COMMODITIES: INCOMPLETE MARKETS

Roger J-B Wets

PIMS, . . . Managing Risk, Natural Resources

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1. Pure Exchange
   - Deterministic model
   - Uncertain environment
2. Incomplete markets
   - Framework
   - Equilibrium
3. Numerical Approach
4. Firms and production

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Classical Arrow-Debreu Model

- $\mathcal{E}$ = exchange of goods $\in R^L$
- (economic) agents: $i \in \mathcal{I}$, $|\mathcal{I}|$ finite
  consumption by agent i: $x_i \in R^L$
  endowment: $e_i \in R^L$
  utility: $u_i : R^L \to [-\infty, \infty)$,
  survival set: $X_i = \text{dom}\, u_i = \{x_i \,|\, u_i(x_i) > -\infty\}$
- exchange at market prices: p
- i-budgetary constraint: $\langle p, x_i \rangle \leq \langle p, e_i \rangle$
- *market clearing*: $\sum_{i\in\mathcal{I}} x_i^* \leq \sum_{i\in\mathcal{I}} e_i$

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The agents: $i \in \mathcal{I}$, $|\mathcal{I}|$ finite

1. information: present state & all potential future states $s \in S$
2. beliefs: agent-i assigns ‘probability’ $b_i(s)$ to (future) state s
3. consumption: $(x_i^0, \boldsymbol{x}_i^1) = (x_i^0, (x_i(s), s \in S))$
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4. delivery contracts (commodities) z_k $[= (z_k^+, z_k^-)]$
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Agent's decisions & resources

- decision criterion: $U_i(x_i^0, \boldsymbol{x}_i^1)$

  for example: $\max u_i^0(x_i^0) + E_i\{u_i^1(\boldsymbol{s}, x_i^1(\boldsymbol{s}))\}$
  $= \max u_i^0(x_i^0) + \sum_{s \in S} b_i(s) u_i^1(s, x_i^1(s)),$
- survival set (feasible consumption): $X_i = \operatorname{dom} U_i$
  $= \{x_i^0, (x_i^1(s), s \in S)) \mid U_i(x_i^0, (x_i^1(s), s \in S)) > -\infty\}$
- U_i usc and concave $\Longrightarrow X_i$ convex, $\not\Longrightarrow X_i$ closed
- U_i increasing $\Longrightarrow X_i + [\, R_+^n \times (R_+^n)^S \,] \subset X_i$, int $X_i \neq \emptyset$,
- insatiability: $\forall (x_i^0, \boldsymbol{x}_i^1) \in X_i$,
  $\exists\, (\tilde{x}_i^0, \boldsymbol{x}_i^1)$ with $U(x_i^0, \boldsymbol{x}_i^1) < U(\tilde{x}_i^0, x_i^1)$
  $\forall\, s \in S,\ \exists\, \tilde{x}_i^1(s)$ such that $U(x_i^0, \boldsymbol{x}_i^1) < U(x_i^0, \tilde{\boldsymbol{x}}_i^1)$.
- endowments: $e_i^0, (e_i^1(s), s \in S) = (e_i^0, \boldsymbol{e}_i^1)$
- *strict survivability* (assumption): $(e_i^0, \boldsymbol{e}_i^1) \in \operatorname{int} X_i,\ i \in \mathcal{I}$

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Real assets: Shifting resources

1. real assets = contracts for <u>delivery</u> of goods
2. contract types $k = 1, \ldots, K$ @ price q_k, bought or sold
3. $D_k(s, p^1(s)) \geq 0$ delivery in state 's' per unit of contract k
   $D_{k,l}(s, p^1(s)) > 0$ some state $s \in S$ some good 'l'
4. Delivery matrix: $D(s, p^1(s)) = [\cdots D_k(s, p_1(s)) \cdots]$
   & some agent is l-insatiable in state s
5. dependence on $p^1(s)$ via price <u>ratios</u>
   $D(s, \lambda p^1(s)) = D(s, p^1(s))$ insensitive to price scaling
6. $p^1(s) \mapsto D_k(s, p^1(s))$ continuous

Not included **for now**: equity contracts
cf. later 'firms and production'

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Example: price-based option ($\approx$ exotic option)

