C*-dynamical systems from number theory

Day 2: C*- dynamical systems and KMS states: More examples *p*-adic numbers, adeles, ideles and all that

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Example (an infinite system based on the Toeplitz algebra)

- **The Toeplitz algebra represented as operators on $\ell^2(\mathbb{N})$:** Let $\{\varepsilon_n\}$ be the standard orthonormal basis of $\ell^2(\mathbb{N})$, and let $S : \varepsilon_n \mapsto \varepsilon_{n+1}$, be the usual unilateral shift on $\ell^2(\mathbb{N})$. Define $\mathcal{T}$ to be the C*-algebra generated by S.
- **The Toeplitz algebra as the universal C*-algebra of an isometry:** There exists a unital C*-algebra generated by an isometry V (i.e. $V^*V = 1$) such that whenever W satisfies $W^*W = 1$ there is a C*-algebra homomorphism $h : C^*(V) \to C^*(W)$ such that $h(V) = W$. (Such V is a universal isometry, and it is unique up to canonical isomorphism).
- Coburn's classical result can be interpreted as saying that the canonical homomorphism mapping $V \mapsto W$ is an isomorphism if and only if $WW^* \neq 1$. In particular this happens when $W = S$.

Example (an infinite system based on the Toeplitz algebra)

- If V is a universal isometry so is $e^{it}V$ for $t \in \mathbb{R}$, and the universal property gives (a continuous group of) automorphisms σ_t of $\mathcal{T}$ determined by what they do to S: $\quad \sigma_t(S) = e^{it}S$ for $t \in \mathbb{R}$.
- (hw: prove this and also verify that $\{\sigma_t\}$ is implemented by the 1-parameter unitary group $t \mapsto e^{itH}$ on $\ell^2(\mathbb{N})$ with Hamiltonian $H\varepsilon_n = n\varepsilon_n$.)

Example (an infinite system based on the Toeplitz algebra)

- Using $S^*S = 1$ we may 'Wick order' the products on S and S^* and have all the S^*'s appear to the right; thus the set $\{S^m S^{*n} : m, n \in \mathbb{N}\}$ spans a dense *-subalgebra of $\mathcal{T}$.
- Notice that $t \mapsto \sigma_t$ is periodic, so it can be viewed as an action of the circle $\mathbb{T}$. Averaging over $\mathbb{T}$ gives a faithful conditional expectation E_σ of $\mathcal{T}$ onto the fixed-point algebra $\mathcal{T}^\sigma = \overline{\text{span}}\{S^n S^{*n} : n \in \mathbb{N}\}$ of σ:

$$E_\sigma(S^m S^{*n}) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{i(m-n)t} S^m S^{*n} dt = \begin{cases} S^m S^{*m} & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

- Since $S^m S^{*m} S^n S^{*n} = S^{\max(m,n)} S^{*\max(m,n)}$ this fixed point algebra is commutative; its spectrum is $\mathbb{N} \cup \{\infty\}$. One way to see this is to prove directly that the map $\delta_n \mapsto S^n S^{*n} - S^{n+1} S^{*n+1}$ extends to an isomorphism of $c = C(\mathbb{N} \cup \{\infty\})$ onto $\mathcal{T}^\sigma$ (hw: do it!).

Example (KMS$_\beta$ states of the Toeplitz algebra for $0 < \beta < \infty$)

1. $z \mapsto \sigma_z(S^m S^{*n}) = e^{i(m-n)z} S^m S^{*n}$ is entire so the spanning elements are analytic.
2. By the KMS$_\beta$ condition (twice): $\varphi(S^m S^{*n}) = e^{-(m-n)\beta}\varphi(S^m S^{*n})$
3. φ is a KMS$_\beta$-state $\iff \begin{cases} \varphi(S^m S^{*n}) = 0 & \text{for } m \neq n \\ \varphi(S^n S^{*n}) = e^{-n\beta} & \text{for } m = n. \end{cases}$
4. Since $\mathcal{T} = \overline{\text{span}}\{S^m S^{*n}\}$, there is at most one KMS$_\beta$ state for each β.
5. Is there one for each β?
   i.e. does the above condition determine a bona-fide state of $\mathcal{T}$?

