STRUCTURE OF NUCLEAR C*-ALGEBRAS: LECTURES 6-10

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1. Day 6: Highlights of classification up to 2006

1.1. **The Kirchberg-Phillips Theorem.** Unfortunately, stating the Kirchberg-Phillips theorem requires a fair amount of terminology. Here are a few requisite definitions.

Definition 1.1. A C*-algebra A is *nuclear* if there exist c.c.p. maps $\varphi_n\colon A \to \mathbb{M}_{k(n)}(\mathbb{C})$ and $\psi_n\colon \mathbb{M}_{k(n)}(\mathbb{C}) \to A$ such that $\psi_n \circ \varphi_n \to \mathrm{id}_A$ in the point-norm topology:

$$\|\psi_n \circ \varphi_n(a) - a\| \to 0,$$

for all $a \in A$.

Nuclearity is also called the *completely positive approximation property* (CPAP) because it means there exist diagrams

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, dashed, "\varphi_n"'] & & A \\ & \mathbb{M}_{k(n)}(\mathbb{C}) \arrow[ur, dashed, "\psi_n"'] & \end{tikzcd}$$

which asymptotically commute pointwise. See [3] for more on approximation properties of operator algebras.

Following Cuntz [6], we restrict to simple C*-algebras in the next definition. See [19] for the nonsimple case.

Definition 1.2. A simple C*-algebra A is *purely infinite* if every hereditary subalgebra $B \subset A$ contains an infinite projection p (i.e., p is Murray-von Neumann equivalent to a proper subprojection).

Next is a notion of equivalence for $*$-homomorphisms.

Definition 1.3. We say two $*$-homomorphisms $\pi, \sigma\colon A \to B$ are *asymptotically unitarily equivalent* if there exists a (norm) continuous family of unitaries $u_t \in D$ (or the unitization $\tilde{D}$, if D is nonunital), $0 \le t < \infty$, such that

$$\lim_{t\to\infty} \|u_t \pi(a) u_t^* - \sigma(a)\| = 0$$

for all $a \in A$.

One of the many remarkable aspects of KK-theory is the number of equivalent definitions that can be given. Specializing to separable unital nuclear simple C*-algebras, $\mathrm{KK}^0(A, D)$ has a beautiful and extraordinarily useful description (see [35, Theorem 4.1.3] for a proof).

Theorem 1.4. *For a separable unital nuclear simple C*-algebra A and a separable unital simple C*-algebra D, $\mathrm{KK}^0(A, D)$ is (isomorphic to) the set of asymptotic unitary equivalence classes of homomorphisms from $\mathbb{K} \otimes \mathcal{O}_\infty \otimes A$ to $\mathbb{K} \otimes \mathcal{O}_\infty \otimes D$ with addition given by direct sum of homomorphisms.*[1]

Moreover, the Kasparov product $\mathrm{KK}^0(A, B) \times \mathrm{KK}^0(B, D) \to \mathrm{KK}^0(A, D)$ is simply composition of homomorphisms.

We say an element $\eta \in \mathrm{KK}^0(A, B)$ is *invertible* if there exists and element $\gamma \in \mathrm{KK}^0(B, A)$ such that the Kasparov product $\eta \cdot \gamma$ equals $[\mathrm{id}_A] \in \mathrm{KK}^0(A, A)$.

Theorem 1.5 (Kirchberg-Phillips)**.** *Assume both A and B are separable nuclear unital purely infinite and simple, and that there exists an invertible element $\eta \in \mathrm{KK}^0(A, B)$. Then there exists an isomorphism $\pi \colon \mathbb{K} \otimes A \to \mathbb{K} \otimes B$ such that $\eta = [\pi]$.*

Separable nuclear unital purely infinite simple C*-algebras are often referred to as *Kirchberg algebras*, and we shall follow this custom.

Here are a few refinements of the Kirchberg-Phillips Theorem. We first get rid of the compacts by assuming that our invertible element η maps the unit of A to the unit of B (where we identify η with a homomorphism).

Corollary 1.6 (Kirchberg-Phillips)**.** *Assume A and B are Kirchberg algebras and there exists a invertible element $\eta \in \mathrm{KK}^0(A, B)$ such that $[1_A] \times \eta = [1_B]$. Then there exists an isomorphism $\pi \colon A \to B$ such that $\eta = [\pi]$.*

Next, we pass from KK-theory to the Elliott invariant by assuming the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet ([42]). A proper discussion of the UCT would take awhile, so just trust that it's a technical assumption that allows one to pass from KK-classification to K-classification. It should also be mentioned that most examples of nuclear C*-algebras (e.g., AF algebras, Cuntz algebras, irrational rotation algebras, crossed products of the form $C(X) \rtimes \mathbb{Z}$, C*-algebras of amenable discrete groups, Bunce-Deddens algebras, graph C*-algebras, etc.) are known to satisfy the UCT; and there is no known example of a nuclear C*-algebra that doesn't satisfy the UCT. (In fact, it is a major open question whether every nuclear C*-algebra satisfies the UCT.)

Corollary 1.7 (Kirchberg-Phillips)**.** *Assume both A and B are Kirchberg algebras satisfying the UCT. If $\alpha \colon \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ is an isomorphism, then there exists an isomorphism $\pi \colon A \to B$ such that $\pi_* = \alpha$.*

1.2. **Lin's Theorem.**

To state Lin's Theorem we need to define a new class of algebras.

Definition 1.8. A unital simple C*-algebra A is called *tracially AF* if for each finite set $\mathcal{F} \subset A$, $\varepsilon > 0$ and nonzero positive element $a \in A$, there is a finite-dimensional algebra $B \subset A$ with unit 1_B such that

(1) $\|[x, 1_B]\| < \varepsilon$ for all $x \in \mathcal{F}$;
(2) $\mathrm{d}(1_B x 1_B, B) < \varepsilon$ for all $x \in \mathcal{F}$ (where d means the distance in norm);
(3) $1_A - 1_B$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra generated by a.

