Stochastic Parameterisation Schemes Based on Rigorous Limit Theorems

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- Most stochastic climate models are specific to the modeled system, but system-specific SDE models in particular implicitly apply limit theorems, which give general formulas

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$\rightarrow$ the equations of motion are linearised about a mean state,

$$\frac{dX}{dt} = LX,$$

white noise is added to account for the (fast-evolving) error in linearisation, and a damping term is added for stability:

$$\frac{dX}{dt} = (L + D)X + \frac{dW}{dt}$$

Theorem-based reduction methods

Hasselmann(1976)

Kurtz(1973) $\longrightarrow$ MTV(1999-2006) $\longrightarrow$ Franzke et al.(2005)

Papanicolaou(1976)

Khasminskii(1966)

Hasselmann(1976) $\longrightarrow$ Arnold et al.(2003)

Fatkullin and Vanden-Eijnden(2004)

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$$\frac{dx}{dt} = f(x,y) \quad \text{(slow climate mode)}$$

$$\frac{dy}{dt} = \frac{1}{\epsilon} g(x,y) \quad \text{(fast weather mode)}$$

As $\epsilon \rightarrow 0$, $x \rightarrow X$ in distribution, where X satisfies:

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$\star$ **Simple to implement: do not have to resolve fast mode,** y

Online closure

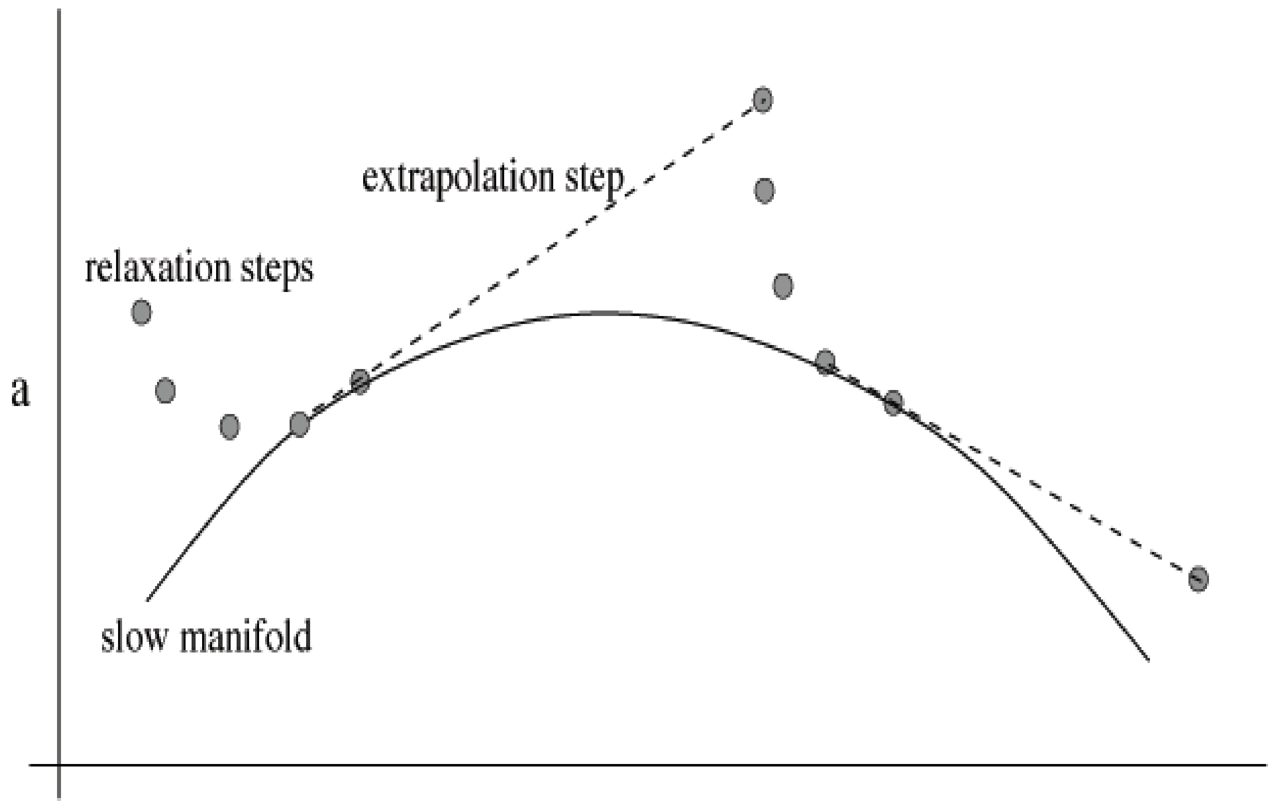

Figure 3: Schematic illustration of the projective integration scheme

. . . ⟶Franzke et al.(2005)