Remark: We will see two techniques to deal with the recurring theme of proving that a linear functional is a state.

Example (Existence of KMS_β states for $0 < \beta < \infty$)

1) Spatially: Recall that the dynamics σ has a diagonal Hamiltonian $H\varepsilon_n = n\varepsilon_n$ with respect to the standard basis of $\ell^2(\mathbb{N})$. The partition function $\mathrm{Tr}(e^{-\beta H}) = \sum_n e^{-n\beta} = \frac{1}{1-e^{-\beta}}$ is defined for every $\beta > 0$, and thus $\varphi_\beta(T) = (1 - e^{-\beta})\,\mathrm{Tr}(Te^{-\beta H})$ is a KMS_β state.

Exercise: Verify that φ_β satisfies $\begin{cases} \varphi(S^m S^{*n}) = 0 & \text{for } m \neq n \\ \varphi(S^n S^{*n}) = e^{-n\beta} & \text{for } m = n. \end{cases}$

2) Via the conditional expectation onto $\mathcal{T}^\sigma$:

Recall the conditional expectation E_σ mapping $\mathcal{T}$ onto the fixed-point algebra $\mathcal{T}^\sigma = \overline{\mathrm{span}}\{S^n S^{*n} : n \in \mathbb{N}\}$ of σ, and recall that $\mathcal{T}^\sigma$ is isomorphic to $C(\mathbb{N} \cup \{\infty\})$. Define a p.l.f. on $C(\mathbb{N} \cup \{\infty\})$ by $P_\beta(\delta_n) := (1 - e^{-\beta})e^{-n\beta}$ and then induce P_β from $\mathcal{T}^\sigma$ up to $\mathcal{T}$ via the conditional expectation:

$$\varphi_\beta(T) = P_\beta \circ E_\sigma(T).$$

Exercise: Verify that this is the same state as above.

Example (KMS$_\beta$ states of the Toeplitz algebra for $\beta = 0$)

- Recall the exact sequence of C*-algebras $0 \to \mathcal{K} \to \mathcal{T} \to C(\mathbb{T}) \to 0$ where $\mathcal{K}$ is the ideal of compact operators, obtained as the closed linear span of the elements $S^m(1 - SS^*)S^{*n}$.
- From the KMS$_0$ condition, $\varphi(SS^*) = \varphi(S^*S) = 1$ and hence $\varphi(S^m(1 - SS^*)S^{*n}) = 0$ (this requires the Cauchy -Schwarz inequality), so a KMS$_0$ state φ vanishes on $\mathcal{K}$ and must be a lifting from a state of $C(\mathbb{T})$.
- States of $C(\mathbb{T})$ correspond to probability measures on $\mathbb{T}$, but because of the extra assumption of σ-invariance, only (normalized) Lebesgue measure will do. So there is exactly one KMS$_0$ state of $\mathcal{T}$;
  it is given by

$$\varphi(S^m S^{*n}) = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n. \end{cases}$$

Examples: Cuntz algebras and their Toeplitz extensions

The **Toeplitz-Cuntz algebra** $\mathcal{TO}_n$ is the universal unital C*-algebra generated by isometries $S_1, \ldots, S_n$ with mutually orthogonal ranges.

Let $\mathbf{F}_n^+$ denote the free monoid on n generators, then $\mathcal{TO}_n$ is faithfully represented on $\ell^2(\mathbf{F}_n^+)$ via $S_j \delta_\mu = \delta_{j\mu}$, where μ is the word $\mu_1 \mu_2 \cdots \mu_k$ whose length k is denoted by $|\mu|$, and $j\mu$ is simply concatenation.

The Cuntz algebra $\mathcal{O}_n$ is the quotient of $\mathcal{TO}_n$ by the ideal generated by the projection $1 - S_1 S_1^* - \cdots - S_n S_n^*$. It is universal for n isometries satisfying $\sum_{i=1}^n S_i S_i^* = 1$.

The gauge action on $\mathcal{TO}_n$ (and on $\mathcal{O}_n$) is the dynamics defined by

$$\sigma_t(S_j) = e^{it} S_j, \quad j = 1, \ldots, n.$$

The elements $S_\mu S_\nu^* = S_{\mu_1} \ldots S_{\mu_k} S_{\nu_l}^* \ldots S_{\nu_1}^*$ and the identity (which corresponds to the empty word) span a dense *-subalgebra of $\mathcal{TO}_n$, and are σ-analytic because $\sigma_t(S_\mu S_\nu^*) = e^{it(|\mu| - |\nu|)} S_\mu S_\nu^*$.

