Lecture 2: Rational curves and the canonical divisor

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Introduction

Low dimensions

High dimensions

Bend-and-Break
Guiding principle

Recall from last time that the canonical line bundle of a smooth projective variety $X$ is

$$
\omega_X = \bigwedge^{\dim X} \Omega_X
$$

and the canonical divisor $K_X$ is any divisor representing $\omega_X$.

**Principle**

The geometry/arithmetic of a smooth projective variety $X$ over a field is controlled by the positivity of $K_X$.

We will discuss this principle in the context of rational curves. We work over the ground field $\mathbb{C}$ unless otherwise specified.
There are different ways of interpreting the “positivity” of a divisor. To start with we will focus on the three types of “pure” positivity:

<table>
<thead>
<tr>
<th>negative</th>
<th>torsion</th>
<th>positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-K_X$ ample</td>
<td>a multiple of $K_X$ is 0</td>
<td>$K_X$ ample</td>
</tr>
</tbody>
</table>

Of course, most projective varieties will not have one of these three “pure” curvature types. However, the Minimal Model Program predicts that any smooth projective variety can be decomposed into a sequence of fibrations whose fibers have “pure” type.
Low dimensions
Let’s start by analyzing our guiding principle when $X$ is a curve. The basic invariant for classifying curves is the genus, but for our purposes it is better to use (the negative of) the Euler characteristic

$$\deg(K_X) = 2g(C) - 2.$$  

With this definition it becomes clear that there is a trichotomy of curves:

<table>
<thead>
<tr>
<th>$\deg(K_X)$</th>
<th>$&lt; 0$</th>
<th>$= 0$</th>
<th>$&gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>genus</td>
<td>0</td>
<td>1</td>
<td>$\geq 2$</td>
</tr>
<tr>
<td>$\text{Mor}(\mathbb{P}^1, X)_d$</td>
<td>open subset of $\mathbb{P}^{2d+1}$</td>
<td>empty</td>
<td>empty</td>
</tr>
</tbody>
</table>

Note that this same trichotomy occurs in other areas of mathematics as well (Riemann Uniformization Theorem, behavior of rational points, etc.).
We next consider the case when $X$ is a surface. We analyze the behavior of rational curves separately for surfaces with the three types of positivity for the canonical divisor.

A surface $X$ with $-K_X$ ample is known as a del Pezzo surface. These surfaces have been completely classified: with the exception of $\mathbb{P}^1 \times \mathbb{P}^1$, a del Pezzo surface is the blow-up of $\mathbb{P}^2$ along at most 8 points in general position. In particular, each del Pezzo surface is birationally equivalent to $\mathbb{P}^2$.

We can find rational curves through any general point of $X$ by taking the strict transforms of rational curves on $\mathbb{P}^2$. We conclude that a del Pezzo surface $X$ is uniruled.
Surfaces

In the Kodaira-Enriques classification there are four types of surface with $K_X$ torsion.

1) Abelian surfaces.
An abelian surface cannot contain any rational curves. Consider any morphism $f : \mathbb{P}^1 \to X$ and its differential $T_{\mathbb{P}^1} \to f^* T_X$. We have $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$ and (since an abelian surface has trivial tangent bundle) $f^* T_X \cong \mathcal{O}_{\mathbb{P}^1} \oplus 2$. Thus the map on tangent bundles is the zero map and $f$ contracts $\mathbb{P}^1$ to a point.

2) Hyperelliptic surfaces.
A hyperelliptic surface cannot contain any rational curves. The Albanese map $alb : X \to B$ maps $X$ to an elliptic curve and the fibers of $alb$ are irreducible curves of genus $\geq 1$. 
3) K3 surfaces.
A K3 surface $X$ can contain a rational curve. (For example, a quartic surface in $\mathbb{P}^3$ can contain a line.) But $X$ is not uniruled: for any non-trivial $f : \mathbb{P}^1 \to X$ the pullback $f^* T_X$ is a rank 2 bundle of degree 0. Since $f^* T_X$ must admit a non-zero map from $\mathcal{O}(2)$, it also must have a negative summand.