Here is Lin's Theorem (cf. [25]).

[1] Don't forget that this KK-picture does *not* hold for more general algebras!

Theorem 1.9 (Lin)**.** *Let A and B be simple unital nuclear tracially AF algebras satisfying the UCT. Then $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

This was a breakthrough in classification because the definition of tracially AF doesn't presuppose an inductive limit decomposition (among other reasons). It turns out, though, that they are inductive limits of a very special type – and you must prove this before deducing Lin's Theorem.

Definition 1.10. An *approximately homogenous* (AH) algebra is an inductive limit of algebras of the form

$$\bigoplus_{i=1}^{k} p_i \mathbb{M}_{m(i)}(C(X_i))p_i,$$

where X_i is a connected compact metric space[2] and $p_i \in \mathbb{M}_{m(i)}(C(X_i))$ is a projection.

Definition 1.11. A is said to have *real rank zero* if every self-adjoint element in A can be approximated by self-adjoints with finite spectrum.

Definition 1.12. An AH algebra has *slow dimension growth* if it is the inductive limit of algebras

$$A_n = \bigoplus_{i=1}^{k(n)} p_i^{(n)} \mathbb{M}_{m(n,i)}(C(X_i^{(n)}))p_i^{(n)},$$

where

$$\lim_{n\to\infty} \left(\max_{1\le i\le k(n)} \frac{\dim(X_i^{(n)})}{\mathrm{rank}(p_i^{(n)})} \right) = 0,$$

and dim denotes the topological covering dimension and $\mathrm{rank}(p_i^{(n)})$ means the rank at any point in $X_i^{(n)}$ (since $X_i^{(n)}$) is connected). In other words, the ranks of the projections should grow faster than the dimensions of the spaces.

It turns out that every tracially AF algebra classified by Lin is isomorphic to a unital simple AH algebra with real rank zero and slow dimension growth, and conversely all such AH algebras are tracially AF. The story is a little complicated, and to tell it properly requires more definitions.

Definition 1.13. We say $\mathrm{K}_0(A)$ is *weakly unperforated* if for every $x \in \mathrm{K}_0(A)$ and $n \in \mathbb{N}$ we have that $nx \in \mathrm{K}_0(A)_+ \setminus \{0\}$ implies $x \in \mathrm{K}_0(A)_+$.

Note that a weakly unperforated group can have torsion (as opposed to an unperforated one, where $nx \in \mathrm{K}_0(A)_+$ implies $x \in \mathrm{K}_0(A)_+$).

Definition 1.14. We say $\mathrm{K}_0(A)$ has the *Riesz interpolation property* if for all $x_1, x_2, y_1, y_2 \in \mathrm{K}_0(A)$ such that $x_i \le y_j, i, j = 1, 2$, there exists $z \in \mathrm{K}_0(A)$ such that $x_i \le z \le y_j, i, j = 1, 2$.

These definitions are relevant to our discussion because if A is simple and tracially AF, then $\mathrm{K}_0(A)$ is always weakly unperforated and has the Riesz interpolation property (see [23], [24]). So the following theorem of Elliott and Gong provides a candidate AH algebra, for each tracially AF algebra (see [10]).

[2]Without loss of generality, one can assume the spaces X_i are CW complexes.

Theorem 1.15. *For each simple unital tracially AF algebra A, there exists a simple unital AH algebra B with real rank zero and slow dimension growth such that* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

So how does the proof of Lin's Theorem go? Well, start with two algebras A_1 and A_2 as in the statement of the theorem. Then use the previous theorem to find a model AH algebra B. Finally, prove that $A_1 \cong B$ and similarly $A_2 \cong B$; hence $A_1 \cong A_2$.

We should also mention that every simple unital AH algebra B with real rank zero and slow dimension growth is tracially AF, thanks to the Dadarlat-Gong reduction theorem (see [7], [12]). (Dadarlat and Gong showed, independently, that every such AH algebra is isomorphic to one of the special inductive limits considered in [10], and these latter algebras are tracially AF by construction.)

1.3. **The Elliott-Gong-Li Theorem.**

After years of hard work, the real-rank-zero assumption was removed when the following theorem was proved.

Theorem 1.16 (Elliott-Gong-Li, [11])**.** *If A and B are unital simple AH algebras with very slow dimension growth, and if $\eta\colon \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ is an isomorphism, then there is an isomorphism $\pi\colon A \to B$ such that $\pi_* = \eta$.*

Very slow dimension growth is a technical condition that means A is the inductive limit of algebras

$$A_n = \bigoplus_{i=1}^{k(n)} p_i^{(n)} \mathbb{M}_{m(n,i)}(C(X_i^{(n)}))p_i^{(n)},$$

where

$$\lim_{n\to\infty} \left(\max_{1\le i\le k(n)} \frac{\dim(X_i^{(n)})^3}{\mathrm{rank}(p_i^{(n)})} \right) = 0.$$

As in the real rank zero case, a crucial step in proving the Elliott-Gong-Li Theorem is a reduction result (due to Gong, see [13]) which shows that every unital simple AH algebra with very slow dimension growth is isomorphic to another AH algebra with a very special inductive limit decomposition; it is these very special inductive limits that end up being classified in [11].

1.4. **Counterexamples.**

In 2001, Mikael Rørdam solved an old and important problem by constructing a simple C*-algebra that contained an infinite projection and a finite projection [39]. In fact, he even constructed a nuclear example, which definitively ended hope of classifying all simple nuclear C*-algebras via their Elliott Invariants. (Since his example has the Elliott Invariant of a purely infinite algebra – the infinite projection prevents traces – yet can't be purely infinite because of the finite projection.)