$$\frac{dx}{d\tau} = \frac{1}{\epsilon} f_1(x,y) + f_0(x,x)$$
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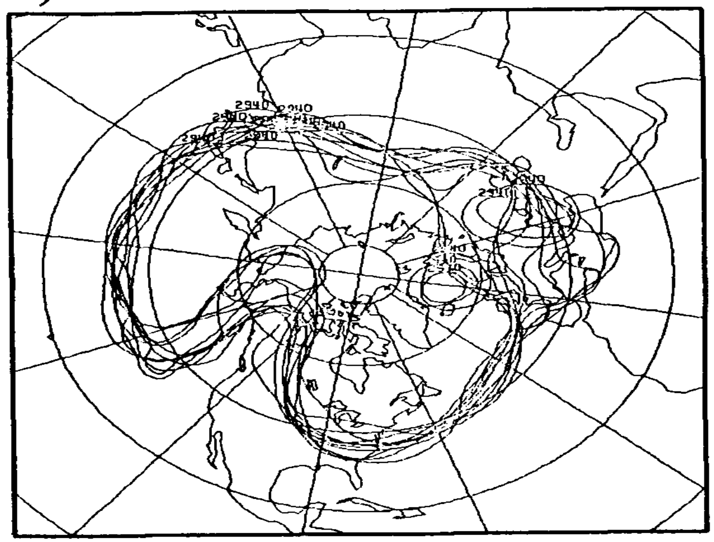
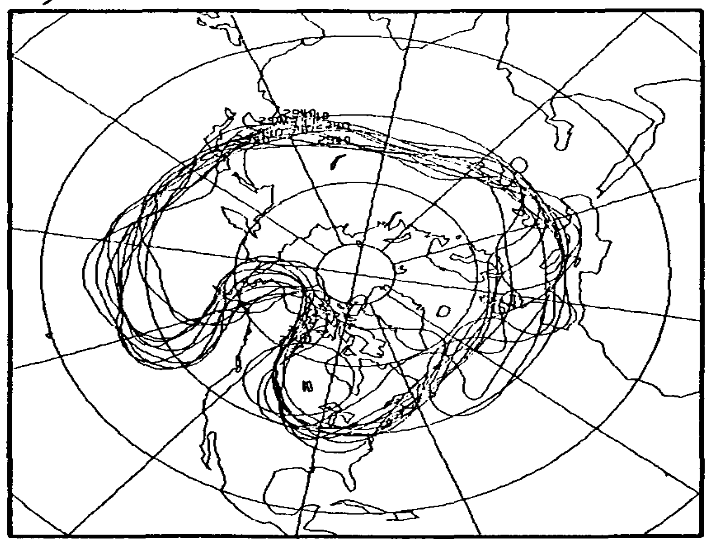
⋆ **the existence of limiting slow dynamics**

Atmospheric low-frequency variability (LFV)