- depends on goods l_0, l_1 and $0 < \kappa < \kappa' < \infty$
  say l_0 = \$, l_1 a commodity (pork bellies)
- contract k delivers in l_0-units,
  depends on $\eta(s) = p^1_{l_1}(s)/p^1_{l_0}(s)$

$$D_{k,l_0}(s) = \begin{cases} 0 & p^1_{l_0}(s) > 0 \text{ and } \eta(s) \leq \kappa \\ p^1_{l_1}(s)/p^1_{l_0}(s) - \kappa & p^1_{l_0}(s) > 0 \text{ and } \kappa \leq \eta(s) \leq \kappa' \\ \kappa' - \kappa & p^1_{l_0}(s) > 0 \text{ and } \kappa' \leq \eta(s) \\ & \quad \text{or } p^1_{l_0}(s) = 0 \;\&\; p^1_{l_1}(s) > 0 \end{cases}$$

- Check: $D_{k,l_0}(s, p^1(s)) > 0$ for some s,
  continuous, insensitive to scaling

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Contracts and deliveries

- z_i^+ contract purchases and z_i^- sales of agent-i
- simultaneous buying/selling allowed
  but won't occur! cf. assumptions: $D_{k,l}(s, p^1(s))$
- (z_i^+, z_i^-) generates $D(s, p^1(s))[z_i^+ - z_i^-]$ goods
- time 0: cost $\langle q, z_i^+ - z_i^- \rangle$
- time 1: value $\langle p^1(s), D(s, p^1(s))[z_i^+ - z_i^-] \rangle$
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- $V_k(s, p^1) = \langle p^1, D_k(s, p^1) \rangle$, $V_k(p^1) = (\ldots V_k(s, p^1) \ldots) \in R^{|S|}$
- $V(p^1) = [\,|S| \times K\,]$-matrix
- $W(p^1) = \text{lin}\, V(p^1)$, linear span of $\{V_k(s, p^1)\}$

Financial market is *complete* for p^1 if $W(p^1) = R^{|S|}$

$\forall t \in R^{|S|}, \exists$ portfolio: $V(p^1)[z^+ - z^-] = t$

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Incomplete market's equilibrium?

The best & the brightest: Arrow, Magill & Quinzii, Radner, Shafer, Dubey, Geanakoplos, Shubik, Zame, Stiglitz, . . .
Existence: contracts types, exogenous bounds, or generic

Theorem

Under these assumptions and no delivery requirements, an equilibrium exists under the following assumption,

$$\text{rank } V(p^1) \text{ is constant on } \{p^1 \mid p^1 > 0\}$$

i.e. $p^1_l > 0$ *for all* l*. Or, equivalently* $p^1 \mapsto W(p^1) = \text{lin } V(p^1)$ *is continuous on the positive orthant of* $(R^L)^{|S|}$*.*

Not generic *** Methodology: Variational Analysis rather than Differential Geometry
A constructive approach: via supply guarantees.

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Deliveries

Our approach: ‘endogenous’, requires deliveries take place

1. sale $z^{-}_{i,k}$ means delivering $D_k(s)z^{-}_{i,k}$
   not just $\langle p^1(s), D_k(s)z_{i,k}\rangle$
2. *the total of all promised deliveries of any good l in state s may not exceed the total amount of good l that is available in state s*, i.e., total endowment $\sum_{i\in\mathcal{I}} e^1_{i,l}(s)$
3. leads to a market of *supply guarantees*
4. per unit $r_l(s) \geq 0$: managed by Walrasian broker
   same process that generates (p^0, p^1)

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Guarantees

1. guarantee price vector
   $$\boldsymbol{r} = (r(s), s \in S) \in (R_+^L)^{|S|}, \quad r(s) = (\ldots r_l(s) \ldots)$$
2. portfolio (z_i^+, z_i^-) cost to agent i:
   $\langle q, z_i^+ - z_i^- \rangle + \sum_{s \in S} \langle r(s), D(s, p^1) z_i^- \rangle$
3. implies adjusted budget constraint(s)
4. different market values for ‘long’ & ‘short’ positions
   long value of contract k is q_k
   short value: $q_k - \sum_{s \in S} \langle r(s), D_k(s, p^1) \rangle$
5. yields existence with no restriction on $V(p^1)$ (rank, etc.)
6. like $[(p^0, \boldsymbol{p}^1), q]$, $\boldsymbol{r}$ is defined endogenously
7. $r_l(s) \neq 0$ only if agents’ portfolios threaten a shortage in good l in state s