A few years later, in [44], Andrew Toms constructed a simple AH algebra with the Elliott Invariant of something covered by the Elliott-Gong-Li Theorem, but which didn't have slow dimension growth.[3] Hence the Elliott Invariant isn't even complete for the very special class of AH algebras, let alone more general stably finite algebras.

[3] More precisely, Andrew's example had a pathology in its Cuntz semigroup that can't occur in a slow-dimension-growth algebra.

1.5. **Coup d'état.** In 2007 Wilhelm Winter began a revolution – singlehandedly, and by force. He defined natural algebras that ought to be classifiable. He developed a coherent program for classifying them. And he successfully carried out the program to a remarkable degree.

It's an exciting time to be studying nuclear C*-algebras and I hope to convey some of this in the pages that follow.

2. Day 7: Order-zero maps, decomposition rank and nuclear dimension.

We now introduce the right class of algebras and become acquainted with definitions, ideas, permanence properties and other basic facts.

2.1. **Order-zero maps.** The theory of c.p. maps and nuclear C*-algebras was largely worked out in the 1970s, but Wilhelm's work exploits the following special c.p. maps.

Definition 2.1. A map $\varphi\colon A \to B$ is said to have *order zero* if it is c.c.p. and preserves orthogonality, meaning $\varphi(a)\varphi(b) = 0$ whenever a and b are self-adjoint and $ab = 0$.[4]

It is clear that $*$-homomorphisms have order zero, however there are other important examples. For example, define a completely positive map $A \to C_0(0,1] \otimes A$ by $a \mapsto x \otimes a$ (where x denotes the linear function with slope 1). This is clearly an order-zero map, but it turns out every other order-zero map arises from the composition of it and a $*$-homomorphism. This follows from an important structure theorem.

Theorem 2.2. [59, Theorem 2.3] *Let $\varphi\colon A \to B$ be an order-zero map and set $C := C^*(\varphi(A))$. Then there is a positive element $h \in C' \cap \mathcal{M}(C)$ (the relative commutant of C inside its multiplier algebra) with $\|h\| = \|\varphi\|$ and a $*$-homomorphism*

$$\pi_\varphi\colon A \to \{h\}' \cap \mathcal{M}(C)$$

such that

$$\varphi(a) = \pi_\varphi(a)h,$$

for all $a \in A$.

If A is unital, then $h = \varphi(1_A)$.

Remark 2.3. A few remarks are in order.

1. The map π_φ is called a *support $*$-homomorphism*.
2. Comparing with Stinespring's Theorem, which essentially says that c.p. maps are compressions of representations, we see that the order-zero ones are simply compressions *by something in the commutant*.
3. Order-zero maps are rarely unital (unlike general c.p. maps). Indeed, if $\varphi\colon A \to B$ is unital, then $h = \varphi(1_A) = 1_B$, so compressing by h doesn't change π_φ.
4. When A is unital, there is a one-to-one correspondence between $*$-homomorphisms $C_0(0,1] \otimes A \to B$ and order-zero maps $A \to B$. Indeed, the operator h determines a $*$-homomorphism $C_0(0,1] \to C^*(h)$, $x \mapsto h$, and the fact that h commutes with the image of π_φ ensures that this map extends to $C_0(0,1] \otimes A$.
5. If A is unital, $\varphi\colon A \to B$ has order zero and $\varphi(1_A) = h$ is an invertible element in B, then the support $*$-homomorphism π_φ takes values in B (since $C^*(\varphi(A))$ contains h and hence contains the unit of B).

[4]We don't really have to insist on contractive maps in the definition, it's just convenient. Also, there is a notion of order-n maps, but we won't need it (cf. [50]).

In summary, one should think of order-zero maps as extremely close to $*$-homomorphisms. When the domain is finite dimensional and the target algebra has tons of projections, this can be sharpened a bit; we can almost arrange that the image of the order-zero map is contained in a finite-dimensional subalgebra. To make this precise, we introduce another definition.

Definition 2.4. An order-zero map $\varphi\colon F \to B$ is *discrete* if $\varphi(p)$ is a multiple of a projection for every minimal central projection $p \in F$.

When F is finite-dimensional, our case of interest, we can write $1 = p_1 + \cdots + p_k$, where p_i are the central projections coming from the matrix summands of F. Hence discreteness of $\varphi\colon F \to B$ implies that

$$\varphi(1) = \varphi(p_1) + \cdots + \varphi(p_k) = \alpha_1 q_1 + \cdots + \alpha_k q_k$$

for scalars α_i and *orthogonal* projections $q_1, \ldots, q_k \in B$. Let $q = q_1 + \cdots + q_k$ and note that $h = \varphi(1)$ is invertible in the corner qBq (if each α_i is non-zero, otherwise cut to a smaller corner). Thus – and here's the crucial point – the map $x \mapsto h^{-1}\varphi(x)$ is a unital order-zero map into qBq; hence it's a $*$-homomorphism (actually, a support $*$-homomorphism). Finally, since this $*$-homomorphism takes p_i to q_i, its image contains h and thus contains the image of φ. Thus, we have proved the following fact.

Lemma 2.5. *If F is finite dimensional and $\varphi\colon F \to B$ is a discrete order-zero map, then there is a support $*$-homomorphism $\pi_\varphi\colon F \to B$ containing $h = \varphi(1_F)$. Thus $\varphi(F) \subset \pi_\varphi(F)$.*

The proof of the following fact is not all that hard, but I won't reproduce the proof – see [52, Lemma 2.4].