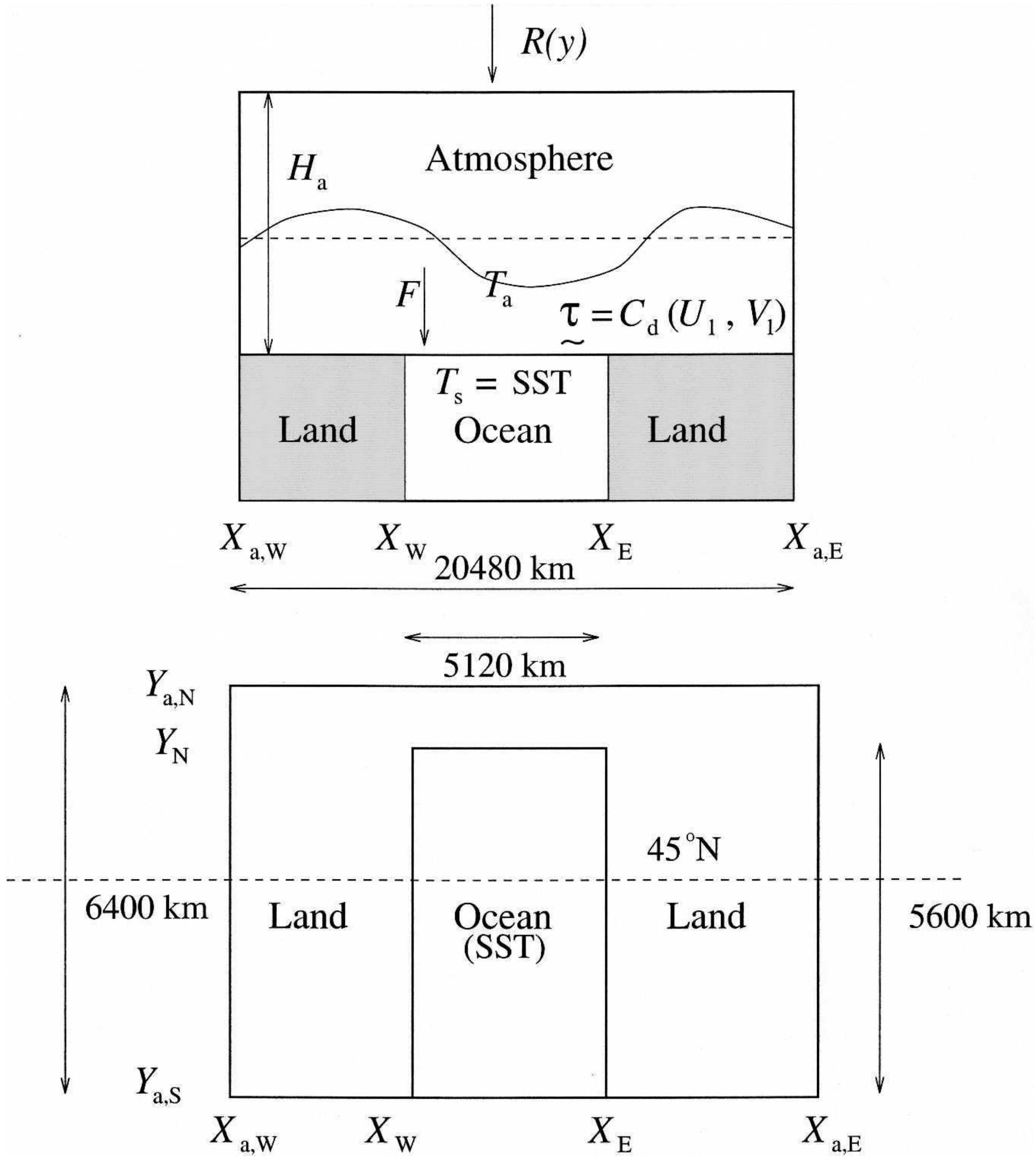
a) 2940M 61 1 20 - 61 1 29 0Z



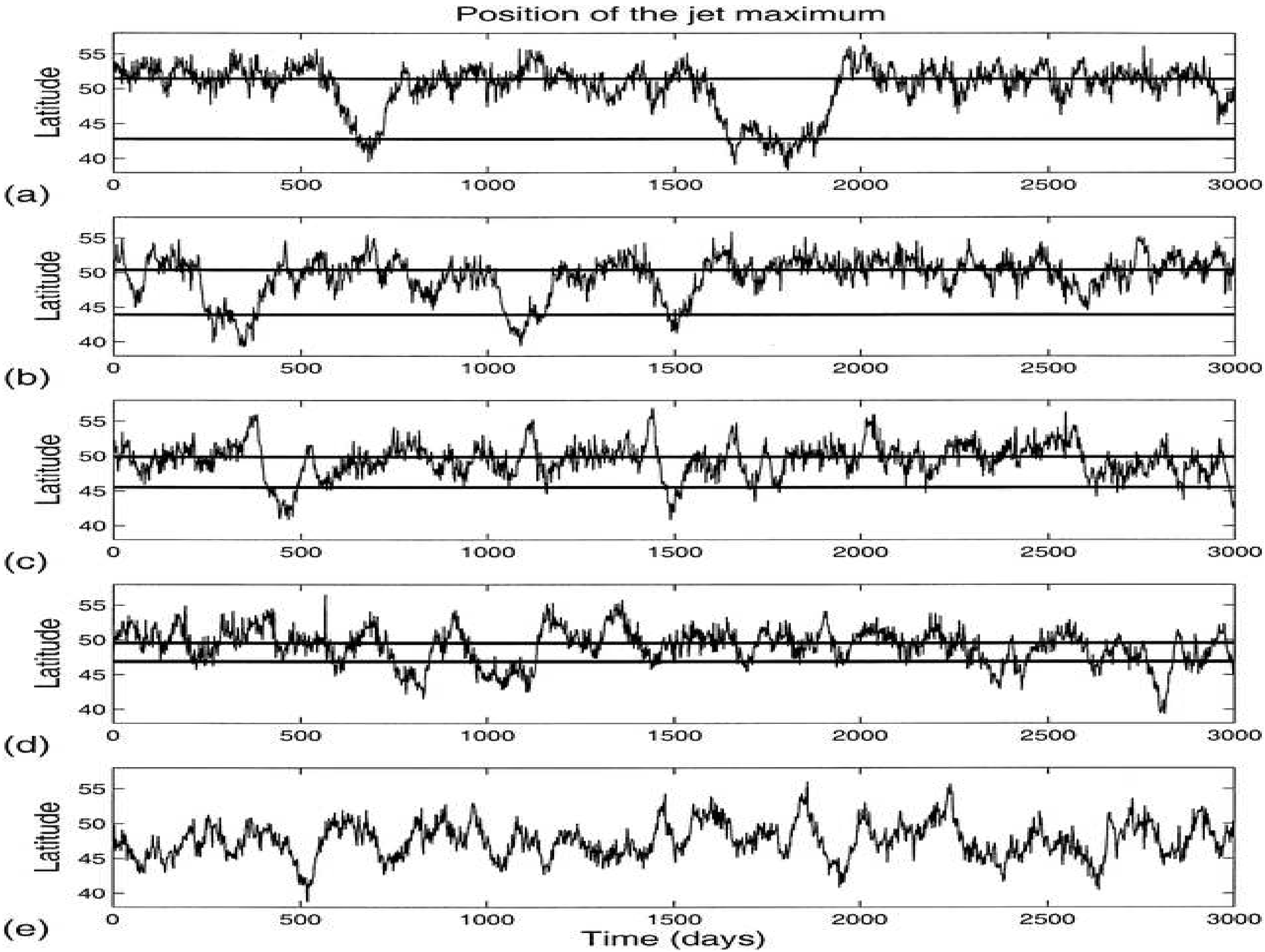
b) 2940M 78 12 31 - 79 1 9 0Z



QG model of LFV (Kravtsov et al. (2005))



Bifurcation: unimodal to bimodal distribution of jet axis



Existence of limiting slow dynamics

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= \frac{1}{\delta\epsilon} g(x,y)\end{aligned}$$

- The fast term is up-scaled by suitable choice of averaging parameters in Hasselmann's deterministic averaging equation