In fact much more is true:

**Theorem (Chen-Gounelas-Liedtke)**

*Every complex K3 surface contains infinitely many non-free rational curves.*

4) Enriques surfaces.
Every Enriques surface is a quotient of a K3 surface and so has similar behavior.
Finally, we consider the case when $K_X$ is ample.

**Conjecture (Algebraic hyperbolicity)**

A smooth projective surface with $K_X$ ample will have only finitely many rational curves.

This conjecture has been verified in some cases. For example, one of the early results is:

**Theorem (Clemens)**

*A very general surface of degree $\geq 5$ in $\mathbb{P}^3$ contains no rational curves.*

Despite some fantastic partial progress, the conjecture remains open in general.
High dimensions
Higher dimensions

When $X$ is a smooth projective variety of dimension $\geq 3$ the picture is similar:

<table>
<thead>
<tr>
<th>$-K_X$ ample</th>
<th>$K_X$ torsion</th>
<th>$K_X$ ample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thm: (Mori) $X$ is uniruled.</td>
<td>“inbetween”</td>
<td>Conj: The rational curves are contained in a proper Zariski closed subset of $X$.</td>
</tr>
</tbody>
</table>

Here “inbetween” covers a range of possibilities: $X$ might admit no rational curves at all (abelian variety) or could admit infinitely many rational curves (K3 surface). However we will soon show that if $K_X$ is torsion then $X$ cannot be uniruled. Thus the rational curves on $X$ sweep out at most a countable union of proper closed subvarieties of $X$. 
Before moving on, we discuss one more notion of “positivity” for the canonical divisor. This notion is based around the behavior of sections of the canonical divisor.

**Definition**

Let $X$ be a smooth projective variety. If $H^0(X, mK_X) = 0$ for every $m > 0$, we say that $X$ has Kodaira dimension $-\infty$. Otherwise, we define the Kodaira dimension to be the smallest non-negative integer $r$ such that

$$\limsup_{m \to \infty} \frac{h^0(X, mK_X)}{m^r} < \infty.$$ 

One can show that the Kodaira dimension of $X$ takes values in the set $\{-\infty, 0, 1, 2, \ldots, \dim(X)\}$. If $K_X$ is ample, torsion, or antiample then $\kappa(X) = \dim(X), 0, -\infty$ respectively.
Higher dimensions

Proposition

If $X$ is uniruled then $\kappa(X) = -\infty$.

Proof: Suppose for a contradiction that $H^0(X, mK_X) \neq 0$ for some $m > 0$. Since $X$ is uniruled, we can find a free rational curve $f : \mathbb{P}^1 \to C \subset X$ and a section $D \in |mK_X|$ such that $D|_C$ does not vanish. In particular $\deg(f^* K_X) \geq 0$.

Consider now the vector bundle $f^* T_X$ of rank $\dim(X)$ on $\mathbb{P}^1$. The calculation above shows that $\deg(f^* T_X) \leq 0$. Since $f$ is free, this must imply that $f^* T_X = \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim(X)}$. However, since $f$ does not contract $\mathbb{P}^1$ to a point there should also be a non-zero map $\mathcal{O}_{\mathbb{P}^1}(2) = T_{\mathbb{P}^1} \to f^* T_X$, yielding a contradiction. \qed
Conversely, the Kodaira dimension should predict the behavior of rational curves. On one extreme, we have:

**Conjecture**

If $\kappa(X) = -\infty$ then $X$ is uniruled.

On the other extreme, we have:

**Conjecture**

If $\kappa(X) = \dim(X)$ then there is a proper closed subset of $X$ which contains all the rational curves on $X$.

This is a birational version of the algebraic hyperbolicity conjecture for rational curves.
Bend-and-Break
For the rest of the lecture, we will focus on Mori’s result: a smooth complex variety with $-K_X$ ample is uniruled. In fact, we will sketch the proof of a stronger theorem:

**Theorem (Mori)**

Let $X$ be a smooth projective variety. Suppose that $C$ is a curve in $X$ satisfying $K_X \cdot C < 0$. Then there is a rational curve in $X$ through every point of $C$.