Proposition 2.6. *Assume F is finite-dimensional, $\varphi\colon F \to B$ is an order-zero map and B has real rank zero.*[5] *For every $\varepsilon > 0$ we can find a larger finite-dimensional algebra F', a unital embedding $\iota\colon F \to F'$ and a* discrete *order-zero map $\varphi'\colon F' \to B$ such that $\varphi(1_F) \le \varphi'(1_{F'})$ and $\|\varphi(x) - \varphi' \circ \iota(x)\| < \varepsilon\|x\|$ for all $x \in F$.*

2.2. **Order-zero approximation of nuclear C*-algebras.** It is a remarkable fact that all nuclear C*-algebras admit approximations by order-zero maps. However, this fact would likely have gone unnoticed had it not been for Wilhelm's work on classification. Put another way, we should regard the following theorem as a deep consequence of the classification program.[6]

Theorem 2.7. *If A is a nuclear C*-algebra, then one can find c.c.p. maps $\varphi_k\colon A \to F_k$ and $\psi_k\colon F_k \to A$ such that $\psi_k \circ \varphi_k \to \mathrm{id}_A$ pointwise, each F_k is finite dimensional and the maps*

$$\psi_k|_{pF_k}\colon pF_k \to A,$$

where $p \in F_n$ is a minimal central projection, all have order zero.

Proof. Here I follow Kirchberg's sketch, but starting from the QD case and pointing out how this leads to approximately order-zero maps φ_k. Or would it be better to outline Ilan's (less technical) argument? □

[5] Meaning every self-adjoint can be approximated by a self-adjoint with finite spectrum.

[6] This result was found by Eberhard Kirchberg and Ilan Hirshberg, independently, and I thank them for showing me their proofs.

It's even possible to arrange that the maps φ_k are "asymptotically" order zero, meaning that the induced map

$$\bar{\varphi}\colon A \to \frac{\Pi F_k}{\oplus F_k},$$

given by $a \mapsto (\varphi_k(a)) + \oplus F_k$ has order zero. I think.

2.3. **Decomposition rank and nuclear dimension.** Theorem 2.7 is more than just an improvement on the approximation property characterizing nuclearity, it points to a natural notion of dimension – one that's distinctly nuclear.

Definition 2.8. We say the *nuclear dimension of A is bounded by n*, or write $\dim_{nuc}(A) \leq n$, if there exist c.c.p. maps $\varphi_k\colon A \to F_k$ and c.p. maps $\psi_k\colon F_k \to A$ such that $\psi_k \circ \varphi_k \to \mathrm{id}_A$ pointwise, each F_k is finite dimensional and there exists a partition $1_{F_k} = p_0 + \cdots + p_n$ where each p_i is a central projection and $\psi_k|_{p_i F_k}\colon p_i F_k \to A$ has order zero.[7]

Comparing with Theorem 2.7, the nuclear dimension simply asks for a uniform bound on the number of summands that we can restrict to and get order-zero maps. Also, note that we didn't require the ψ_k's to be contractive (though $\|\psi_k\| \leq n+1$, since each $\psi_k|_{p_i F_k}$ is contractive). Adding this axiom may seem innocuous, but it actually leads to a much stronger definition.

Definition 2.9. We say the *decomposition rank of A is bounded by n*, or write $\mathrm{dr}(A) \leq n$, if there exist c.c.p. maps $\varphi_k\colon A \to F_k$ and $\psi_k\colon F_k \to A$ such that $\psi_k \circ \varphi_k \to \mathrm{id}_A$ pointwise, each F_k is finite dimensional and there exists a partition $1_{F_k} = p_0 + \cdots + p_n$ where each p_i is a central projection and $\psi_k|_{p_i F_k}\colon p_i F_k \to A$ has order zero.

As usual, one then defines the nuclear dimension, respectively decomposition rank, of A to be the smallest integer n satisfying the appropriate definition. To get started let's analyze the case of nuclear dimension zero.

Proposition 2.10. $\dim_{nuc}(A) = 0$ *if and only if A is AF.*

Proof. It is clear that every AF algebra has nuclear dimension zero, so assume $\dim_{nuc}(A) = 0$. For simplicity, let's also assume A is unital. Then we can find c.c.p. maps $\varphi_k\colon A \to F_k$ and $\psi_k\colon F_k \to A$ such that $\psi_k \circ \varphi_k \to \mathrm{id}_A$ pointwise and ψ_k has order zero. Since φ_k is contractive, $\varphi_k(1_A) \leq 1_{F_k}$; hence $\psi_k(\varphi_k(1_A)) \leq \psi_k(1_{F_k}) \leq 1_A$, the last inequality being due to the contractivity of ψ_k. But this implies $\psi_k(1_{F_k}) \to 1$ (because $\psi_k(\varphi_k(1_A)) \to 1_A$), which in turn implies two things: the support $*$-homomorphisms π_{ψ_k} take values in A, since $\psi_k(1_{F_k})$ is invertible, and for any $a \in A$ we have

$$a \approx \psi_k(\varphi_k(a)) = \psi_k(1_{F_k})\pi_{\psi_k}(\varphi_k(a)) \approx \pi_{\psi_k}(\varphi_k(a)).$$

Finally, since $\pi_{\psi_k}(\varphi_k(a))$ is contained in a finite-dimensional subalgebra of A, we see that A must be AF. $\square$

The following fact is a little harder (see [60, Proposition 2.4]), but one direction is a good exercise. Namely, using partitions of unity, you may want to show that the topological covering dimension dominates the nuclear dimension.

Proposition 2.11. *If X is a second countable compact Hausdorff space, then*

$$\dim_{nuc}(C(X)) = \dim_{top}(X).$$

[7]Of course, if no such maps exist, then $\dim_{nuc}(A) = \infty$.

Here are some fairly straightforward inequalities.

Proposition 2.12. *The following statements make good exercises:*

(1) $\dim_{nuc}(A \oplus B) \leq \max\{\dim_{nuc}(A), \dim_{nuc}(B)\}$;
(2) $\dim_{nuc}(A \otimes B) \leq (\dim_{nuc}(A)+1)(\dim_{nuc}(B)+1)-1$;
(3) $\dim_{nuc}(\varinjlim A_i) \leq \liminf(\dim_{nuc}(A_i))$;
(4) $\dim_{nuc}(A/J) \leq \dim_{nuc}(A)$.[8]

Here are some less trivial permanence-type facts.