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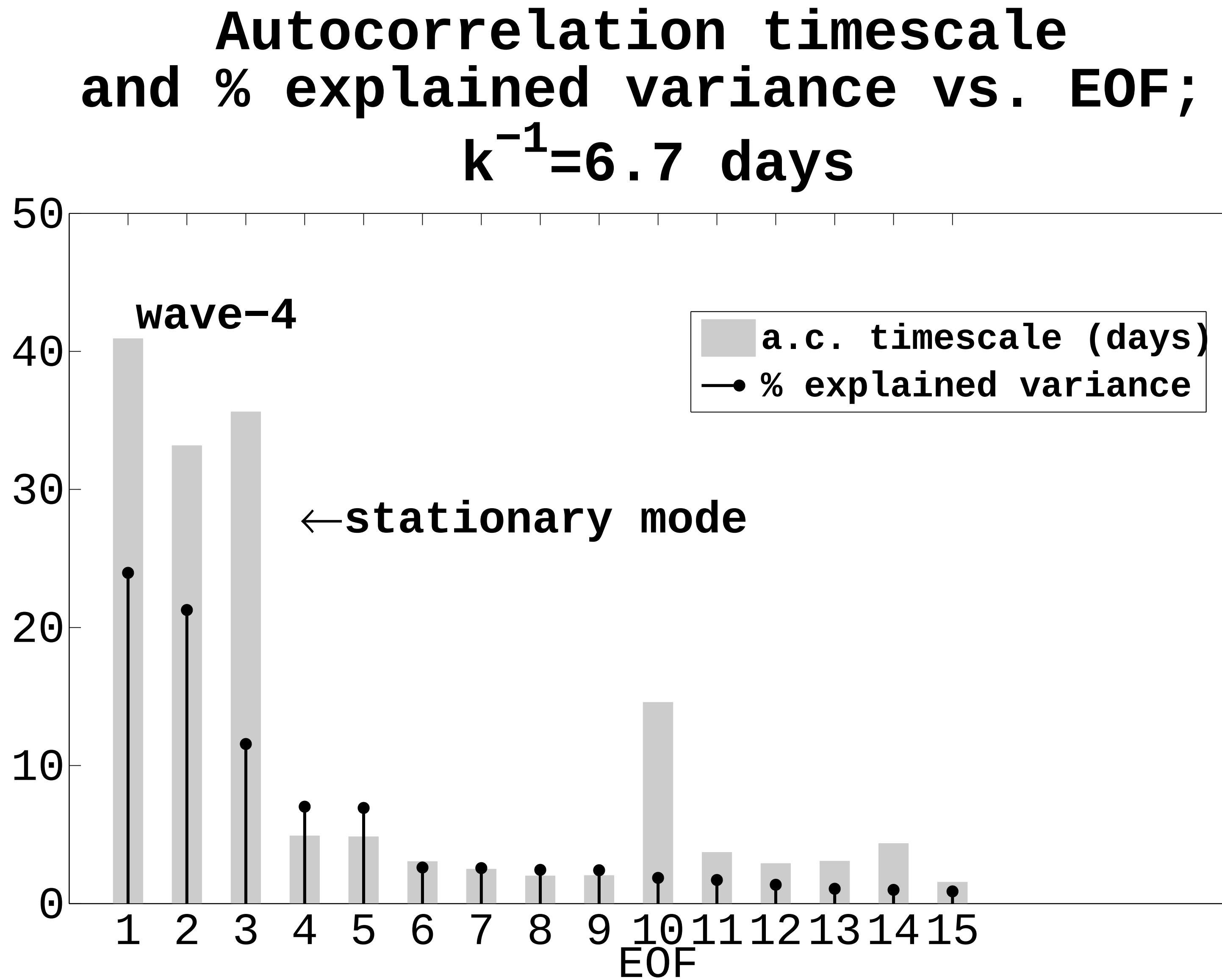
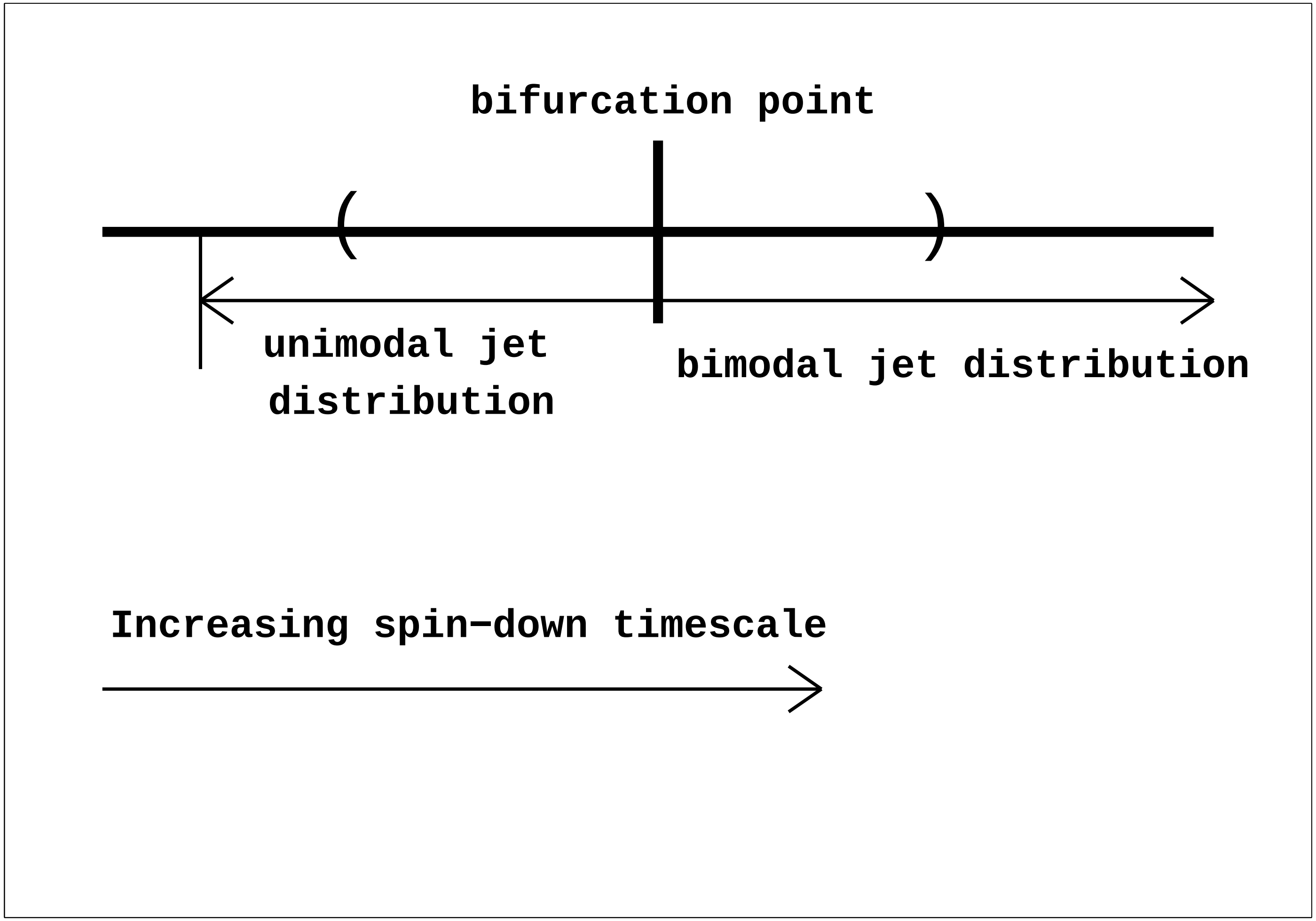
- Speeding up the fast mode is equivalent to changing the bifurcation parameter (the bottom drag parameter)
- In large neighbourhood of bifurcation point, **not every set of slow variables has limiting slow dynamics**

