in \mathbb{N}^\times$ and $e(r)$ for $r \in \mathbb{Q}/\mathbb{Z}$ subject to the relations

1. $\mu_1 = 1, \quad \mu_n^*\mu_n = 1, \quad$ and $\quad \mu_m\mu_n = \mu_{mn};$
2. $\mu_m^*\mu_n = \mu_n\mu_m^*$ if $(m, n) = 1;$
3. $e(0) = 1, \quad e(r)^* = e(-r), \quad$ and $\quad e(r)e(s) = e(r + s);$
4. $\mu_n e(r)\mu_n^* = \frac{1}{n}\sum_{[s:ns=r]} e(s),$ *(since $s \in \mathbb{Q}/\mathbb{Z}$ there are $n$ summands)*;
5. $e(r)\mu_n = \mu_n e(nr).$

Moreover,

- The set $\{\mu_m e(r)\mu_n^* : r \in \mathbb{Q}/\mathbb{Z},\ m, n \in \mathbb{N}^\times,\ (m, n) = 1\}$ is linearly independent and has dense linear span in $\mathcal{C}_{\mathbb{Q}}$.

Note: Relations (2) and (5) from the original presentation, are key properties, but are consequences of the other three.

# Interpreting the presentation of $\mathcal{C}_{\mathbb{Q}}$

- relations (1) and (2) say that $\{\mu_n\}_{n\in\mathbb{N}^\times}$ is a covariant semigroup of isometries, generating a representation of $C^*(\mathbb{N}^\times)$;
- relation (3) says that $r \mapsto e(r)$ is a unitary representation of the group $\mathbb{Q}/\mathbb{Z}$, generating a copy of $C^*(\mathbb{Q}/\mathbb{Z})$;
- relation (4) is a covariance relation: the r.h.s. defines a semigroup of endomorphisms $\alpha_n(e(r)) := \frac{1}{n}\sum_{[s:ns=r]} e(s)$, and the l.h.s. implements it via conjugation with the semigroup of isometries
- the canonical dual action of $(\mathbb{Q}^*_+)\hat{}$ on $C^*(\mathbb{N}^\times)$ respects relation (4) hence extends to $\mathcal{C}_{\mathbb{Q}}$ and the dynamics $\sigma$ is a natural 1-parameter subgroup of this extension.
- The natural action of $\operatorname{Aut}\mathbb{Q}/\mathbb{Z}$ on $C^*(\mathbb{Q}/\mathbb{Z})$ respects relation (4) hence extends to $\mathcal{C}_{\mathbb{Q}}$; moreover the extended action commutes with $\sigma$, so $\operatorname{Aut}\mathbb{Q}/\mathbb{Z}$ acts as symmetries of $(\mathcal{C}_{\mathbb{Q}}, \sigma)$.

## $\mathcal{C}_{\mathbb{Q}}$ as a semigroup crossed product (arithmetic version)

**Theorem (L–Raeburn '99)**

(i) *There is an action* $\alpha$ *of* $\mathbb{N}^{\times}$ *by endomorphisms of* $C^*(\mathbb{Q}/\mathbb{Z})$ *such that*

$$\alpha_n(e(r)) = \frac{1}{n}\sum_{[s:ns=r]} e(s) \quad \text{for } n \in \mathbb{N}^{\times} \text{ and } r \in \mathbb{Q}/\mathbb{Z};$$

(ii) *the map* $\gamma_n : e(r) \mapsto e(nr)$ *defines an endomorphism of* $C^*(\mathbb{Q}/\mathbb{Z})$ *such that* $\gamma_n \circ \alpha_n = \mathrm{id}$ *while* $\alpha_n \circ \gamma_n =$ *multiplication by* $\alpha_n(1)$;

(iii) *there is a canonical isomorphism* $\mathcal{C}_{\mathbb{Q}} \cong C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_{\alpha} \mathbb{N}^{\times}$.

**Idea of proof:** One verifies directly that $\alpha_n$ is a semigroup action and then one recognizes the definition of semigroup crossed product in (a subset of) the relations. □

- this reveals that relations (1) (3) (4) imply (2) and (5).
- $\sigma$ is quasi-periodic and its fixed point algebra is $\mathcal{C}_{\mathbb{Q}}^{\sigma} = C^*(\mathbb{Q}/\mathbb{Z})$.
- $\mathrm{Aut}\,\mathbb{Q}/\mathbb{Z}$ (a profinite group) has fixed point algebra $\mathcal{C}_{\mathbb{Q}}^{\theta} = C^*(\mathbb{N}^{\times})$.