Proposition 2.13. [20, Proposition 3.11] *If A is unital with unitization $\tilde{A}$, then* $\dim_{nuc}(A) = \dim_{nuc}(\tilde{A})$.

Proposition 2.14. [60, Proposition 2.5] $\dim_{nuc}(A) \leq \dim_{nuc}(B)$, *whenever $A \subset B$ is a hereditary subalgebra.*

The assumption that A is hereditary is essential in the previous proposition. Indeed, any abelian C*-algebra can be embed into an AF algebra, so nuclear dimension can decrease in arbitrary subalgebras.

Proposition 2.15. [60, Proposition 2.9] *For any ideal $J \triangleleft A$ we have*

$$\max\{\dim_{nuc}(J), \dim_{nuc}(A/J)\} \leq \dim_{nuc}(A) \leq \dim_{nuc}(J) + \dim_{nuc}(A/J) + 1.$$

2.4. **Examples.** It turns out that most of our favorite nuclear C*-algebras are known to be finite dimensional. Though the proofs are rarely easy, here's a few important examples.

(1) If X is a finite-dimensional topological space and $\alpha\colon X \to X$ is a minimal homeomorphism, then $C(X) \rtimes \mathbb{Z}$ is also finite dimensional. (See [48].)
(2) Finitely generated subhomogeneous algebras are finite dimensional. (See [32].)
(3) Kirchberg algebras satisfying the UCT all have dimension ≤ 5. (See [60].)
(4) All the AH and ASH algebras that have been classified are also finite dimensional (thanks to the various reduction theorems that go into their proofs).

3. Day 8: Classification: A Paradigm

3.1. **A little history.** In [4] Alain Connes proved there is a unique *injective* II$_1$-factor – namely, the hyperfinite II$_1$-factor R. His proof boiled down to two things. First, if M is an injective II$_1$-factor, then $M \bar{\otimes} R \cong R$. Next, Connes showed that M absorbs R tensorially, i.e., $M \cong M \bar{\otimes} R$, and the proof was complete. (See part 2 of [4, Theorem 5.1].)

Two decades later Kirchberg and Phillips followed the same pattern. It was shown that if A and B are Kirchberg algebras and there is an invertible element $\eta \in \mathrm{KK}^0(A,B)$, then

$$A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathbb{K}.$$

Indeed, Elliott's approximate intertwining argument together with Definition 1.4 (which isn't really a definition, it's a hard theorem!) yield this fact without too much trouble. But then Kirchberg's absorption theorem – i.e., the fact that $A \cong A \otimes \mathcal{O}_\infty$ for every Kirchberg algebra [18] – completes the proof of the Kirchberg-Phillips Theorem.

There are two lessons to take away from these examples.

[8] For this quotient result, you need to know the Choi-Effros lifting theorem,i.e., that there is a c.c.p. splitting $A/J \to A$ whenever A or A/J is nuclear.

(1) It is easier to classify after tensoring with nice algebras. (R in Connes's case, and $\mathcal{O}_\infty$ in Kirchberg and Phillips's case.) Very roughly, enlarging algebras in this way gives more space, more room to move, and this helps a lot. As a simple example of this phenomenon, two non-orthogonal projections in A can be made orthogonal in $A \otimes \mathcal{O}_\infty$ by tensoring with orthogonal projections from $\mathcal{O}_\infty$.
(2) Tensorial absorption of certain algebras is to be *expected* for large and natural classes of operator algebras. For example, since $\mathcal{O}_\infty$ is KK-equivalent to $\mathbb{C}$, it follows that $\mathrm{Ell}(A) \cong \mathrm{Ell}(A \otimes \mathcal{O}_\infty)$ for every Kirchberg algebra A. Hence, Kirchberg's absorption theorem was *predicted* by classification (and turned out to be a necessary step in proving classification because of lesson (1) above).

With one additional wrinkle, Wilhelm Winter followed the same pattern as Connes and Kirchberg-Phillips to prove the following remarkable theorem.

Theorem 3.1. *Assume A and B are unital simple separable C^*-algebras with finite decomposition rank and which satisfy the UCT. If both A and B have a unique trace (or, more generally, projections separate traces), then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.*

The remainder of this paper is devoted to outlining the proof and discussing the future.

3.2. **A three-step program.** Rather than two steps, like Connes, Kirchberg and Phillips, Wilhelm took three steps to his classification theorem.

$$\boxed{\text{Existing results}} \longrightarrow \boxed{\text{UHF-stable classification}}$$

The first step is to use existing classification results (Lin's Theorem, actually) to show that if A and B are as in Theorem 3.1 and have isomorphic Elliott Invariants, then $A \otimes \mathcal{U} \cong B \otimes \mathcal{U}$, where $\mathcal{U}$ is a UHF algebra. (See [55].) At first blush, this step appears to be of limited value since $A \otimes \mathcal{U}$ has tons of projections – hence can't be isomorphic A, in general – but it is just a precursor to the right stable-classification theorem.

$$\boxed{\text{UHF-stable classification}} \longrightarrow \boxed{\mathcal{Z}\text{-stable classification}}$$

In step two, one has to show that the UHF-stable classification theorem implies $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang-Su algebra – the finite analogue of $\mathcal{O}_\infty$. This was accomplished in [54] (with help, in an appendix, from Huaxin Lin).

$$\boxed{\mathcal{Z}\text{-absorption theorem}}$$

Finally, in a show of extraordinary technical strength, Wilhelm showed $A \cong A \otimes \mathcal{Z}$, thereby completing the proof ([57]).

3.3. **Introduction to the Jiang-Su algebra $\mathcal{Z}$.** For a supernatural number $\mathfrak{p}$ we let $\mathbb{M}_\mathfrak{p}$ denote the corresponding UHF algebra.