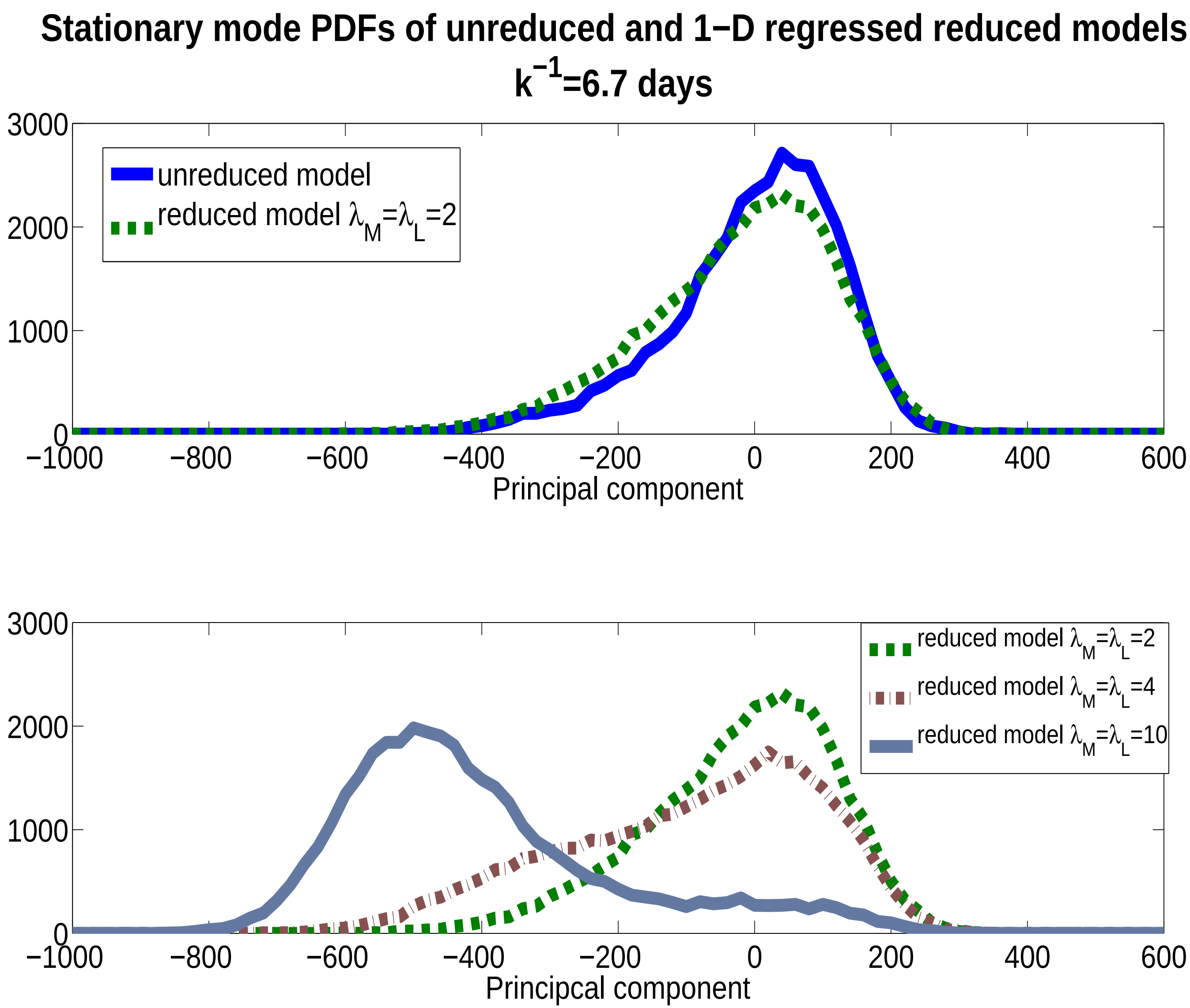
EOFs in region of jet bimodality



Schematic of KRG05 dynamics