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   $$\boldsymbol{r} = (r(s), s \in S) \in (R_+^L)^{|S|}, \quad r(s) = (\dots r_l(s) \dots)$$
2. portfolio (z_i^+, z_i^-) cost to agent i:
   $$\langle q, z_i^+ - z_i^- \rangle + \sum_{s \in S} \langle r(s), D(s, p^1) z_i^- \rangle$$
3. implies adjusted budget constraint(s)
4. different market values for 'long' & 'short' positions
   long value of contract k is q_k
   short value: $q_k - \sum_{s \in S} \langle r(s), D_k(s, p^1) \rangle$
5. yields existence with no restriction on $V(p^1)$ (rank, etc.)
6. like $[\,(p^0, \boldsymbol{p}^1), q\,]$, $\boldsymbol{r}$ is defined endogenously
7. $r_l(s) \neq 0$ only if agents' portfolios threaten a shortage in good l in state s

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Admissible prices

Definition

$(p^0, \boldsymbol{p}^1, q, \boldsymbol{r})$ *admissible price system* when

- $p^0 \geq 0, \boldsymbol{p}^1 \geq 0, q \geq 0, \boldsymbol{r} \geq 0$
- $p^0 \neq 0, p^1(s) \neq 0$ for all $s \in S$

Definition

Good l^* can serve as a *numéraire* if $p^0_{l^*} > 0, p^1_{l^*}(s) > 0, \forall\, s \in S$.
Re-scaling so that $p^0_{l^*} = 1, p^1_{l^*}(s) = 1, \forall\, s \in S$
leads to *numéraire prices*.

Agents' optimization problems

Given an admissible price system, agent i solves:

$$\begin{aligned} \max\ & U_i(x_i^0, \boldsymbol{x}_i^1) \qquad \text{subject to} \\ & \langle p^0, x_i^0 - e_i^0 \rangle + \langle q, z_i^+ - z_i^- \rangle \\ & \qquad + \sum_{s \in S} \langle r(s), D(s, p^1(s)) z_i^- - e_i^1(s) \rangle \leq 0 \\ & \forall\, s \in S:\ \langle p^1(s), x_i^1(s) - e_i^1(s) + D(s, p^1(s))[\, z_i^+ - z_i^- \,] \rangle \leq 0, \\ & \qquad (x_i^0, \boldsymbol{x}_i^1) \in X_i, \quad z_i^+ \geq 0,\ z_i^- \geq 0 \end{aligned}$$

note: '≤ 0' constraints consistent with free disposal

Equilibrium

Definition

An admissible price system $(\bar{p}^0, \bar{\boldsymbol{p}}^1, \bar{q}, \bar{\boldsymbol{r}})$ is an *equilibrium* when $(\bar{x}^0, \bar{\boldsymbol{x}}^1, \bar{z}^+, \bar{z}^-)$ solve the corresponding agents' problems and

- $\sum_{i \in \mathcal{I}} (\bar{x}_i^0 - e_i^0) \leq 0$, $=_l$ if $\bar{p}_l^0 > 0$
- $\sum_{i \in \mathcal{I}} (\bar{\boldsymbol{x}}_i^1 - \boldsymbol{e}_i^1) \leq 0$, $=_{l,s}$ if $\bar{p}_l^1(s) > 0$
- $\sum_{i \in \mathcal{I}} \bar{z}_i^+ = \sum_{i \in \mathcal{I}} \bar{z}_i^-$
- $\forall\, s :\; D(s, \bar{p}^1(s) \sum_{i \in \mathcal{I}} \bar{z}_i^- \leq \sum_{i \in \mathcal{I}} e_1^1(s)$, $=_l$ when $\bar{r}_l(s) > 0$.

