Categorical Heisenberg Actions on Hilbert schemes of points

Background: S smooth surface

$S^{[n]}$ = Hilbert scheme of n pts.

$S \rightsquigarrow$ Heisenberg algebra $\mathcal{H}_S$.

eg $S = \mathbb{C}^2 \rightsquigarrow \mathcal{H}_{\mathbb{C}^2}$ gens. $a(m)$ $m \in \mathbb{Z}$

$[a(m), a(n)] = \delta_{m+n,0}\, m$.

Fact: (Nak. & Groj.). $\exists$ action of $\mathcal{H}_S$ on.

$$\bigoplus_{n \geq 0} H^*(S^{[n]}).$$

Idea: gens are given by.

$$W = \{ (J_1, J_2) : J_2 \subset J_1 \ \dim(\underbrace{J_1/J_2}_{\text{supported at 1 pt.}}) = k \}.$$

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ S^{[n]} & & S^{[n+k]} \end{tikzcd}$$

Question: can you lift this to.

K-theory $K(S^{[n]})$ and $D(S^{[n]})$

derived cat of coh sheafs.

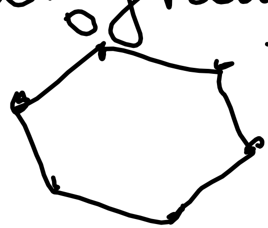
today: lifting this action.

when $S = \widehat{\mathbb{C}^2/\Gamma}$ $\quad \Gamma \subset SL_2\mathbb{C}$ finite

min. resolution

Recall: $\Gamma \subset SL_2\mathbb{C} \xleftrightarrow{\text{McKay}}$ affine Dynkin diagram.

eg $\Gamma = \mathbb{Z}/n\mathbb{Z} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \rightsquigarrow$



$\Big\}$ n vertices

$$\begin{tikzcd} S_\Gamma = \widehat{\mathbb{C}^2/\Gamma} \arrow[d, "\pi"] & \supset & \pi^{-1}(0) \arrow[d] \\ \mathbb{C}^2/\Gamma & \ni & 0 \end{tikzcd}$$

chain of $\mathbb{P}^1$'s indexed by vertices of Dynkin $I_\Gamma \setminus 0$

eg $\Gamma = \mathbb{Z}/4\mathbb{Z} \rightsquigarrow \pi^{-1}(0) =$ $E_1 \cong \mathbb{P}^1$, $E_2 \cong \mathbb{P}^1$, $E_3 \cong \mathbb{P}^1$

$i \in I_\Gamma$

$\uparrow$

Dynkin vertex:

$$W_i = \{ (I_2 \subset I_1 : I_1/I_2 \simeq \mathcal{O}_z \} \xrightarrow{p} E_i \cong \mathbb{P}^1$$

where $z \in E_i$

$\pi_1 : W_i \to S_p^{[n]}$, $\pi_2 : W_i \to S_p^{[n+1]}$

functors: $D(S_p^{[n]}) \underset{Q_i}{\overset{P_i}{\rightleftarrows}} D(S_p^{[n+1]})$

$$P_i(\cdot) \simeq R\pi_{2*}(L\pi_1^*(\cdot) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-1))$$

$$Q_i(\cdot) = \pi_{1*}(\pi_2^*(\cdot) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-1))$$

lemma $P_iP_j \overset{\text{isom.}}{\simeq} P_jP_i \quad \forall\, i,j$

$Q_iQ_j \simeq Q_jQ_i$

$$Q_jP_i \simeq P_iQ_j \oplus \begin{cases} 0 & \text{if } \overset{i}{\bullet} \quad \overset{j}{\bullet} \\ Id & \text{if } \overset{i}{\bullet} \longrightarrow \overset{j}{\bullet} \\ Id[-1] \oplus Id[1] & \text{if } i=j \end{cases}$$

$$Id[-1] \oplus Id[1] = \bigoplus_{[z]} Id$$

[P_i & Q_i are left & right adjoints of each other]

Higher structure

eg $\mathrm{End}(P_i^n) = \,?$ . . . .

<u>lemma:</u> $\mathbb{C}[S_n] \hookrightarrow \mathrm{End}(P_i^n)$

$(\forall\, i \in I_\Gamma)$ symmetric grp

<u>Cor</u> for any $\lambda \vdash n$ (partition).

can define $P_i^{(\lambda)} = \mathrm{Im}(e_\lambda)$

Idemp. corresp. to λ in S_n

<u>Lemma</u> $Q_i^{(n)} P_i^{(m)} \simeq \bigoplus_{k \geq 0} \bigoplus_{[k+1]} P_i^{(m-k)} Q_i^{(n-k)}$

<u>eg</u> ...

<u>Summary</u>: this gives a categorical action of $\underline{\mathcal{H}_{S_\Gamma}}$ on $\bigoplus_{n \geq 0} D(S_\Gamma^{[n]})$

heisenberg gen. by all P's & Q's above.

(you can quantize).

Vertex operators:

standard gens. of $\mathcal{H}s_{\Gamma}$ are

$a_i(m)$ $m \in \mathbb{Z}$, $i \in I_\Gamma$ relns: $[a_i(m), a_j(n)]$

$$= \delta_{m+n,0}\, n \langle i, j \rangle$$

$$\langle i,j\rangle = \begin{cases} 0 & \text{if } \overset{i}{\bullet} \quad \overset{j}{\bullet} \\ -1 & \text{if } \bullet\!\!-\!\!\bullet \\ 2 & \text{if } i=j \end{cases}$$

$$\sum_{m\geq 0} P_i^{(m)} z^m = \exp\Big(\underbrace{\sum \frac{a_i(-m)}{m} z^m}_{\text{(half) vertex operators}}\Big).$$

and similarly for Q's.

<u>Moral</u>: geometry tells you to use "vertex ops".

Frenkel-Kac-Segal Constr.

define: $E_i = \exp\left(\sum \frac{a_i(-m)}{m} z^m \right) \cdot \exp\left(- \sum \frac{a_i(m)}{m} z^m \right).$

$i \in I_p$.

F_i = similar

these give you action of

$U(\mathfrak{g}_\Gamma)$ quantum

affine algebra of $\Gamma = \mathbb{Z}/n$.

$U(\mathfrak{g}_\Gamma) = \widehat{\mathfrak{sl}_n}$.

in our generators.

$$\underset{\lambda+\alpha_i}{}E_i\underset{\lambda}{} = \sum_{k\geq 0} (-1)^k P_i^{(k)} Q_i^{(1^{m+k})}.$$

F_j similar

Thm (L.-Licata). can define complexes of functors.

$$\mathcal{E}_i = \left[\cdots \xrightarrow{d} P_i^{(2)} Q_i^{(1^{m+2})}\langle -2\rangle \xrightarrow{d} P_i Q_i^{(1^{m+1})}\langle -1\rangle \to Q_i^{(1^m)} \right]$$

$\mathcal{F}_i$ = similar

which give a "categorical action" of $U_q(\mathfrak{g}_\Gamma)$. $\Gamma \neq \mathbb{A}$