# The Bost-Connes phase transition theorem

## Theorem (Bost-Connes, '95)

*Define a dynamics* $\sigma$ *on* $\mathcal{C}_{\mathbb{Q}}$ *by* $\sigma_t(\mu_m e(r)\mu_n^*) = (m/n)^{it}\mu_m e(r)\mu_n^*$.

1. *For each* $0 < \beta \leq 1$ *there is a unique KMS*$_\beta$ *state of* $(\mathcal{C}_{\mathbb{Q}}, \sigma)$. *It is an injective type III*$_1$ *factor state, invariant under the action of* $\operatorname{Aut}\mathbb{Q}/\mathbb{Z}$.
2. *For each* $1 < \beta \leq \infty$ *the extremal KMS*$_\beta$ *states* $\phi_{\beta,\chi}$ *are parametrized by the complex embeddings* $\chi : \mathbb{Q}^{cycl} \to \mathbb{C}$. *These are type I factor states, on which the action of* $\operatorname{Aut}\mathbb{Q}/\mathbb{Z}$ *is free and transitive.*
3. *The partition function of the system is the Riemann zeta function* $\zeta(\beta)$ *for each of the states in part (2).*

We shall outline a proof along the following lines:

- Realize $\mathcal{C}_{\mathbb{Q}}$ as a semigroup crossed product $C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times$.
- Show that KMS states correspond 1-1 to scaling measures on $C(\hat{\mathbb{Z}})$.
- Construct the scaling measures explicitly.
- Classify the states according to type.

# Bost-Connes system, **adelic version.**

Recall the adeles and the ideles

- $\hat{\mathbb{Z}} := \operatorname{proj\,lim}_n(\mathbb{Z}/n\mathbb{Z}) \cong \prod_p \mathbb{Z}_p$ be the ring of integral adeles, and let
- $\hat{\mathbb{Z}}^* = \operatorname{proj\,lim}_n(\mathbb{Z}/n\mathbb{Z})^* \cong \prod_p \mathbb{Z}_p^*$ be its group of units, (the integral ideles).
- There is a duality pairing of $\hat{\mathbb{Z}}$ to $\mathbb{Q}/\mathbb{Z}$ through which $\hat{\mathbb{Z}}^*$ (acting on $\hat{\mathbb{Z}}$ by multiplication) corresponds to $\operatorname{Aut}(\mathbb{Q}/\mathbb{Z})$.
- $C^*(\mathbb{Q}/\mathbb{Z}) \cong C(\hat{\mathbb{Z}})$ and the endomorphisms of $C(\hat{\mathbb{Z}})$ are given by

$$\alpha_n(f)(x) = \begin{cases} f(x/n) & \text{if } x \in n\hat{\mathbb{Z}} \qquad \text{“\textit{division by } } n \\ 0 & \text{otherwise.} \qquad \textit{when possible in } \hat{\mathbb{Z}}\text{”.} \end{cases}$$

# Bost-Connes system, **adelic version.**

- the Bost Connes C*-algebra $\mathcal{C}_{\mathbb{Q}}$ is isomorphic to

$$C(\hat{\mathbb{Z}}) \rtimes_\alpha \mathbb{N}^\times,$$

- the dynamics $\sigma$ is given by

$$\sigma_t(\mu_m f \mu_n^*) = (m/n)^{it} \mu_m f \mu_n^*,$$

Note: the products $\mu_m f \mu_n^*$ are $\sigma$-analytic and span a dense subset.

- the symmetries are now given by $\{\theta_w : w \in \hat{\mathbb{Z}}^*\}$ :

$$\theta_w(\mu_m f \mu_n^*) = \mu_m\, f_w\, \mu_n^* \quad \text{where} \quad f_w(x) := f(xw).$$

Note: Both endomorphisms and symmetries are by multiplication in $\hat{\mathbb{Z}}$.