Examples: KMS states of Toeplitz-Cuntz algebras

Suppose φ is a KMS$_\beta$ state of $\mathcal{TO}_n$

- if $|\mu| \neq |\nu|$, then $\varphi(S_\mu S_\nu^*) = 0$ by σ-invariance.
- If μ and ν are finite words, then $\varphi(S_\mu S_\nu^*) =$

$$= \varphi(S_{\mu_2} \dots S_{\mu_k} S_{\nu_k}^* \dots S_{\nu_1}^* \sigma_{i\beta}(S_{\mu_1})) = \delta_{\mu_1,\nu_1} e^{-\beta} \varphi(S_{\mu_2} \dots S_{\mu_k} S_{\nu_k}^* \dots S_{\nu_2}^*).$$

Repeating the process we see something stronger than σ-invariance:

- $$\varphi(S_\mu S_\nu^*) = \begin{cases} e^{-|\mu|\beta} & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu. \end{cases}$$
- $\overline{\text{span}}\{S_\mu S_\mu^* : \mu \in \mathbf{F}_n^+\}$ is a commutative C*-algebra with spectrum $\Omega_n =$ compactification of (finite) path space $\mathbf{F}_n^+$.
- There is a canonical (dual) coaction of the free group $\mathbf{F}_n$ on $\mathcal{TO}_n$, and $\overline{\text{span}}\{S_\mu S_\mu^* : \mu \in \mathbf{F}_n^+\} \cong C(\Omega_n)$ is its fixed point algebra.
- The KMS state factors through the corresponding conditional expectation $E : \mathcal{TO}_n \to C(\Omega_n)$ determined by $E : S_\mu S_\nu^* \mapsto \delta_{\mu,\nu} S_\mu S_\mu^*$, and is thus more symmetric than one would expect from σ-invariance.

Examples: KMS states of Toeplitz-Cuntz algebras

- If a KMS_β state φ exists, it is uniquely determined by the values
$$\varphi(S_\mu S_\nu^*) = \begin{cases} e^{-|\mu|\beta} & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu. \end{cases}$$
- Since $0 \leq \varphi(1 - \sum_{j=1}^n S_j S_j^*) = 1 - ne^{-\beta}$, we must have $\beta \geq \log n$, and in the case of $\mathcal{O}_n$ we must have equality.

Do such states exist?

- It is not difficult to construct the unique state φ_β that satisfies the above condition by inducing the probability measure supported on the finite paths in Ω_n given by $P_\beta(\delta_\mu) = (1 - ne^{-\beta})e^{-\beta|\mu|}$ through the conditional expectation $E : \mathcal{TO}_n \to C(\Omega_n)$: $\quad \varphi_\beta = P_\beta \circ E$.

In the case of $\mathcal{O}_n$ a KMS_β state φ exists only for $\beta = \log n$. It is induced from a measure on Ω_n supported on the boundary $\{1, 2, \cdots n\}^\infty$ of Ω_n. The probability that gives rise to the unique KMS_β state is the product of the uniform distribution $p_j = 1/n$ for $j = 1, \cdots, n$.

Summarizing, we have:

Theorem (Olesen-Pedersen, Evans)

- *For $\beta = \log n$ there exists a unique σ-KMS$_\beta$-state on $\mathcal{O}_n$; there are no KMS$_\beta$-states for $\beta \neq \log n$.*
- *For each $\beta \geq \log n$ there exists a unique σ-KMS$_\beta$-state on $\mathcal{T}\mathcal{O}_n$; there are no KMS$_\beta$-states for $\beta < \log n$.*

- In the standard representation of $\mathcal{T}\mathcal{O}_n$ on $\ell^2(\mathbf{F}_n^+)$ the dynamics has a diagonal Hamiltonian: $H\varepsilon_\mu = |\mu|\varepsilon_\mu$.
- The partition function is $\mathrm{Tr}(e^{-\beta H}) = \sum_{\mu \in \mathbf{F}_n^+} e^{-|\mu|\beta} = \frac{1}{1-ne^{-\beta}}$ and is defined for every $\beta > \log n$.
- The state φ_β is of type I for $\beta > \log n$ and of type III$_{1/n}$ for $\beta = \log n$.