Definition 3.2. Given supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$, define the *generalized dimension drop algebra* to be

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}} := \{f \in C([0,1], \mathbb{M}_\mathfrak{p} \otimes \mathbb{M}_\mathfrak{q}) : f(0) \in \mathbb{M}_\mathfrak{p} \otimes 1 \text{ and } f(1) \in 1 \otimes \mathbb{M}_\mathfrak{q}\}.$$

It is a fact that $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ has a *trace-collapsing* unital endomorphism, so long as $\mathfrak{p}$ and $\mathfrak{q}$ are relatively prime and infinite. That is, there exists a unital $*$-homomorphism $\rho\colon \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ with the property that the image of the induced map on traces $\rho_*\colon \mathrm{T}(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}) \to \mathrm{T}(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$ is a single point. Put another way, the image of ρ lies in some sliver of $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ where all traces restrict to the same thing.

I will not explain how Rørdam and Winter constructed a trace-collapsing endomorphism (see [41]) because it would be very nice to have a different proof.

Problem 3.3. Assume $\mathfrak{p}$ and $\mathfrak{q}$ are relatively prime and construct a trace-collapsing endomorphism of $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ (without reference to the Jiang-Su algebra).

Though it's actually a theorem (see [41, Theorem 3.4]), we define the Jiang-Su algebra as follows.

Definition 3.4. Let $\rho\colon \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ be a trace collapsing unital endomorphism. Then the *Jiang-Su algebra* is the inductive limit of the (stationary) system

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\rho} \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\rho} \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\rho} \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to \cdots .$$

One should prove this doesn't depend on the choice of endomorphism. But we won't. The most important facts about the Jiang-Su algebra are summarized below (see [16] and [14] for proofs).

Theorem 3.5. *The following statements are not false.*

1. $\mathcal{Z}$ *is unital simple and nuclear with no nontrivial projections and a unique trace;*
2. $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots$*;*
3. *every unital endomorphism of* $\mathcal{Z}$ *is approximately inner;*
4. $K_0(\mathcal{Z}) = \mathbb{Z}$ *and* $K_1(\mathcal{Z}) = 0$*;*[9]
5. $K_0(A \otimes \mathcal{Z}) = \mathrm{K}_0(A)$*,* $K_1(A \otimes \mathcal{Z}) = \mathrm{K}_1(A)$ *and* $\mathrm{T}(A \otimes \mathcal{Z}) \cong \mathrm{T}(A \otimes \mathcal{Z})$*, for every* C^**-algebra* A*;*[10]
6. $\mathrm{Ell}(A \otimes \mathcal{Z}) \cong \mathrm{Ell}(A)$ *if and only if* $K_0(A)$ *is weakly unperforated.*

3.4. **Why "UHF-stable classification" might imply "$\mathcal{Z}$-stable classification".** One of Wilhelm's most brilliant innovations was the second step in the program outlined above, because classification becomes significantly easier when you tensor with a UHF algebra (this often forces real rank zero, for example). Though the technical issues are serious and sometimes subtle, the essence is as follows.

1. Assume A and B have the property that $A \otimes \mathbb{M}_{\mathfrak{p}} \cong B \otimes \mathbb{M}_{\mathfrak{p}}$ for every supernatural number $\mathfrak{p}$. Then one can hope to show $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \cong B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$, because these algebras are continuous fields with isomorphic fibers.[11]
2. Since $A \otimes \mathcal{Z}$ is the limit of

$$A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\mathrm{id}_A \otimes \rho} A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\mathrm{id}_A \otimes \rho} A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\mathrm{id}_A \otimes \rho} \cdots$$

and $B \otimes \mathcal{Z}$ is the limit of

$$B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\mathrm{id}_B \otimes \rho} B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\mathrm{id}_B \otimes \rho} B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\mathrm{id}_B \otimes \rho} \cdots ,$$

[9] Also, $\mathcal{Z}$ is KK-equivalent to $\mathbb{C}$.

[10] This is just the Kunneth formula for K-theory, together with the general fact that tensoring with a unique-trace algebra doesn't change the tracial state space.

[11] Emphasis on *hope*; it isn't true that all continuous fields with isomorphic fibers are isomorphic! cite Marius's paper...

one could hope to use an isomorphism $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \cong B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$, together with facts like "endomorphisms of $\mathcal{Z}$ are approximately inner," to get an approximate intertwining and deduce $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$.

In [54] Wilhelm nearly made this pipe-dream a reality for a large class of algebras with finite decomposition rank, only requiring some technical support from Lin in [27]. Shortly thereafter, a cleaner result was established: The Lin-Niu Theorem.

3.5. **The Lin-Niu theorem.** Following Wilhelm's original strategy, but overcoming significant technical difficulties, Lin and Niu proved the following general $\mathcal{Z}$-stable classification theorem.

Theorem 3.6. [29, Theorem 5.4] *Assume A and B are simple, separable, nuclear, unital C*-algebras satisfying the UCT and with the property that $A \otimes \mathbb{M}_{\mathfrak{p}}$ and $B \otimes \mathbb{M}_{\mathfrak{p}}$ are tracially AF for every supernatural number $\mathfrak{p}$.*[12] *If $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$, then $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$.*

4. Day 9: Proving Winter's classification theorem

Sorry, but I can't type up this lecture in its entirety. It contained too many misleading oversimplifications (read: lies) to commit to paper. ;-) My main goal was to convey the spirit of the proofs, so rigor was sacrificed at the alter of lucidity. However, here are a few things worth writing down.

4.1. **Step 3: $\mathcal{Z}$-stability is automatic.** Just because I described Wilhelm's work as a three-step program doesn't mean that we have to prove things in that order. In fact, the first step to be completed in complete generality was the third one.