Theorem

Under our assumptions, $\exists$ an equilibrium price system contract prices $\bar{q} > 0$ and $\bar{p}_l^1(s) > 0$ for all $s \in S$ in which good l is to be delivered

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Arbitrage

Definition

Admissible prices $(p^0, \boldsymbol{p}^1, q, \boldsymbol{r})$ affords *arbitrage* if $\exists$ portfolio (z^+, z^-) such that

1. $\langle q, z^+ - z^- \rangle + \sum_{s \in S} \langle r(s), D(s, p^1(s)) z^- \rangle \leq 0$
2. $\forall\, s\colon\ \langle p^1(s), D(s, p^1(s))[\, z_i^+ - z_i^- \,\rangle \leq 0$ & $<_i$ for some i

Theorem

Given an admissible price system $(p^0, \boldsymbol{p}^1, q, \boldsymbol{r})$, the agents' problems are solvable if and only if the price system doesn't afford arbitrage

Discounting to the present

Theorem

A necessary and sufficient condition for no-arbitrage:
$\exists$ discount factors $\boldsymbol{\rho} = (\rho(s), s \in S)$ subject to *for* $k = 1, \ldots, K$,
$$\sum_{s \in S} \rho(s) \langle p^1(s), D_k(s, p^1(s)) \rangle \in \left[q_k - \sum_{s \in S} \langle r(s), D_k(s, p^1(s)) \rangle,\ q_k \right]$$

Definition

1. *consolidated discount factor* $\rho^0 = \sum_{s \in S} \rho(s)$
2. *imputed probabilities*: $\pi(s) = \rho(s)/\rho^0$
3. $\boldsymbol{\rho}$ is the *discount bundle* associated with $(p^0, \boldsymbol{p}^1, q, \boldsymbol{r})$ if it satisfies the no-arbitrage NS-conditions.

for $\boldsymbol{t} \in R^{|S|} \implies \langle \boldsymbol{\rho}, \boldsymbol{t} \rangle = \rho^0 \sum_{s \in S} \pi(s) t(s) = \rho^0 E_\pi\{\boldsymbol{t}\} \ \forall \boldsymbol{t}$

Market valuations

Definition

$(p^0, \boldsymbol{p}^1, q, \boldsymbol{r})$-admissible & $g_k(\boldsymbol{p}^1, \boldsymbol{r}) = \sum_{s \in S} \langle r(s), D_k(s, p^1(s)) \rangle$

- *long value of* $\boldsymbol{t} = (\dots, t(s), \dots) \in R^{|S|}$
  $v^+(t) = \min \left(\langle q, z^+ \rangle - \langle q - g(\boldsymbol{p}^1, \boldsymbol{r}), z^- \rangle \right)$ subject to
  $z^+ \geq 0,\ z^- \geq 0,\ D(s, p^1(s)[z^+ - z^-] \geq t(s)$ for all s
- *short value of* $\boldsymbol{t}$
  $v^-(t) = \max \left(\langle q - g(\boldsymbol{p}^1, \boldsymbol{r}), z^- \rangle - \langle q, z^+ \rangle \right)$ subject to
  $z^+ \geq 0,\ z^- \geq 0,\ D(s, p^1(s)[z^+ - z^-] \leq t(s)$ for all s

Market valuations via discounts

Theorem

Let $\boldsymbol{\rho}$ be a discount bundle (that satisfies the no-arbitrage NS-conditions) given $(p^0, \boldsymbol{p}^1, q, \boldsymbol{r})$-admissible, one has

$$v^+(t) = \max_{\boldsymbol{\rho}} \sum_{s \in S} \rho(s) t(s), \quad v^-(t) = \min_{\boldsymbol{\rho}} \sum_{s \in S} \rho(s) t(s)$$

The functions $t \mapsto v^+(t)$ and $t \mapsto v^-(t)$ are, respectively, sublinear and suplinear on $R^{|S|}$. Moreover $v^-(t) = -v^+(t)$,

Relaxing delivery requirements

- α_i guaranteed fraction of delivery obligation by i
- budget constraint for i becomes:
  $\langle p^0, x_i^0 - e_i^0 \rangle + \langle q, z_i^+ - z_i^- \rangle$
  $+ \sum_{s \in S} \langle r(s), D(s, p^1(s)) \alpha_i z_i^- - e_i^1(s) \rangle \leq 0$
- adjusted equilibrium condition,
  $D(s, \bar{p}^1(s)) \sum_{i \in \mathcal{I}} \alpha_i \bar{z}_i^- \leq \sum_{i \in \mathcal{I}} e_i^1(s)$ with $=_l$ when $r_l(s) > 0$