Remark: The KMS$_\beta$ state of $\mathcal{O}_n$ was originally obtained as $\varphi = \tau \circ E_\sigma$ where τ is the unique tracial state on the fixed point algebra $\mathcal{O}_n^\sigma$, which is the UHF-algebra of type n^∞, via the corresponding conditional expectation

$$E_\sigma : \mathcal{O}_n \to \mathcal{O}_n^\sigma, \quad E_\sigma(a) = \frac{1}{2\pi}\int_0^{2\pi} \sigma_t(a)dt.$$

Examples: Cuntz-Krieger algebras

Let A be an $n \times n$ matrix of zeros and ones having no zero rows. The Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal C*-algebra generated by partial isometries s_k for $k = 1, 2, \ldots n$, (*partial isometry* means $ss^*s = s$) such that

$$\text{(CK1):} \;\; 1 = \sum_j s_j s_j^* \qquad \text{and} \qquad \text{(CK2):} \;\; s_k^* s_k = \sum_j A(k,j) s_j s_j^*.$$

We define a time evolution σ on $\mathcal{O}_A$ by $\sigma_t(s_j) = e^{it} s_j$.

Theorem (Enomoto-Fujii-Watatani)

A KMS_β state on $\mathcal{O}_A$ exists iff there exists a non negative vector v such that $Av = e^\beta v$.
If A is irreducible, this happens only for $\beta = \log r_A$, where r_A = spectral radius of A and the KMS_β state φ_β is unique and determined by $v = \{\varphi_\beta(s_j s_j^*)\}_{j=1}^n$ = normalized Perron-Frobenius eigenvector corresponding to the largest eigenvalue e^β of A.

As is customary, when $\mu = \mu_1\mu_2 \cdots \mu_n$ is a finite word in the symbols $\{1, 2, \ldots n\}$ we write s_μ for the product $s_{\mu_1} s_{\mu_2} \cdots s_{\mu_n}$.

hw project: prove the E-F-W theorem

here are the key steps.

1. Show that the elements $s_\mu s_\nu^*$ are analytic and have dense linear span.
2. Identify the vector v in the E-F-W theorem in terms of the values of a state on convenient expressions in the s_j, and use (CK2) to show that the condition $Av = e^\beta v$ is necessary if the state is KMS_β.
3. Prove that the condition is also sufficient for KMS_β (this is a bit harder).
4. When the matrix A is irreducible apply the Perron-Frobenius Theorem to get the uniqueness result.

□

We have seen several examples,

Finite quantum systems: $M_n(\mathbb{C})$; $\sigma(A) = e^{itH} a e^{-itH}$;
Gibbs state $\varphi_G(a) = \frac{1}{\mathrm{Tr}(e^{-\beta H})} \mathrm{Tr}(ae^{-\beta H})$ partition function $\mathrm{Tr}(e^{-\beta H})$

Toeplitz system: $C^*(S)$, (S = shift); $\sigma_t(S) = e^{it}$; φ_β induced from geometric density $(1 - e^{-\beta})e^{-\beta n}$; partition function $\frac{1}{1-e^{-\beta}}$

Toeplitz-Cuntz system: $\mathcal{TO}_n = C^*(S_1, S_2, \cdots, S_n \,|\, S_k^* S_j = \delta_{k,j} 1)$; $\sigma_t(S_j) = e^{it} S_j$; a KMS_β state exists for each $\beta \geq \log n$; φ_β induced from probability measure on rooted $(n+1)$-tree with density $(1 - ne^{-\beta})e^{-\beta|\mu|}$ (μ a path of length $|\mu|$), partition function $= \frac{1}{1-ne^{-\beta}}$.