Theorem 4.1. [58] *Let A be a simple unital separable C*-algebra of finite nuclear dimension. Then $A \cong A \otimes \mathcal{Z}$.*

The proof of this fact is a technical monstrosity, as difficult as anything I've ever read. However, I will point out that a crucial ingredient is essentially a C*-analogue of McDuff's celebrated characterization of II_1-factors that absorb the hyperfinite II_1-factor tensorially. Though not explicitly stated, the following result can be deduced from [41].

Lemma 4.2. *$A \cong A \otimes \mathcal{Z}$ if and only if for every $s \in \mathbb{N}$, there exists an order-zero map $\varphi\colon \mathbb{M}_s \to A' \cap A_\infty$ (the central sequence algebra) such that $(1 - \varphi(1)) \precsim \varphi(e_{11})$ in $A' \cap A_\infty$ (meaning there exist $x_n \in A' \cap A_\infty$ such that $x_n^* \varphi(e_{11}) x_n \to (1 - \varphi(1))$).*

4.2. **Step 2: proving classification up to $\mathcal{Z}$-stability.** See Theorem 3.6.

4.3. **Step 1: proving classification up to UHF-stability.** So Theorem 3.1 is reduced to showing that if $\dim_{nuc}(A) < \infty$ and A has a unique trace, then $A \otimes \mathbb{M}_{\mathfrak{p}}$ is tracially AF. To do this we need a couple theorems. See [37] for the following theorem.

Theorem 4.3. *If A is simple, unital, nuclear and has a unique trace, then the following assertions hold.*

1. *$K_0(A \otimes \mathbb{M}_{\mathfrak{p}})$ is weakly unperforated;*
2. *$A \otimes \mathbb{M}_{\mathfrak{p}}$ has stable rank one;*
3. *$A \otimes \mathbb{M}_{\mathfrak{p}}$ has real rank zero (this is the only part that requires a unique trace);*

[12] Actually, one only needs to know this for a pair of relatively prime supernatural numbers whose product yields the universal UHF algebra; see [29].

Next we need a result of Lin (see [23]).

Theorem 4.4. *Assume C has real rank zero, stable rank one, $K_0(C)$ is weakly unperforated and for every finite $\mathfrak{F} \subset C$ and $\varepsilon > 0$ there exists a finite-dimensional $B \subset C$ with unit p such that*

1. *$\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$*
2. *$d(pxp, B) < \varepsilon$ for all $x \in \mathcal{F}$*
3. *$\tau(p) > 1 - \varepsilon$ for every $\tau \in \mathrm{T}(A)$.*

Then C is tracially AF.

The following exercise is an excellent introduction to an important geometric series argument.

Exercise 4.5. Assume C has real rank zero, stable rank one, $K_0(C)$ is weakly unperforated and a unique trace τ. Also, assume there exists $\delta > 0$ such that A and all of its hereditary subalgebras have the following property: for every finite set $\mathfrak{F}$ and $\varepsilon > 0$ there exists a finite-dimensional B with unit p such that

1. $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$
2. $d(pxp, B) < \varepsilon$ for all $x \in \mathcal{F}$
3. $\tau(p) > \delta$ (one replaces τ with an appropriately rescaled version when considering hereditary subalgebras).

Then C is tracially AF.

Put another way, if you have a unique trace and can find a *uniform* lower bound on the sizes of the finite-dimensional algebras B, then you can use a geometric series argument to construct a finite-dimensional algebra that's large in trace.

Here's another good exercise that follows from Lemma 2.5 and Proposition 2.6.

Exercise 4.6. If A has finite nuclear dimension and real rank zero, then there exist c.c.p. maps $\varphi_k \colon A \to F_k$ and c.p. maps $\psi_k \colon F_k \to A$ such that $\psi_k \circ \varphi_k \to \mathrm{id}_A$ pointwise, each F_k is finite dimensional and there exists a partition $1_{F_k} = p_0 + \cdots + p_n$ where each p_i is a central projection, $\psi_k|_{p_i F_k} \colon p_i F_k \to A$ has order zero and – here's the point – there are support $*$-homomorphisms $\pi_i^{(k)} \colon p_i F_k \to A$.

We can now sketch the proof of the following result of Winter.

Theorem 4.7. *If A is separable simple and unital, has finite decomposition rank and a unique trace, then $A \otimes \mathbb{M}_{\mathfrak{p}}$ is tracially AF for every infinite supernatural number $\mathfrak{p}$.*

To prove this, we first note that $A \otimes \mathbb{M}_{\mathfrak{p}}$ has real rank zero, stable rank one, weakly unperforated K-theory and a unique trace τ; hence we can hope to use Exercise 4.5. Indeed, Exercise 4.6 provides candidate finite-dimensional subalgebras – the images of the support homomorphisms. And we can let δ be something like $\frac{1}{n+1}$ since $1_A \approx \psi_k(\varphi_k(1))$ implies that the trace of at least one of the order-zero blocks has to be bigger than $\frac{1}{n+1}$.

So far this reasoning works fine for finite nuclear dimension and it's only at the final stage that we have to use decomposition rank. Namely, we don't yet know how to arrange the approximate commutativity required by Exercise 4.5 if one only assumes finite nuclear dimension; but this can be arranged with finite decomposition rank (see [52]).

5. Day 10: Applications and Winter's Program.

5.1. **Applications.** There is a notion of "slow dimension growth" for ASH algebras, but it would require defining "recursive subhomogeneous algebras" and this would take us too far afield (see [45], for example).

