Long time behavior of solutions of Hamilton-Jacobi in the plane

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The problem

- Long time behaviour of the average $u(\cdot, t)/t$, where u is the solution to

$$\begin{cases} u_t(x,t) + F(x, Du(x,t)) = 0 & \text{in } \mathbb{R}^N \times (0, +\infty) \\ u(x,0) = 0 & \text{in } \mathbb{R}^N \end{cases}$$

- Limit as $\lambda \to 0$ of λv_λ where v_λ is the solution to

$$\lambda v_\lambda(x) + F(x, Dv_\lambda(x)) = 0 \qquad \text{in } \mathbb{R}^N$$

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Motivations

- Ergodic control and differential games
- Homogenization

The classical setting

If

- $F(x, \cdot)$ is coercive : $\lim_{|p|\to+\infty} F(x,p) = +\infty$ uniformly in x
- $F(\cdot, p)$ is $\mathbb{Z}^N-$periodic

Theorem [Lions-Papanicolau-Varadhan]

There is a constant $c \in \mathbb{R}$ s.t.

$$\lim_{T\to+\infty} \frac{1}{T} u(x,T) = \lim_{\lambda\to 0^+} \lambda v_\lambda(x) = c$$

Relies on

1. uniform bounds on $\frac{1}{T}u(x,T)$ and on $\lambda v_\lambda(x)$
2. uniform Lipschitz bounds for v_λ
3. uniqueness of the constant c for which there is a continuous periodic solution to

$$F(x, D\chi(x)) = -c \qquad \text{in } \mathbb{R}^N$$

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Relaxation of the coercivity condition

What happens when H is not coercive ?

Without coercivity condition on F

- No uniform Lipschitz bounds for v_λ
- Equation

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Examples

For $N = 2$ and

$$F(x_1, x_2, p_1, p_2) = -|p_1| + |p_2| - \ell(x_1, x_2) \qquad \forall (p_1, p_2), (x_1, x_2) \in \mathbb{R}^2 .$$

- Resonnance : If $\ell(x_1, x_2) = \bar{\ell}(x_1 - x_2)$, then

$$\frac{1}{T} u(x_1, x_2, T) = \lambda v_\lambda(x_1, x_2) = \bar{\ell}(x_1 - x_2) .$$

  The limit exists but is not constant.

- Saddle point : [Alvarez-Bardi] If

$$\bar{\ell} := \min_{x_1} \max_{x_2} \ell(x_1, x_2) = \max_{x_2} \min_{x_1} \ell(x_1, x_2) ,$$

  then

$$\lim_{T \to +\infty} \frac{1}{T} u(x, T) = \lim_{\lambda \to 0^+} \lambda v_\lambda(x) = \bar{\ell} .$$

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Examples (continued)

Assume

$$F(x,p) = a(x)|p+p_0| \qquad (x,p) \in \mathbb{R}^N \times \mathbb{R}^N$$

where

- $p_0 \in \mathbb{R}^N \backslash \{0\}$
- $a : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous, $\mathbb{Z}^N-$periodic and changes sign

Proposition

The limits

$$w(x) := \lim_{T\to+\infty} \frac{1}{T} u(x,T) = \lim_{\lambda\to 0^+} \lambda v_\lambda(x)$$

exist.

However w is discontinuous unless it is identically zero.

Example borrowed from [C., Lions, Souganidis].

Some references

- non-resonnance conditions (convex case) : Arisawa-Lions (1998)
- special decoupled structure conditions : Barles (2007), Imbert-Monneau (2008), Bardi (2009)
- sub-additive type result (convex case) : Quincampoix-Renault (2009).

The Arisawa-Lions nonresonnance condition

Assumptions :

$$F(x,p) = H(p) - \ell(x), \qquad \forall (x,p) \in \mathbb{R}^N \times \mathbb{R}^N ,$$

where

- $\ell : \mathbb{R}^N \to \mathbb{R}$ is continuous, $\mathbb{Z}^N-$periodic,
- $H : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous, $1-$positively homogeneous and convex.
- Nonresonnance condition :

$$\forall k \in \mathbb{Z}^N \backslash \{0\} \ , \ \exists a \in \partial H(0) \text{ with } k.a \neq 0 \ ,$$

Theorem (special case of [Arisawa-Lions, 98])