## Bost Connes system **as full corner in** $C_0(\mathbb{A}_{\mathbb{Q},f}) \rtimes \mathbb{Q}_+^*$

- Localizing $\hat{\mathbb{Z}}$ at the positive integers $\mathbb{N}^\times$ one obtains the ring

$$\mathbb{A}_{\mathbb{Q},f} = (\mathbb{N}^\times)^{-1}\hat{\mathbb{Z}}$$

  of finite adeles, in which the elements of $\mathbb{N}^\times$ are invertibles and $\hat{\mathbb{Z}}$ is the maximal compact open subring.

- $\mathbb{A}_{\mathbb{Q},f}$ has a factorization as a restricted product:

$$\mathbb{A}_{\mathbb{Q},f} \cong \prod_p (\mathbb{Q}_p, \mathbb{Z}_p)$$

- there is a canonical (diagonal) embedding of $\mathbb{Q}_+^*$ in $\mathbb{A}_{\mathbb{Q},f}$ so $\mathbb{Q}_+^*$ acts by multiplication.

# Bost Connes system **as full corner in** $C_0(\mathbb{A}_{\mathbb{Q},f}) \rtimes \mathbb{Q}^*_+$

- The action of $\mathbb{Q}^*_+$ by multiplication on $C_0(\mathbb{A}_{\mathbb{Q},f})$ determines a crossed product $C_0(\mathbb{A}_{\mathbb{Q},f}) \rtimes \mathbb{Q}^*_+$.
- the characteristic function $\mathbb{1}_{\hat{\mathbb{Z}}}$ of the integral adeles is a full projection in this crossed product and the associated full corner is isomorphic to $\mathcal{C}_\mathbb{Q}$:

$$C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times \cong \mathbb{1}_{\hat{\mathbb{Z}}}(C_0(\mathbb{A}_{\mathbb{Q},f}) \rtimes \mathbb{Q}^*_+)\mathbb{1}_{\hat{\mathbb{Z}}},$$

(by definition, the projection $p \in A$ is *full* if $\overline{\text{span}} ApA = A$.)

- the proof is by a dilation/extension theorem suitable for the localization at $\mathbb{N}^\times$.

  Note: if you missed the definition of $\mathcal{C}_\mathbb{Q}$ or of $C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times$, you can use this corner as definition.

# Fixed point algebras and conditional expectations

The group of symmetries $\{\theta_w\}_{w\in\hat{\mathbb{Z}}^*}$ is compact. The dynamics $\{\sigma_t\}_{t\in\mathbb{R}}$ is not compact but it is quasi-periodic because the periods of the different components are rationally independent, so the real line is wrapped densely in the infinite torus, $\widehat{\mathbb{Q}^*_+}$, i.e. $\overline{\sigma_{\mathbb{R}}} = \hat{\alpha}_{\widehat{\mathbb{Q}^*_+}}$, which is a compact group.
Hence the $\theta$- and $\sigma$-averages are well defined and give conditional expectations onto the respective fixed point algebras, forming a commuting diagram:

$$\begin{array}{ccc} C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times & \xrightarrow{E_\theta} & C^*(\mathbb{N}^\times) \\ \downarrow E_\sigma & \searrow^{E_{\theta,\sigma}} & \downarrow E_\sigma \\ C(\hat{\mathbb{Z}}) & \xrightarrow{E_\theta} & B_{\mathbb{N}^\times} \end{array}$$

where $B_{\mathbb{N}^\times} := \overline{\text{span}}\{\mu_n\mu_n^* : n \in \mathbb{N}^\times\} = \overline{\text{span}}\{\mathbb{1}_{n\hat{\mathbb{Z}}} : n \in \mathbb{N}^\times\} \cong C(\hat{\mathbb{Z}}/\hat{\mathbb{Z}}^*)$

is a commutative C*-algebra with spectrum $\hat{\mathbb{Z}}/\hat{\mathbb{Z}}^* = \prod_p p^{\mathbb{N}\cup\{\infty\}}$.

# KMS states are induced from scaling measures on $\hat{\mathbb{Z}}$

## Proposition (L, JFA 98)

*The map* $\nu \mapsto \varphi := (\nu)_* \circ E_\sigma$ *is an affine isomorphism of the Borel probability measures* $\nu$ *on* $\hat{\mathbb{Z}}$ *that satisfy the* $\beta$*-scaling condition*

$$\nu(nE) = n^{-\beta}\nu(E), \qquad \text{for } E \subset \hat{\mathbb{Z}},$$

*and the* $KMS_\beta$ *states of* $(C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times, \sigma)$.
*The inverse map is given by restriction.*

## Proof.

If $\nu$ is a $\beta$-scaling measure, then a calculation shows that the induced state $\varphi := (\nu)_* \circ E_\sigma$ satisfies $\varphi(\mu_m f \mu_n^* \mu_r g \mu_s^*) = (m/n)^{-\beta}\varphi(\mu_r g \mu_s^* \mu_m f \mu_n^*)$; since the analytic monomials have dense linear span, $\varphi$ is $\text{KMS}_\beta$.