Method of Franzke et al.(2005); 1-D SDE



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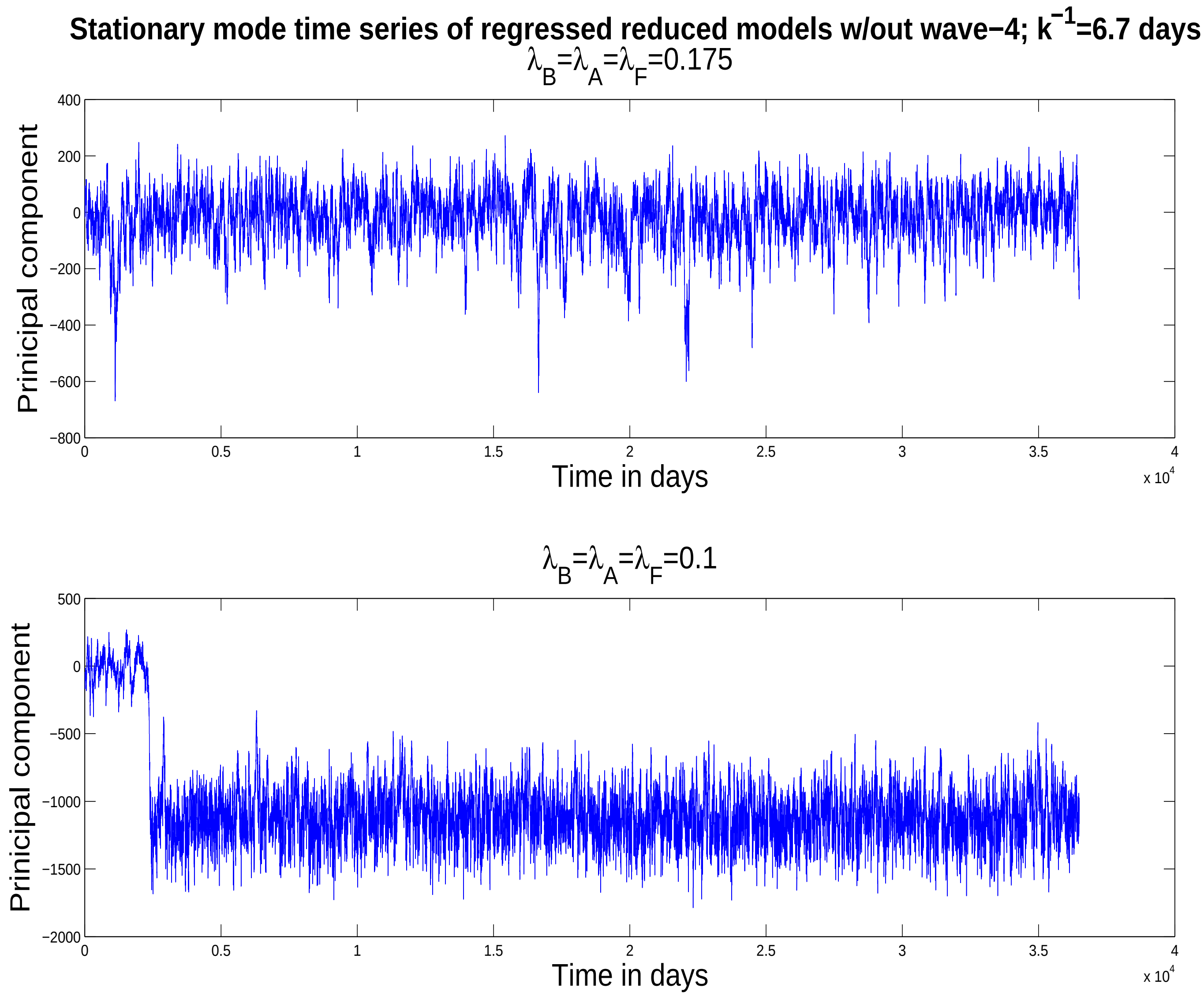