Toeplitz-Cuntz-Krieger system: (ok, we haven't really seen this one, only the [E-F-W] theorem for $\mathcal{O}_A$, but the T-C-K system is similar to the T-C system except that one has a restricted tree of A–admissible paths, [L. Exel, Comm. Math. Phys. 2003]);

Before introducing the systems form number theory, we need some basic constructions from number theory.

the p-adic integers $\mathbb{Z}_p$ via Hensel series

Let p be a prime number. Every positive integer can be written in a unique way as

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k \quad \text{with } a_j \in \{0, 1, 2, \ldots, (p-1)\}.$$

If we now allow formal infinite sums, or *Hensel series*

$$z = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k + \cdots ,$$

and we define sums and products of sequences by mimicking what happens with the finite sums (i.e. with carry-over to the right), then we obtain a compact ring which is usually denoted $\mathbb{Z}_p$ and called the p-adic integers. This way of viewing the infinite product space $\prod_0^\infty \{0, 1, 2, \ldots, (p-1)\}$ is very convenient because the series in powers of p remind us of how to add and multiply. As indicated above, the positive integers correspond to finite expansions. *Exercise: Find the Hensel series of* -1.

$\mathbb{Z}_p$ as a completion of $\mathbb{N}$ and as a projective limit

- $\mathbb{Z}_p$ can also be defined as the completion of $\mathbb{N}$ under the p-adic absolute value, $|n|_p = p^{-k}$ (where p^k is the highest power of p that divides n). To see this, it suffices to verify that $\mathbb{N}$ embeds isometrically in $\mathbb{Z}_p$ as the finite Hensel series, which are dense.
- For each k consider the finite ring $\mathbb{Z}/p^k$ of integers modulo p^k. If $k \leq j$ then *reduction modulo* p^k determines surjective ring homomorphisms $h_{k,j}$ of $\mathbb{Z}/p^j$ to $\mathbb{Z}/p^k$, and produces a projective system

$$\cdots \mathbb{Z}/p^k \to \mathbb{Z}/p^{k-1} \to \cdots \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

By definition $(\operatorname{proj}\lim_j \mathbb{Z}/p^j)$ is the subset of $\prod_j (\mathbb{Z}/p^j)$ consisting of sequences $\{a_j\}$ such that $h_{k,j}(a_j) = a_k$ whenever $k \leq j$. This gives homomorphisms $h_{k,\infty} : (\operatorname{proj}\lim_j \mathbb{Z}/p^j) \to \mathbb{Z}/p^k$ such that when $k \leq j$ $h_{k,j} \circ h_{j,\infty} = h_{k,\infty}$.

Exercise: show that the three definitions of $\mathbb{Z}_p$ (Hensel series, p-adic completion, projective limit) yield the same object.

The dual group of $\mathbb{Z}_p$

- Denote by $\frac{1}{p^k}\mathbb{Z}/\mathbb{Z}$ the group of rationals with denominator p^k, taken modulo $\mathbb{Z}$.
- If $r \in \frac{1}{p^k}\mathbb{Z}/\mathbb{Z}$ and $z \in \mathbb{Z}/p^k$, it makes sense to define a pairing $\langle z, r\rangle := \exp 2\pi irz$ (?!) and the map $r \mapsto \langle \cdot, r\rangle$ gives a concrete realization of the dual of the additive group $\mathbb{Z}/p^k$.
- These pairings (for each $k \in \mathbb{N}$) are compatible with the projective system $(\mathbb{Z}/p^k)_{k\in\mathbb{N}}$ and with the injective system $(\frac{1}{p^k}\mathbb{Z}/\mathbb{Z})_{k\in\mathbb{N}}$.
- The direct limit of the injective system is simply the group $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = \bigcup_k \frac{1}{p^k}\mathbb{Z}/\mathbb{Z}$ of rationals with denominator a power of p, taken modulo $\mathbb{Z}$.

The duality established between $\mathbb{Z}/p^k$ and $\frac{1}{p^k}\mathbb{Z}/\mathbb{Z}$ gives a duality between the respective limits (the dual of an inverse limit is the direct limit of duals), and we conclude that

$$\mathbb{Z}_p \quad \text{is in duality with} \quad \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}$$

through the pairing

$$\langle z, r\rangle = \exp 2\pi i r z$$

(*Exercise: make sure this makes sense to you*).

Specifically, the map $z \in \mathbb{Z}_p \mapsto \langle z, \cdot\rangle = \exp(2\pi i \cdot z)$ gives an isomorphism of compact groups

$$\mathbb{Z}_p \cong (\mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z})\hat{}.$$

The ring of integral adeles as an inverse limit.