If $\varphi$ is a $\text{KMS}_\beta$ state, then $\varphi = \varphi|_{C(\hat{\mathbb{Z}})} \circ E_\sigma$ by $\sigma$-invariance, and $\varphi|_{C(\hat{\mathbb{Z}})}$ is the state of a probability $\nu$ on $\hat{\mathbb{Z}}$ which is $\beta$-scaling because $\varphi(\alpha_n(f)) = \varphi(\mu_n f \mu_n^*) = n^{-\beta}\varphi(f \mu_n^* \mu_n)$ by the $\text{KMS}_\beta$ condition. □

# Ground states and $\text{KMS}_\infty$ states

**Definition**

$\varphi$ is a **ground state** of $(\mathcal{A}, \sigma)$, if the entire function $z \mapsto \varphi(b\sigma_z(a))$ is bounded on the upper half plane $\Im z > 0$ for $a$ and $b$ $\sigma$-analytic.
The **$\text{KMS}_\infty$ states** are the weak*- limits of $\text{KMS}_\beta$ states as $\beta \to \infty$, and are automatically ground states.

**Claim (from Day 1:** In the ground state condition it suffices to check that $z \mapsto \varphi(b\sigma_z(a))$ is bounded on the upper half plane for $a$ and $b$ in a set of analytic elements with dense linear span.

# KMS$_\infty$ states of the BC-system

If $\varphi$ is a ground state of the BC-system, then $z \mapsto (1/n)^{iz}\varphi(\mu_m f \mu_n^*)$ is bounded for $\Im z > 0$, and so it has to vanish for $n > 1$ and (on taking adjoints) also for $m > 1$. Hence $\varphi$ comes from a probability measure $\nu$ on $\hat{\mathbb{Z}}$ satisfying $\nu(p\hat{\mathbb{Z}}) = \varphi(\alpha_p(1)) = \varphi(\mu_p\mu_p^*) = 0$ for each $p \in \mathcal{P}$, that is, **each ground state is induced from a probability supported on**

$$\bigcap_p (\hat{\mathbb{Z}} \setminus p\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^*.$$

Conversely any probability measure $\nu$ supported on $\hat{\mathbb{Z}}^*$ gives a ground state.

From the formulas obtained (later) for KMS$_\beta$ states, e.g. for extremal ones, we will see that $\varphi_{\beta,w}(f) = \sum_{n\in\mathbb{N}^\times} \frac{n^{-\beta}}{\zeta(\beta)} f(nw) \;\to\; f(w)$ as $\beta \to \infty$, so every ground state of $(\mathcal{C}_\mathbb{Q}, \sigma)$ is a KMS$_\infty$ state.

**Key Observation:** Evaluating extremal ground states $\varphi_\chi$ with $\chi \in \hat{\mathbb{Z}}^*$ at the characters $\langle r, \cdot\rangle \in C(\hat{\mathbb{Z}})$ for $r \in \mathbb{Q}/\mathbb{Z}$, yields the roots of unity i.e.

$$\varphi_\chi(\langle r, \cdot\rangle) = \langle r, \chi\rangle = \exp(2\pi i r\chi).$$

# Orbit representations of $\mathcal{C}_{\mathbb{Q}}$.

Fix $w \in \hat{\mathbb{Z}}^*$ and define operators on $\ell^2(\mathbb{N}^\times)$:

$$T_m \varepsilon_n = \varepsilon_{mn} \qquad m \in \mathbb{N}^\times$$

$$\pi_w(f)\varepsilon_n = f(nw)\varepsilon_n \qquad f \in C(\widehat{\mathbb{Z}})$$

Then $\{T_m : m \in \mathbb{N}^\times\}$ is a semigroup of isometries, $\pi_w$ is a representation of $C(\hat{\mathbb{Z}})$, and $\pi_w$ is faithful because the orbit $\mathbb{N}^\times \cdot w$ is dense in $C(\hat{\mathbb{Z}})$.