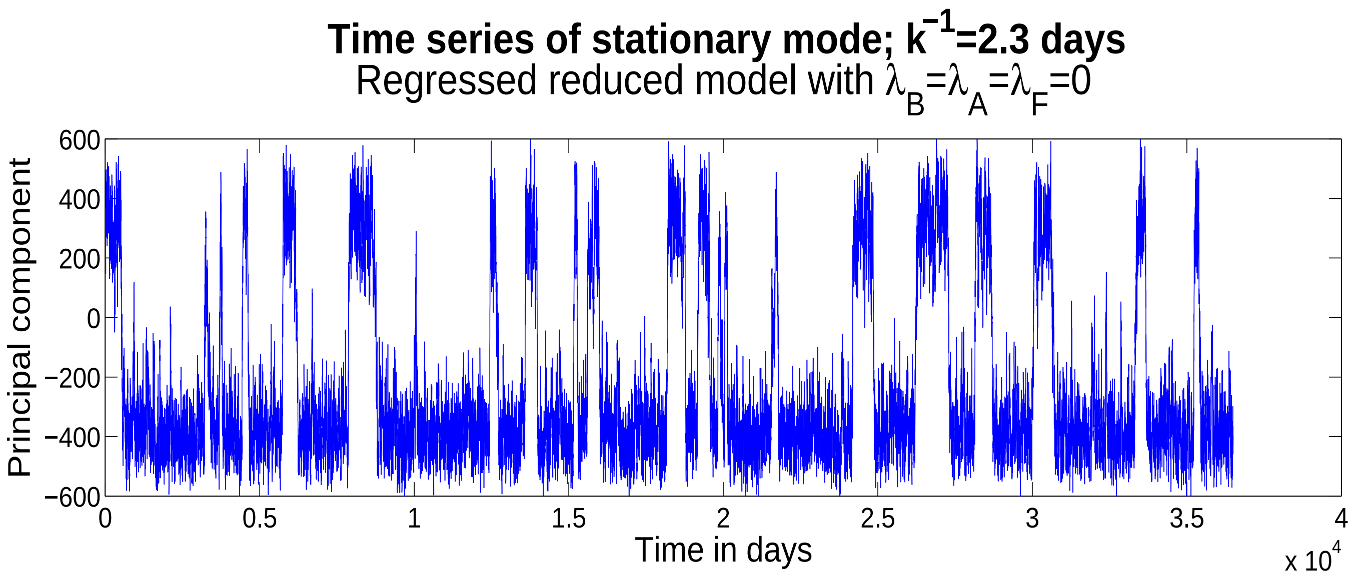
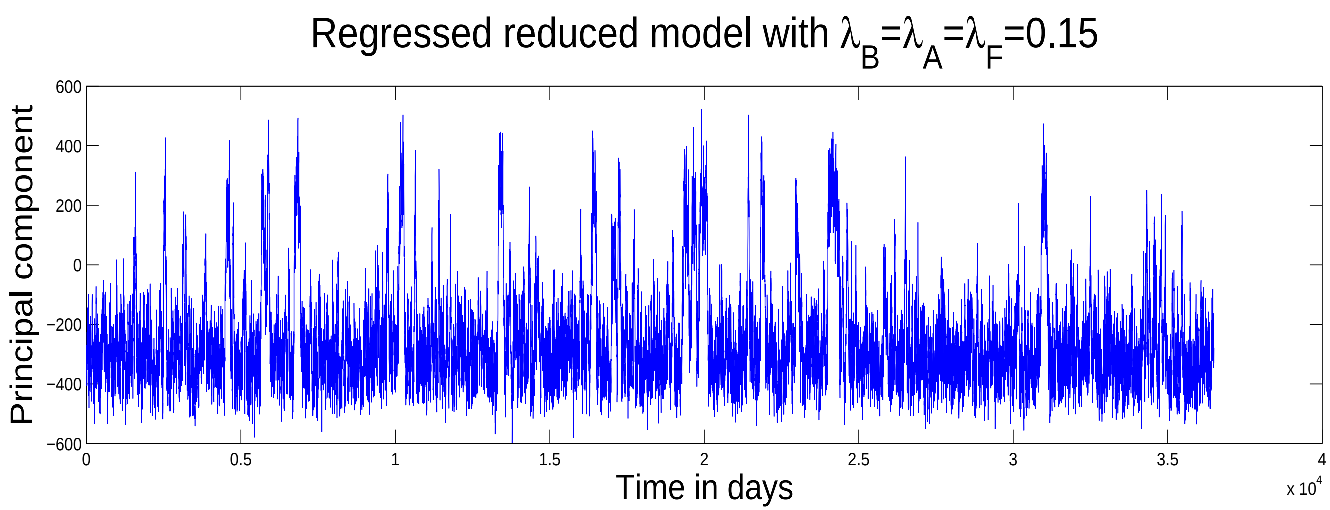
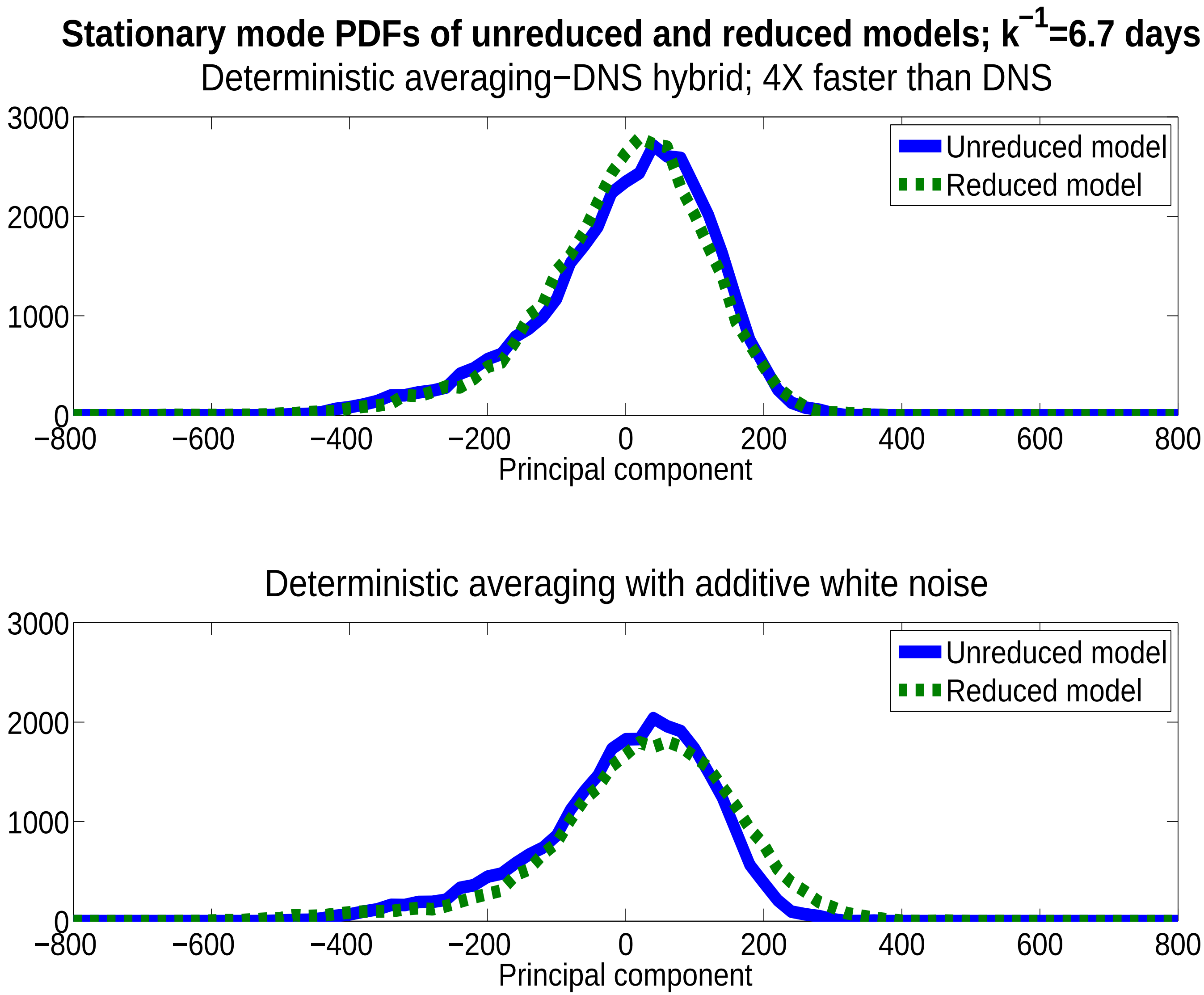