Theorem 5.1. [58, Corollary 6.6] *Let A and B be simple, unital, ASH algebras with slow dimension growth and assume that projections separate traces.*[13] *Then $A \cong B$ if and only if* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$*.*

Note that we haven't assumed finite nuclear dimension. However, our algebras do have finite nuclear dimension locally, and so the main result of [58] would apply if we knew that such ASH algebras have the Cuntz semigroup of a $\mathcal{Z}$-stable algebra. Luckily, they do (see [45]), and hence we deduce that such algebras are $\mathcal{Z}$-stable; i.e., step 3 of the classification program is done. So, by the Lin-Niu Theorem it suffices to show $A \otimes \mathbb{M}_{\mathfrak{p}}$ is tracially AF. But this was accomplished in [54] when Wilhelm proved that local finite decomposition rank plus $\mathcal{Z}$-stable plus projections separating traces is enough to imply that UHF-amplifications are tracially AF.

It is possible to formulate the next result as a classification amongst crossed product algebras, but we'll state it in terms of ASH algebras instead.

Theorem 5.2. *Let (X, α) be a minimal, uniquely ergodic dynamical system with X finite dimensional. Then, $C(X) \rtimes \mathbb{Z}$ is an ASH algebra with no dimension growth.*[14]

Here's a sketch of the proof. First, by [48], $C(X) \rtimes \mathbb{Z}$ is $\mathcal{Z}$-stable. (This fact only depends on the finite-dimensionality of X.) Hence, it suffices to prove classification up to UHF-stability; i.e., it suffices to show $(C(X) \rtimes \mathbb{Z}) \otimes \mathcal{U}$ is TAF (again, by the Lin-Niu Theorem). But showing that $(C(X) \rtimes \mathbb{Z}) \otimes \mathcal{U}$ is TAF is very similar to [30] (see [48] for details).

5.2. **Winter's Program.** I'll now outline a classification program that has been suggested by Wilhelm Winter. (Many thanks, Wilhelm, for letting me reproduce it here!)

Since step 3 is complete in breathtaking generality, we only have to consider analogues of the first two steps. But it turns out that Huaxin Lin has already generalized the Lin-Niu Theorem to algebras with tracial rank ≤ 1 (these are like TAF algebras, except finite dimensional algebras are replaced with 1-dimensional homogeneous algebras, see [24]).

Theorem 5.3. [28, Theorem 11.7] *Assume A and B are simple, separable, nuclear, unital C*-algebras satisfying the UCT and with the property that $A \otimes \mathbb{M}_{\mathfrak{p}}$ and $B \otimes \mathbb{M}_{\mathfrak{p}}$ have tracial rank ≤ 1 for every supernatural number $\mathfrak{p}$. If* $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$*, then $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$.*

For the remainder of these notes, A **denotes a simple unital separable C*-algebra with finite nuclear dimension**.

So, here's what we have to do.....

(1) First we should find a proof of Theorem 3.1 that only assumes finite nuclear dimension. As I explained in the last section, decomposition rank is only needed at one technical point of the proof. And we should correct this.

[13]I.e., for every pair of distinct traces τ_1, τ_2, there is a projection p such that $\tau_1(p) \neq \tau_2(p)$.

[14]In fact, the base spaces have dimension bounded by 2.

(2) It would be nice if $A \otimes \mathbb{M}_{\mathfrak{p}}$ always had the Elliott invariant of something with tracial rank ≤ 1. But unfortunately it's K_0 group could fail the Riesz interpolation property, though this is the only problem. So, the next thing we should try to do is assume $\mathrm{K}_0(A \otimes \mathbb{M}_{\mathfrak{p}})$ has the Riesz property, and show that it has tracial rank ≤ 1.

Karen Strung and Wilhelm have made progress on this problem, assuming restrictions on the number of traces and that the algebra is ASH. But if the general case is proved, then we've classified all finite-nuclear-dimension algebras whose K-theory satisfies Riesz interpolation (and the UCT). And that would be awesome.

(3) The final step in the program will be to pass from the 1-dimensional homogeneous case above, to 1-dimensional **sub**homogeneous algebras. That is, we will have to find a class $\mathcal{S}$ of 1-dimensional subhomogeneous algebras such that TA$\mathcal{S}$ algebras exhaust the unperforated Elliott invariants (Guihua Gong has already proposed a candidate for this); and you can generalize [28] to TA$\mathcal{S}$ algebras (warning: this step will require generalizing the classification theorem of [26] before you can even attempt to generalize [28]); and you can prove the analogue of the previous problem (i.e., tensoring with a UHF algebra lands one in the TA$\mathcal{S}$ class).

Do this, and you've completed the classification of simple, unital, finite nuclear dimensional algebras satisfying the UCT in terms of the Elliott Invariant. Then it'll rain champagne – because I'm buying!

5.3. **Other problems.** Wilhelm also suggested the following problems: Give a direct proof (without assuming the UCT and appealing to classification results) that locally finite nuclear dimension and $\mathcal{Z}$-stable implies finite nuclear dimension. Also, compute the decomposition rank of $C(X) \otimes \mathcal{U}$ when X is infinite dimensional; this concrete, non-simple example remains quite mysterious.

We should also continue to search for examples of algebras with local finite nuclear dimension (e.g., more general cross products) and which have the Cuntz semigroup of a $\mathcal{Z}$-stable algebras.

I hesitate to mention it, but there is one more problem that I would love to see resolved (though I don't recommend it as a thesis problem!). Namely, can we find a "better" invariant for classifying general nuclear C*-algebras? Of course "better" isn't well defined, but here are a couple litmus tests I would apply. Whatever the new invariant is, it should lead to simpler proofs of existing classification theorems (meaning it will probably be some huge invariant carrying tons of structural information). Ideally, it would also treat the purely infinite and stably finite cases at the same time (though this may be too much to ask for, and I'd happily overlook a failure of this litmus test). But, this new invariant should still be functorially equivalent to the Elliott invariant for algebras with finite nuclear dimension. (E.g. in the stably finite case you may consider the Cuntz semigroup plus K_1, or the Thomsen semigroup plus K_1, but these will not do in the purely infinite case.....and I'm really hoping for an invariant that works equally well in both cases.)

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