$(\pi_w, T)$ is a covariant pair: $\pi_w(\alpha_n(f)) = T_n \pi(f) T_n^*$, [hw verify this].
Hence they give a representation of the crossed product:

$$\pi \times T : C(\hat{\mathbb{Z}}) \rtimes_\alpha \mathbb{N}^\times \to \mathcal{B}(\ell^2(\mathbb{N}^\times)),$$

such that $\pi_w \times T(\mu_n) = T_n$ and $\pi_w \times T(f) = \pi(f)$.
As it turns out, $\pi_w \times T$ is faithful, so you may think of any of these representations (say the one for $w = 1$) as a concrete realization of $\mathcal{C}_{\mathbb{Q}}$ as operators on a Hilbert space.

# Orbit representations of $\mathcal{C}_{\mathbb{Q}}$.

- Next define a unitary group on $\ell^2(\mathbb{N}^\times)$ by

$$U_t \varepsilon_n = n^{it} \varepsilon_n$$

  The associated Hamiltonian satisfies $H\varepsilon_n = \log n \, \varepsilon_n$, and is an (unbounded) positive operator with 0 in its spectrum.
- $U_t = e^{itH}$ implements the dynamics $\sigma_t$ on $C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times$ *[Exercise: Verify this directly from the definitions of $T$, $\pi_w$, and $U$.]*
- Since $\mathrm{Tr}(e^{-\beta H}) = \sum_n n^{-\beta} = \zeta(\beta)$, we may define a generalized Gibbs state for $\beta > 1$:

$$\varphi_{\beta,w}(X) = \zeta(\beta)^{-1} \, \mathrm{Tr}(X e^{-\beta H}) \qquad (X \in \mathcal{C}_{\mathbb{Q}})$$

  using the density operator $\zeta(\beta)^{-1} e^{-\beta H}$.
- $\varphi_{\beta,w}(X)$ is KMS$_\beta$ for $\sigma$. This can be verified directly or proving the scaling property on functions.
- These are states arising from the evaluation functionals on $\hat{\mathbb{Z}}^*$, but it is easy to average them and obtain one for each probability measure on $\hat{\mathbb{Z}}^*$.

# A symmetric KMS$_\beta$ state for each $\beta \in (0,\infty)$

- If a KMS$_\beta$ state $\varphi$ is $\theta$-invariant, then it factors through $E_{\theta,\sigma}$ and hence is **uniquely determined by** $\varphi(\alpha_n(1)) = \nu(n\hat{\mathbb{Z}}) = n^{-\beta}$, because the span of these projections is dense in the range of $E_{\theta,\sigma}$.
- $C(\hat{\mathbb{Z}})^\theta = C(\hat{\mathbb{Z}}/\hat{\mathbb{Z}}^*) \cong C(\prod_p p^{\mathbb{N}\cup\{\infty\}})$ has a product space for spectrum, on which a concrete symmetric $\beta$-scaling measure is easy to construct for each $\beta \in (0,\infty)$: first for each $p \in \mathcal{P}$ take the probability measure

$$P_{\beta,p}(p^k) = (1 - p^{-\beta})p^{-k\beta} \text{ on the set } p^{\mathbb{N}\cup\{\infty\}},$$

  then form the product measure $P_\beta := \prod_p P_{\beta,p}$ on $\prod_p p^{\mathbb{N}\cup\{\infty\}}$.
- since $\varphi_\beta := (P_\beta)_* \circ E_{\theta,\sigma}$ is $\beta$-scaling, it is the **unique symmetric KMS$_\beta$ state**.
- Using an explicit formula for the expectation $E_\theta$ we can compute $\varphi_\beta$ on characters $\langle r,\cdot\rangle$ of $\hat{\mathbb{Z}}$:

$$\varphi_\beta(\langle r,\cdot\rangle) = b^{-\beta}\prod_{p|b}\frac{1-p^{\beta-1}}{1-p^{-1}} \qquad r = a/b \in \mathbb{Q}/\mathbb{Z} \text{ in reduced form.}$$

# Type $\mathrm{III}_1$ for the symmetric system $(C^*(\mathbb{N}^\times), \sigma)$

**Proposition (Boca-Zaharescu '00, Blackadar, JFA 77)**

*For each $\beta \in (0,1]$, the state $\varphi_\beta$ of $C^*(\mathbb{N}^\times)$ is of type $III_1$.*

**Sketch of the proof.**

This follows Boca and Zaharescu's with simplifications 'borrowed' from Tzanev (unpubl.), [Jacob, J. Noncommut. Geom. 06] (for function fields), and [Neshveyev, Proc. AMS 02].