This immediately implies the desired result for varieties with $-K_X$ ample.
In order to prove this theorem, we will need to understand the space of morphisms $\text{Mor}(B, X)$ where $B$ is a smooth projective curve of arbitrary genus. Fortunately, the situation is exactly the same:

- $\text{Mor}(B, X)$ can be constructed as a subscheme of $\text{Hilb}(B \times X)$ and thus admits a universal family.
- Given a morphism $f : B \to X$, the tangent space to the morphism scheme at $f$ is $H^0(B, f^* T_X)$.
- The expected dimension

$$\chi(f^* T_X) = -K_X \cdot f_* B + (1 - g(B)) \dim(X)$$

gives a lower bound for the dimension of $\text{Mor}(B, X)$ near $f$. 

Bend-and-Break
We will also need a slight modification: we will consider morphisms $f : B \to X$ which send a fixed point in $B$ to a fixed point in $X$.

Suppose we fix a map $f : B \to X$ and a point $p \in B$. We denote by $\text{Mor}(B, X; f|_p)$ the sublocus of maps $g \in \text{Mor}(B, X)$ such that $g(p) = f(p)$. We will also need to analyze the tangent space of this subscheme:

- Given a morphism $f : B \to X$ and a point $p \in B$, the tangent space to $\text{Mor}(B, X; f|_p)$ at $f$ is $H^0(B, f^* T_X \otimes \mathcal{O}_B(-p))$.

- The expected dimension

$$
\chi(f^* T_X \otimes \mathcal{O}_B(-p)) = -K_X \cdot f^* B - g(B) \cdot \dim(X)
$$

gives a lower bound for the dimension of $\text{Mor}(B, X; f|_p)$ near $f$. 


**Theorem (Mori’s Bend-and-Break)**

Let $X$ be a smooth projective variety and let $B$ be a smooth projective curve of genus $\geq 1$. Fix a non-trivial map $f : B \to X$ and a point $p \in B$ and suppose we have a curve $T \subset \text{Mor}(B, X; f|_p)$ containing $f$. Then there is a rational curve through $f(p)$ in $X$.

**Proof:** Let $T' \to T$ denote the normalization and let $U'$ denote the base-change of the universal family to $T'$. Thus $U' \cong B \times T'$ and we have a map $ev : U' \to X$ that contracts the section $\{p\} \times T'$.

We next compactify: we let $\overline{T}$ denote a smooth projective curve containing $T'$ and let $\overline{U}$ denote $B \times \overline{T}$. We now have a rational map $ev : \overline{U} \dashrightarrow X$. 

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**Introduction**

**Low dimensions**

**High dimensions**

**Bend-and-Break**
The next step is to appeal to:

**Rigidity Lemma:** Suppose that $ev : B \times \overline{T} \rightarrow X$ is well-defined at every point of the section $\{p\} \times \overline{T}$ and contracts this section to a point. Then $ev$ factors through the projection map to $B$.

**Proof of lemma:** A projective curve is contracted by $ev$ if and only if it has vanishing intersection against the pullback of an ample divisor on $X$. But this is a numerical property; if it is true for one section, it will be true for all of them.

Since by assumption $T$ parametrizes a family of morphisms which vary in moduli, we see that $ev : B \times \overline{T} \rightarrow X$ must fail to be defined at some point $(p, t)$. 
Bend-and-Break

The last step is to appeal to the birational geometry of surfaces.

We know that the rational map $ev$ can be resolved. That is, there is a birational map $\phi : S \to B \times \overline{T}$ obtained by a sequence of point blow-ups and a morphism $ev_S : S \to X$ which agrees with $ev$ on the common locus of definition.