When $m|n$ let $r_{m,n} : \mathbb{Z}/n \to \mathbb{Z}/m$ be the ring homomorphism given by reduction modulo m. The multiplicative order on $\mathbb{N}^\times$ is not total but it is directed (given k and m take $n = km$ to get an element that follows both k and m). These connecting maps are coherent in the sense that if $k|m|n$, then $r_{k,m} \circ r_{m,n} = r_{k,n}$, so

$$\{\mathbb{Z}/m : m \in \mathbb{N}^\times\}$$

is an inverse system of rings indexed by the multiplicatively ordered semigroup $\mathbb{N}^\times$. The inverse limit

$$\widehat{\mathbb{Z}} = \varprojlim_m (\mathbb{Z}/m)$$

is thus a compact (profinite) ring, called the ring of **finite adeles**.

The multiplicative order in $\mathbb{N}^\times$ is not linear, but it is directed, and the technical definition of inverse limit is the usual one: $\widehat{\mathbb{Z}}$ consists of sequences $(a_n)_{n\in\mathbb{N}^\times}$ such that $a_n \in \mathbb{Z}/n$ for each n and $a_m = r_{m,n}a_n$ whenever $m|n$.

This tells us how to add and multiply in $\widehat{\mathbb{Z}}$, and it also tells us that $\mathbb{Z}$ embeds as a dense subring of $\widehat{\mathbb{Z}}$: for $z \in \mathbb{Z}$ choose $a_n = z \pmod{n}$.

The inverse limit can also be characterized (up to canonical isomorphism) by a universal property.

The ring of integral adeles as a product.

Let $n = \prod_p p^{v_p(n)}$ be the prime factorization of $n \in \mathbb{N}^\times$.
The Chinese Remainder Theorem gives a decomposition

$$\mathbb{Z}/n = \prod_p \mathbb{Z}/p^{v_p(n)}$$

As n tends multiplicatively to infinity, all the $v_p(n)$ go to infinity, and taking limits on both sides gives

$$\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p.$$

Recall that $\mathbb{Z}_p = (\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})\hat{}$.
Exercise (due now): guess what the Pontryagin dual of $\widehat{\mathbb{Z}}$ *is. Or rather, to keep the hats apart guess the group of which* $\widehat{\mathbb{Z}}$ *is the dual.*

Using the pairing of the inverse system $\{\mathbb{Z}/n : n \in \mathbb{N}^\times\}$ giving rise to $\widehat{\mathbb{Z}}$ to the directed system $\{\frac{1}{n}\mathbb{Z}/\mathbb{Z} : n \in \mathbb{N}^\times\}$ giving rise to $\mathbb{Q}/\mathbb{Z}$, one proves that

$$\widehat{\mathbb{Z}} = (\mathbb{Q}/\mathbb{Z})\hat{}$$

Sorry about this, but the $\widehat{\ }$ on the left denotes the adeles, while the ()ˆ on the right indicates the Pontryagin dual, i.e. the continuous homomorphisms (of $\mathbb{Q}/\mathbb{Z}$ in this case) into the circle group.

invertibles and zero divisors in $\widehat{\mathbb{Z}}$

- The invertible elements of the ring $\widehat{\mathbb{Z}}$ are the integral ideles:

$$\widehat{\mathbb{Z}}^* = \varprojlim (\mathbb{Z}/n)^* = \prod_p \mathbb{Z}_p^*.$$

- Notice that $z \in \mathbb{Z}_p$ is invertible if and only if its first Hensel coefficient is nonzero (in which case long division is possible and gives the inverse), so
$(z_p)_{p\in\mathcal{P}} \in \prod_{p\in\mathcal{P}} \mathbb{Z}_p$ is invertible iff $(z_p)_0 \neq 0$ for all p (equivalently $z_p \notin pZ_p$) for all p.

- $\widehat{\mathbb{Z}}$ has lots of zero divisors:
$(z_p)_{p\in\mathcal{P}} \in \prod_{p\in\mathcal{P}} \mathbb{Z}_p$ is a zero divisor iff $z_p = 0$ for some p.