Recall that $C^*(\mathbb{N}^\times) \cong \bigotimes_p C^*(\mu_p)$ where $\mu_p$ is a shift for each prime $p$ and $C^*(\mu_p)$ is a copy of the Toeplitz C*-algebra of $\mathbb{N}$. The state $\varphi_\beta$ factorizes as $\varphi_\beta = \bigotimes_p \varphi_{\beta,p}$ with $\varphi_{\beta,p}$ a type I factor state of $C^*(\mu_p)$ having eigenvalue list $\{(1-p^{-\beta})p^{-k\beta}\}_{k\in\mathbb{N}}$.
So $\pi_\beta(C^*(\mathbb{N}^\times))''$ is an $\mathrm{ITPFI}_\infty$, (infinite tensor product of type $\mathrm{I}_\infty$ factors) with eigenvalue lists depending on $p \in \mathcal{P}$.

To prove this ITPFI is type $\text{III}_1$ using the Araki-Woods classification, it suffices to produce, given any $\lambda > 0$, two sequences of prime numbers $\{p_n\}_{p\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ such that

- $\left(\frac{p_n}{q_n}\right)^\beta \to \lambda$, and
- $\sum_n \frac{1}{p_n^\beta} = \infty$ (so that $\sum_n q_n^{-\beta}(1 - p_n^{-\beta})(1 - q_n^{-\beta}) = \infty$).

Changing $\lambda$ to $\lambda^{1/\beta}$, it suffices to find the sequences for $\beta = 1$. To produce such sequences Blackadar and Boca-Zaharescu use the Prime Number Theorem.

Recall the diagram

$$\begin{array}{ccc} C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times & \xrightarrow{E_\theta} & C^*(\mathbb{N}^\times) \\ \downarrow E_\sigma & \searrow^{E_{\theta,\sigma}} & \downarrow E_\sigma \\ C(\hat{\mathbb{Z}}) & \xrightarrow{E_\theta} & B_{\mathbb{N}^\times} \end{array}$$

To prove that $\varphi_\beta \circ E_\theta$ (the state of the B-C algebra) is type $\text{III}_1$, Neshveyev proves what amounts to "permanence of $\text{III}_1$ through a conditional expectation of profinite index".

# Partial zeta functions for subsets of primes

- Let $\mathbb{N}_B^\times$ be the set of numbers that factorize within $B \subset \mathcal{P}$, and
- let $\zeta_B(\beta) := \sum_{n \in \mathbb{N}_B^\times} n^{-\beta} = \prod_{p \in B}(1 - p^{-\beta})^{-1}$.
  Note that if $B$ is finite, then $\zeta_B(\beta) < \infty$ for every $\beta > 0$.
- The set $W_B$ of integral adeles that are not multiples of any $p \in B$,

$$W_B := \bigcap_{p \in B} (\hat{\mathbb{Z}} \setminus p\hat{\mathbb{Z}}) \cong \prod_{p \in B} \mathbb{Z}_p^* \times \prod_{q \notin B} \mathbb{Z}_q,$$

  is the support of the projection $Q_B := \prod_{p \in B}(1 - \alpha_p(1))$.

Suppose $\varphi$ is a $\beta$-scaling probability measure on $\hat{\mathbb{Z}}$. If $(m, n) = 1$, then $\varphi(m\hat{\mathbb{Z}} \cap n\hat{\mathbb{Z}}) = \varphi(mn\hat{\mathbb{Z}}) = (mn)^{-\beta} = m^{-\beta} n^{-\beta} = \varphi(m\hat{\mathbb{Z}})\varphi(n\hat{\mathbb{Z}})$, hence

$$\varphi(W_B) = \prod_{p \in B}(1 - \varphi(p\hat{\mathbb{Z}})) = \prod_{p \in B}(1 - p^{-\beta}) = \zeta_B(\beta)^{-1}.$$