Method of Franzke et al.(2005); 3-D SDE





Hasselmann’s method



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→ leading fast synoptic eddies are of first-order importance and wave-4 facilitates transitions between states

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- For larger models, off-line calculations would be impractical without further simplifications

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INDIANA JONES
and the
IVORY TOWER
WELL, HENRY, YOUR PERFORMANCE REVIEW IS IN—AND IT DOESN'T LOOK TOO GOOD.
IT SEEMS YOU HAVEN'T BEEN PUBLISHING ENOUGH ARTICLES IN REFERRED JOURNALS LATELY...
THIS MIGHT HAVE SOMETHING TO DO WITH YOUR REPEATED AND SUDDEN "LEAVES OF ABSENCE."
YES - ALSO, YOUR THEORETICAL FRAMEWORK IS COMPLETELY OUT OF DATE. ONE WOULD THINK YOU HAVEN'T EVEN HEARD OF LEVI-STRAUSS?
THE DENIM GUY?
WE'VE ALSO HAD REPORTS OF INAPPROPRIATE CONTACT WITH AN UNDERGRADUATE.
WELL, YOU KNOW...
OH, WE DO, DR. JONES. I CAN ASSURE YOU, WE DO.
BUT- SHE PAINTED "I LOVE YOU" ON HER EYELIDS!
OH, SO THE YOUNG WOMAN WAS "ASKING FOR IT"?!?
LADY—THAT "YOUNG WOMAN" SHOWED ME A THING OR TWO.
WITH THE WHIP, I MAY ADD.
THIS IS POINTLESS. IT IS CLEAR TO US THAT MAYBE THIS INSTITUTION AND YOU ARE NOT THE BEST FIT.
WHAT?! LISTEN —
NO - YOU LISTEN - YOU WILL BE TAKING ONE OF YOUR "LEAVES OF ABSENCE" EFFECTIVE IMMEDIATELY—AND WE'LL SEE WHAT WE CAN DO ABOUT YOUR CRUMBLING CAREER.
DON'T STRESS TOO MUCH - I HEAR THERE'S A PART-TIME LECTURESHIP AT NEBRASKA STATE ...
YOU'RE AN OUTDOORSMAN, AREN'T YOU?
THE IVY LEAGUE IS NOT FOR EVERYONE, DR. JONES.
WE WISH YOU THE VERY BEST.
SOUTH DAKOTA
IOWA
NEBRASKA
COLORADO
KANSAS