There is a constant $c \in \mathbb{R}$ such that

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Sketch of proof

- Under the structure condition, (λv_λ) is unif. bounded and Lipschitz continuous.
- Hence up to a subsequence, (λv_λ) converges to some $\bar{w}$ Lipschitz continuous, $\mathbb{Z}^N-$periodic solution to

$$H(D\bar{w}) = 0 \qquad \text{in } \mathbb{R}^N$$

- From the homogeneity and the convexity of H,

$$\langle a, D\bar{w}\rangle \leq 0 \qquad \text{in } \mathbb{R}^N \qquad \forall a \in \partial H(0)$$

  Integrating over $[0,1]^N$ leads to

$$\langle a, D\bar{w}\rangle = 0 \qquad \text{in } \mathbb{R}^N \qquad \forall a \in \partial H(0)$$

- Therefore $\bar{w}$ is constant along the lines $t \to x + ta$ for any $x \in \mathbb{R}^N$, $a \in \partial H(0)$.

A sub-additive type result

Assumptions :

$$F(x,p) = H(p) - \ell(x), \qquad \forall (x,p) \in \mathbb{R}^N \times \mathbb{R}^N \,,$$

where

- $\ell : \mathbb{R}^N \to \mathbb{R}$ is continuous, $\mathbb{Z}^N-$periodic,
- $H : \mathbb{R}^N \to \mathbb{R}$ is $1-$positively homogeneous and convex.

Theorem (special case of [Quincampoix-Renault, preprint])

The limits

$$\lim_{T\to+\infty} \frac{1}{T} u(x,T) = \lim_{\lambda\to 0^+} \lambda v_\lambda(x)$$

always exist (but need not be constant).

Outline

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1. Non-resonnance conditions
2. The resonnant case
3. General Hamiltonians

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We work in the plane ($N = 2$) and with F of the form

$$F(x,p) = H(p) - \ell(x) \qquad \forall (x,p) \in \mathbb{R}^2 \times \mathbb{R}^2 .$$

where

- ℓ is $\mathbb{Z}^2-$periodic and Lipschitz continuous,
- $H : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous, $1-$positively homogeneous,
- For any $p \in \mathbb{R}^2\backslash\{0\}$ with $H(p) = 0$, $DH(p)$ exists and is nonzero.

Standard consequence of the assumptions

Lemma

Under the above assumptions, $\frac{1}{T}u(\cdot, T)$ and λv_λ are uniformly Lipschitz continuous and bounded.

We note that $w_\lambda := \lambda v_\lambda$ solves

$$\lambda w_\lambda + H(Dw_\lambda) - \lambda \ell = 0 \qquad \text{in } \mathbb{R}^2 .$$

Lemma

Any limit w of a converging subsequence of (λv_λ) is a Lipschitz continuous, periodic viscosity solution to

$$H(Dw(x)) = 0 \qquad \text{in } \mathbb{R}^2$$

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Rigidity result : Heuristic arguments

Characteristic method

Assume w is a smooth solution to $H(Dw) = 0$ in $\mathbb{R}^N$.
Then the map $t \to w(x_0 + tDH(Dw(x_0))$ is constant in $\mathbb{R}$.

Indeed

1. Let $x'(t) = DH(Dw(x(t)),\ t \in \mathbb{R}, \qquad x(0) = x_0$.
2. $\frac{d}{dt} w(x(t)) = \langle Dw(x(t)), x'(t)\rangle = \langle Dw(x(t)), DH(Dw(x(t)))\rangle = H(Dw(x(t)) = 0$.
   Hence $w(x(t)) = w(x_0)$ for all $t \in \mathbb{R}$.
3. $H(Dw(x)) = 0$ for all x. Hence $D^2 w(x) DH(Dw(x)) = 0$.
4. So $\frac{d}{dt} Dw(x(t)) = D^2 w(x(t)) x'(t) = D^2 w(x(t)) DH(Dw(x(t))) = 0$.
   Hence $x(t) = x_0 + tDH(Dw(x_0))$ for all $t \in \mathbb{R}$.

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   Hence $x(t) = x_0 + tDH(Dw(x_0))$ for all $t \in \mathbb{R}$.

Rigidity result : a counter-example

If w is a Lipschitz continuous solution to $H(Dw) = 0$, the map $t \to w\left(x_0 + tDH(Dw(x_0))\right)$ is not constant on $\mathbb{R}$ in general.