# Reconstruction from the conditional probability

If $\zeta_B(\beta) < \infty$, we define the **conditional state** $\varphi_{Q_B}$ ($\varphi$ given $Q_B$) by

$$\varphi_{Q_B}(\cdot) := \zeta_B(\beta)\varphi(Q_B \cdot Q_B).$$

## Lemma (L-Raeburn '10, cf. Neshveyev '02)

*If $\varphi$ is a KMS$_\beta$ state of $(\mathcal{C}_\mathbb{Q}, \sigma)$ and $B$ is a subset of $\mathcal{P}$ such that $\zeta_B(\beta) < \infty$, then*

$$\varphi(T) = \sum_{n \in \mathbb{N}_B^\times} \frac{n^{-\beta}}{\zeta_B(\beta)} \varphi_{Q_B}(\mu_n^* T \mu_n) \quad \textit{for } T \in \mathcal{C}_\mathbb{Q}.$$

## Proof.

The sets $nW_B$ for $n \in \mathbb{N}_B^\times$ are mutually disjoint and

$$\varphi\Big(\bigcup_{n\in\mathbb{N}_B^\times} nW_B\Big) = \sum_{n\in\mathbb{N}_B^\times} \varphi(nW_B) = \sum_{n\in\mathbb{N}_B^\times} n^{-\beta}\varphi(W_B) = 1.$$

Then $\quad \varphi(E) = \sum_{n\in\mathbb{N}_B^\times} \varphi(nW_B \cap E) = \sum_{n\in\mathbb{N}_B^\times} n^{-\beta}\varphi(n^{-1}E \cap W_B).$ □

# Symmetry-breaking for $\beta \in (1, \infty]$

## Proposition (L, JFA 98)

*Supose $\beta \in (1, \infty)$ and for each probability measure $\nu$ on $\hat{\mathbb{Z}}^*$ define a probability $T_\beta \nu$ on $\hat{\mathbb{Z}}$ by*

$$T_\beta \nu(E) := \tfrac{1}{\zeta(\beta)} \textstyle\sum_{n \in \mathbb{N}^\times} n^{-\beta} \nu(n^{-1} E \cap \hat{\mathbb{Z}}^*) \quad \text{for } E \subset \hat{\mathbb{Z}}.$$

*Then the map $\nu \mapsto (T_\beta \nu)_* \circ E_\sigma$ is an affine isomorphism of the probability measures on $\hat{\mathbb{Z}}^*$ onto the KMS$_\beta$ states, in which the extremal KMS$_\beta$ states correspond to the unit point masses on $\hat{\mathbb{Z}}^*$ (and the orbit representations).*

## Proof.

$T_\beta$ is clearly affine, and since $T_\beta \nu(E) = (1/\zeta(\beta))\nu(E)$ for $E \subset \hat{\mathbb{Z}}^*$ the map is injective; the $\beta$-scaling condition is built into the formula because $\cup_n n\hat{\mathbb{Z}}^*$ is a disjoint union with total mass 1, so we get KMS$_\beta$ states.
To prove $T_\beta$ is surjective we need to reconstruct $\varphi$ from $\varphi|_{\hat{\mathbb{Z}}^*} \ldots$ □

# The map $\nu \to T_\beta \nu$ is surjective

Since for $\beta \in (1, \infty)$ the full zeta-series converges, we may take $B = \mathcal{P}$ and $W_B = \hat{\mathbb{Z}}^*$ in the Lemma to recover a $\beta$-scaling probability $\varphi$ on $\hat{\mathbb{Z}}$ from its conditioning to $\hat{\mathbb{Z}}^*$.
Since the Lemma then says $\varphi = T_\beta \, \varphi_{\hat{\mathbb{Z}}^*}$, this proves that $T_\beta$ is surjective.
The unit point mass at $w \in \hat{\mathbb{Z}}^*$ gives an extremal $\mathrm{KMS}_\beta$ state

$$\varphi_{\beta,w}(f) = \sum_{n \in \mathbb{N}_B^\times} \frac{n^{-\beta}}{\zeta_B(\beta)} f(nw).$$

This completes the proof of part (2) of the BC-theorem, i.e. the case $\beta \in (1, \infty)$. □

Another (harder) application of the Lemma, with $B$ finite, shows that for $\beta \in (0, 1]$ any $\mathrm{KMS}_\beta$ state (or any $\beta$-scaling probability) is necessarily symmetric, and hence equal to the state $\varphi_\beta$ constructed earlier as the induced state from a product probability.