Equilibrium: same (formal) argument
!! α_i (reliability of vendor?) enters model *exogenously*

Limit case: no guarantees

Now, let $\alpha_i \to 0$

Theorem

Under our assumptions, if the requirement for the delivery is dropped that in turns leads to the price supply guarantees $r \equiv 0$, an equilibrium still exists under the following assumption,

$$\text{rank } V(p^1) \text{ is constant on } \{p^1 \mid p^1 > 0\}$$

i.e. $p_l^1 > 0$ for all l. Or, equivalently $p^1 \mapsto W(p^1) = \text{lin } V(p^1)$ is continuous on the positive orthant of $(R^L)^{|S|}$.

Variational Inequality I

$$\max_{x_i} u_i(x_i) \quad \text{s.t.} \quad \langle p, x_i \rangle \leq \langle p, e_i \rangle, \ x_i \in C_i \quad i \in \mathcal{I}$$

$$\sum_i (e_i - x_i) = s(p) \geq 0.$$

KKT-conditions and Market clearing conditions:

$\bar{x}_i \in C_i$ optimal $\iff \exists \bar{\lambda}_i \geq 0$ (linear constrait)

(a) $\langle p, e_i - \bar{x}_i \rangle \geq 0$ (feasibility)

(b) $\bar{\lambda}_i \Big(\langle p, e_i - \bar{x}_i \rangle \Big) = 0$ (compl.slackness)

(c) $\nabla u_i(\bar{x}_i) = \bar{\lambda}_i p \quad (e_i \in \text{int}\, C_i)$

(d) $\sum_i (e_i - \bar{x}_i) \geq 0$ (market clearing)

Variational Inequality II

$$\max_{x_i} u_i(x_i) \quad \text{such that} \quad \langle p, x_i \rangle \leq \langle p, e_i \rangle, \; x_i \in C_i \quad i \in \mathcal{I}$$

$$\sum_i (e_i - x_i) = s(p) \geq 0.$$

$$G(p, (x_i), (\bar{\lambda}_i)) = \left[\sum_i (e_i - x_i); \left(\bar{\lambda}_i p - \nabla u_i(x_i) \right); \langle p, e_i - x_i \rangle \right]$$

$$D = \Delta \times \left(\prod_i C_i \right) \times \left(\prod_i R_+ \right)$$

$$N_D(\bar{z}) = \left\{ v \,\middle|\, \langle v, z - \bar{z} \rangle \leq 0, \; \forall\, z \in D \right\}$$

$$-G(\bar{p}, (\bar{x}_i), (\bar{\lambda}_i)) \in N_D(\bar{p}, (\bar{x}_i), (\bar{\lambda}_i)).$$

Replacing D by $\hat{D}$ bounded: explicit bound on λ_i via duality.
D polyhedral leads to efficient algorithmic procedures.

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Actually . . .

Geomtric Variational Inequality:

find $\bar{x} \in C$ such that $-G(\bar{x}) \in N_C(\bar{x})$

where $N_C(\bar{x}) = \{v \,|\, \langle v, x - \bar{x}\rangle \leq 0, \forall\, x \in C\}$

Functional Variational Inequality:

find $\bar{x}$ such that $-G(\bar{x}) \in \partial f(\bar{x})$

or equivalently,

$$f(x) \geq f(\bar{x}) - \langle G(\bar{x}), x - \bar{x}\rangle \quad \forall\, x \in R^n.$$

Production, firms and shares

1. Activities (at time 0): $\{y_i, i \in \mathcal{I}\}$
2. resources input: $T_{i0}y_i$, goods output: $T_{i1}(s)y_i$
3. auxiliary goods $y^{0\prime}$: endowment $e^{0\prime}_{j,l'_j}$, traded @ time 0
4. $Y_j = \{(y^0_j, y^{0\prime}_j, \mathbf{y}^1_j\}$ *technology* set for activity $j \in \mathcal{J}$
   closed convex cone
5. Share ownership: $\theta_j = y^{0\prime}_{j,l'_j}$ and $\theta_{i,j}$ ownership by agent i
6. Examples: production, **savings** and storage, pre-existing securities and investments (bonds, equity shares), . . .

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