For instance, $w(x_1, x_2) = -|x_1| - x_2$ is a solution for

$$H(p_1, p_2) = |p_1| + p_2$$

However, if $x_0 = (1, 1)$, then $Dw(1, 1) = (-1, -1)$ and $DH(Dw(\bar{x})) = (-1, 1)$, while

$$w((1, 1) + tDH(Dw(\bar{x})) = \min\{-2, -2t\} \ < \ -2 = w(1, 1) \qquad \forall t > 1 \ .$$

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Rigidity result

Lemma

Let w be a Lipschitz continuous viscosity solution to

$$H(Dw(x)) = 0 \qquad \text{in } \mathbb{R}^2$$

If x_0 is a point of differentiability of w, then

$$t \to w\left(x_0 - tDH(Dw(x_0))\right) \qquad \text{is constant on } [0, +\infty) .$$

Rigidity result (continued)

Let

$$\mathcal{P}_0 = \left\{ p \in \mathbb{R}^2 \backslash \{0\} \; , \; H(p) = H(-p) = 0 \right\} \; .$$

Theorem

Equation

$$H(Dw(x)) = 0 \qquad \text{in } \mathbb{R}^2$$

has a Lipschitz continuous, $\mathbb{Z}^2-$periodic and non constant solution w iff there is some $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathcal{P}_0$ with either $\bar{p}_2 = 0$ or $\bar{p}_1/\bar{p}_2 \in \mathbb{Q}$.
Moreover w is $1-$dim., i.e.,

$$w(x) = \bar{w}(\langle p, x \rangle) \qquad \forall x \in \mathbb{R}^2 \; ,$$

for some p as above, where $\bar{w} : \mathbb{R} \to \mathbb{R}$ is periodic.

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Example

If

$$H(p_1, p_2) = -|p_1| + \alpha|p_2| \qquad \forall (p_1, p_2) \in \mathbb{R}^2 \ ,$$

where $\alpha > 0$, then

$$\mathcal{P}_0 = \Big\{(p_1, p_2) \in \mathbb{R}^2 \backslash \{0\} \ , \ |p_1| = \alpha|p_2|\Big\} \ .$$

Hence equation

$$H(Dw(x)) = 0 \qquad \text{in } \mathbb{R}^2$$

has a Lipschitz continuous, $\mathbb{Z}^2-$periodic and non constant solution w iff $\alpha \in \mathbb{Q}$.

Ergodic behaviour

Let us set

$$\mathcal{P} = \left\{ p \in \mathbb{R}^2 \backslash \{0\} \;,\; H(p) = H(-p) = 0 \text{ and } [\, p_2 = 0 \text{ or } p_1/p_2 \in \mathbb{Q} \,] \right\} .$$

Theorem (Non-resonnance conditions)

Assume that $\mathcal{P} = \emptyset$. Then the (λv_λ) and the $u(\cdot, T)/T$ converge to the same constant as $\lambda \to 0$ and $T \to +\infty$.

Proof :

- Any converging subsequence of (λv_λ) converges to a constant.
- This constant c must be independant of the subsequence by comparison.
- For $\lambda > 0$ small, $ct + v_\lambda$ is "almost" a solution of

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Outline

Convergence result

We now assume that $\mathcal{P} \neq \emptyset$. Then equation

$$(*) \qquad H(Dw) = 0 \qquad \text{in } \mathbb{R}^2$$

has continuous, $\mathbb{Z}^2-$periodic solutions.

Theorem (Resonnant case)

Under the above assumptions, there is a continuous $\mathbb{Z}^2-$periodic solution $w : \mathbb{R}^2 \to \mathbb{R}$ to $(*)$ such that

$$w(x) = \lim_{T\to+\infty} \frac{u(x,T)}{T} = \lim_{\lambda\to 0^+} \lambda v_\lambda(x) \quad \text{uniformly w.r.t. } x \in \mathbb{R}^2.$$

Idea of proof (I)

Let

$$\mathcal{W} = \{ \ w \text{ cluster points of } (\lambda v_\lambda) \ \} \ .$$

Lemma

There is some $\bar{p} \in \mathcal{P}$ s.t. for any $w \in \mathcal{W}$,

$$w(x) = \bar{w}(\langle \bar{p}, x \rangle)$$

for some $\bar{w} : \mathbb{R} \to \mathbb{R}$.

Idea of proof (II)

Let us assume to fix the ideas that $\bar{p} = (1, -1)$.

Lemma

Let $w \in \mathcal{W}$ and $\bar{w} : \mathbb{R} \to \mathbb{R}$ be $1-$periodic such that

$$w(x_1, x_2) = \bar{w}(x_1 - x_2) \qquad \forall x = (x_1, x_2) \in \mathbb{R}^2 .$$

Then at any point $x = (x_1, x_2)$ in a neighbourhood of which w is not constant,

$$w(x_1, x_2) = \int_0^1 \ell(t, x_2 - x_1 + t) dt$$

Remark : The above equality does not hold for any $x \in \mathbb{R}^2$ in general.

Idea of proof (III) : local corrector

To fix the ideas suppose that

$$H(p_1, p_2) = -|p_1| + |p_2| \qquad \text{and} \qquad \bar{p} = (1, -1)$$

Then

$$DH(s\bar{p}) = (-1, -1) \qquad \forall s > 0 \; .$$

Let

$$\bar{\ell}(s) = \int_0^1 \ell(t, t-s)dt$$

and

$$\chi(x_1, x_2) = \int_0^{x_1} \ell(t, x_2 - x_1 + t)dt - x_1 \bar{\ell}(x_1 - x_2)$$

Then χ is continuous and $\mathbb{Z}^2-$periodic.

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Then χ is continuous and $\mathbb{Z}^2-$periodic.

Lemma

If $\bar{\ell}'(\bar{x}_1 - \bar{x}_2) > 0$, then

$$z_\lambda(x_1, x_2) = \frac{1}{\lambda}\bar{\ell}(x_1 - x_2) + \chi(x_1, x_2)$$

is an “approximate solution” of

$$\lambda z_\lambda + H(Dz_\lambda) - \ell = 0$$

in a neighborhood of the line $x_1 - x_2 = \bar{x}_1 - \bar{x}_2$.

This means that χ is an local corrector.

Outline

Assumptions

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We still work in the plane ($N = 2$) and with F of the form

$$F(x,p) = H(p) - \ell(x) \qquad \forall (x,p) \in \mathbb{R}^2 \times \mathbb{R}^2 .$$

where

- ℓ is $\mathbb{Z}^2-$periodic and Lipschitz continuous,
- there are $k \geq 1$ and $H_\infty : \mathbb{R}^2 \to \mathbb{R}$, $k-$positively homogeneous and locally Lipschitz continuous, such that

$$\lim_{s \to 0^+} s^k H(p/s) = H_\infty(p) \quad \text{locally uniformly in } p$$

- For any $p \in \mathbb{R}^2 \backslash \{0\}$ with $H_\infty(p) = 0$, $DH_\infty(p)$ exists and is nonzero.

Ergodicity

Let

$$\mathcal{P} = \left\{ p \in \mathbb{R}^2 \ , \ H_\infty(p) = H_\infty(-p) = 0 \text{ and } \left[p_2 = 0 \text{ or } \frac{p_1}{p_2} \in \mathbb{Q} \right] \right\} .$$

Theorem

If either $\mathcal{P} = \emptyset$ or

$$\forall p \in \mathcal{P} \qquad \lim_{s \to 0} \left| H\left(\frac{p}{s}\right) \right| = +\infty \ ,$$

then there is $c \in \mathbb{R}$ with

$$\lim_{T \to +\infty} \frac{u(x,T)}{T} = \lim_{\lambda \to 0^+} v_\lambda(x) = c \quad \text{uniformly w.r.t. } x \in \mathbb{R}^2.$$

Example

Let $H_\infty : \mathbb{R}^2 \to \mathbb{R}$ satisfy

- H_∞ is Lipschitz continuous and $k-$positively homogeneous $(k > 1)$,
- for any $p \in \mathbb{R}^2 \backslash \{0\}$ with $H(p) = 0$, $DH(p)$ exists and is nonzero.

Set $H(p) = H_\infty(p + a)$ for some $a \in \mathbb{R}^2$. If $\mathcal{P} \neq \emptyset$ but

$$\langle DH_\infty(p), a\rangle \neq 0 \text{ for any } p \in \mathcal{P},$$

then ergodicity holds because

$$\begin{aligned} \left| H\left(\frac{p}{s}\right) \right| &= (1/s)^k \mid H_\infty (p + sa) \mid \\ &= (1/s)^k \left|H_\infty(p) + sDH_\infty(p).a + o(s)\right| \\ &= (1/s)^{k-1} \left|DH_\infty(p).a + o(1)\right| \to +\infty \qquad \text{as } s \to 0^+. \end{aligned}$$

Application to homogeneization

Let us consider the problem :

$$\left\{ \begin{array}{ll} z^\epsilon_t + H(Dz^\epsilon(x)) - \ell(x/\epsilon) = 0 & \text{in } \mathbb{R}^2 \times (0,T) \\ z^\epsilon(x,0) = z_0(x) & \text{in } \mathbb{R}^2 \end{array} \right.$$

where H is $1-$positively homogeneous and ℓ as above.

Corollary

If $\mathcal{P} = \emptyset$, then there is a Lipschitz continuous Hamiltonian $\bar{H} : \mathbb{R}^2 \to \mathbb{R}$ such that, for any bounded uniformly continuous map $z_0 : \mathbb{R}^2 \to \mathbb{R}$, the solution z^ϵ uniformly converges to the solution z of

$$\left\{ \begin{array}{ll} z_t + \bar{H}(Dz(x)) = 0 & \text{in } \mathbb{R}^2 \times (0,T) \\ z(x,0) = z_0(x) & \text{in } \mathbb{R}^2 \end{array} \right.$$

Assumption for the resonnant case

We assume $k = 1$.

Further assumption

We assume that $\mathcal{P} \neq \emptyset$ and that, for any $\bar{p} \in \mathcal{P}$, there are $\alpha(\bar{p}) \in \mathbb{R}^2 \backslash \{0\}$ and $\beta(\bar{p}) \in \mathbb{R}$ s.t.,

$$\lim_{s \to 0^+} H\left(\frac{\theta \bar{p}}{s} + b\right) = \alpha(\bar{p}).b + \beta(\bar{p})$$

where the limit is uniform w.r.t. $\theta \geq 1/M$ and $|b| \leq M$ for any $M > 0$.

Example

Let $H_\infty : \mathbb{R}^2 \to \mathbb{R}$ satisfy

- H_∞ is Lipschitz continuous and $1-$positively homogeneous,
- for any $p \in \mathbb{R}^2 \backslash \{0\}$ with $H(p) = 0$, $DH(p)$ exists and is nonzero,
- $\mathcal{P} \neq \emptyset$

Set $H(p) = H_\infty(p + a)$ for some $a \in \mathbb{R}^2$. Then the "further assumption" holds :

$$
\begin{array}{rcl}
H\left(\dfrac{\theta \bar{p}}{s} + b\right) & = & \dfrac{\theta}{s} H_\infty\left(\bar{p} + \dfrac{s}{\theta}(a+b)\right) \\
& = & \dfrac{\theta}{s}\left(H_\infty(\bar{p}) + \dfrac{s}{\theta} DH_\infty(\bar{p}).(a+b) + o(\dfrac{s}{\theta})\right) \\
& = & \alpha(\bar{p}).b + \beta(\bar{p}) + o(1)
\end{array}
$$

where $\alpha(\bar{p}) = DH_\infty(\bar{p}) \neq 0$ and $\beta(\bar{p}) = \langle DH_\infty(\bar{p}), a\rangle$.

The resonnent case

Theorem

Under the above assumptions,

$$w(x) = \lim_{t\to+\infty} \frac{u(x,t)}{t} = \lim_{\lambda\to 0^+} v_\lambda(x) \quad \text{uniformly w.r.t. } x \in \mathbb{R}^2.$$

where $w(x) = \bar{w}(\langle \bar{p}, x\rangle)$ *for some* $\bar{p} \in \mathcal{P}$.

At any "point of increase" (resp. "point of decrease") s of $\bar{w}$ *we have :*

$$\bar{w}(s) = \frac{1}{T(\bar{p})}\int_0^{T(\bar{p})} \ell(s\bar{p} + t\alpha(\bar{p}))ds - \beta(\bar{p}) \qquad (\text{resp. } -\beta(-\bar{p}))\ ,$$

where $T(\bar{p}) > 0$ *is s.t.* $T(\bar{p})\alpha(\bar{p}) \in \mathbb{Z}^2$.

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Open problems

- Uniform value
  In differential games one would like to know if the "optimal strategies" are independant of the (large) time horizon.

- Characterization of the limit
  Where are the flat parts of the limit w ?

- $N-$dimensional case
  Completely open for $N \geq 3$.

Thank you for your attention !