Consider the fiber of $\phi$ over $(p, t)$; this is a union of rational curves on $S$. Not all of these curves can be contracted by $ev_S$; if they were, then our original map $ev$ would have been defined at $(p, t)$. Furthermore, the image of this fiber must intersect the image of the strict transform in $S$ of $\{p\} \times \overline{T}$. Altogether, we see that at least one of the rational curves in the fiber of $\phi$ over $(p, t)$ must survive on $X$ and go through $f(p)$. $\square$
Figure 4: The 1-cycle $f_*C$ degenerates to a 1-cycle with a rational component $e(E)$.

Picture from Debarre, “Bend and Break”
Bend-and-Break

There is also a Bend-and-Break theorem for rational curves.

Given a morphism $f : \mathbb{P}^1 \to X$ and two different points $p, q \in \mathbb{P}^1$, we denote by $\text{Mor}(\mathbb{P}^1, X; f|_{p,q})$ the set of morphisms $g : \mathbb{P}^1 \to X$ such that $g(p) = f(p)$ and $g(q) = f(q)$.

**Theorem (Mori’s Bend-and-Break)**

*Let $X$ be a smooth projective variety. Fix a non-trivial map $f : \mathbb{P}^1 \to X$ and points $p, q \in \mathbb{P}^1$. Suppose we have a curve $T \subset \text{Mor}(B, X; f|_{p,q})$ containing $f$ such that the maps parametrized by $T$ sweep out a surface in $X$. Then the image cycle $f_*(\mathbb{P}^1)$ deforms to a non-integral curve with rational components which contains $f(p)$ and $f(q)$.***
Figure 5: The rational 1-cycle $f_* C$ bends and breaks

Picture from Debarre, “Bend and Break”
We now return to our original goal:

**Theorem (Mori)**

Let $X$ be a smooth projective variety. Suppose that $C$ is a curve in $X$ satisfying $K_X \cdot C < 0$. Then there is a rational curve in $X$ through every point of $C$.

Let $f : B \to C \subset X$ denote the normalization map. It suffices to consider the case when $g(B) \geq 1$. Fix any point $p \in B$. If we knew that $\text{Mor}(B, X; f|_p)$ had dimension $\geq 1$, then Bend-and-Break would imply the existence of the desired rational curve through $p$.

Of course, there is no reason to assume that $\dim(\text{Mor}(B, X; f|_p)) \geq 1$. In fact the expected dimension

$$-K_X \cdot f_*(B) - g(B) \cdot \dim(X)$$

might be very negative. Mori found an ingenious way around this obstacle by passing to characteristic $p$. 


Sketch of proof:

Step 1: spreading out
We can choose an algebra $\mathcal{Z}$ that is finitely generated over $\mathbb{Z}$ such that every relevant object in our situation is defined over $\mathcal{Z}$. After possibly shrinking $\text{Spec}(\mathcal{Z})$, we can find a smooth map $\mathcal{X} \to \text{Spec}(\mathcal{Z})$ whose fiber over the generic point is isomorphic to $\mathcal{X}$ (after extending the base field). We may also ensure that all our constructions extend over all of $\mathcal{X}$.

Thus for every closed point $z \in \text{Spec}(\mathcal{Z})$ we obtain a fiber $X_z$ and a curve $C_z$ satisfying $K_{X_z} \cdot C_z < 0$. Note that each such $X_z$ is defined over a finite field of characteristic $p > 0$. 
Bend-and-Break

**Sketch of proof:**

Step 2: twisting up

Let \( f_z : B_z \to C_z \subset X_z \) denote the normalization map. Fix a point \( p_z \in B_z \) and consider the scheme \( \text{Mor}(B_z, X_z; f_z|_{p_z}) \). As remarked earlier, there is no reason to assume that the expected dimension

\[
-K_{X_z} \cdot f_z^*(B_z) - g(B_z) \cdot \dim(X_z)
\]

is positive. However, suppose that we now precompose \( f_z \) by \( r \) iterates of the Frobenius map for \( B_z \). If we let \( h_z \) denote the composed map and let \( p \) denote the characteristic of the residue field of \( p_z \), the expected dimension is now

\[
p^r(-K_{X_z} \cdot f_z^*(B_z)) - g(B_z) \cdot \dim(X_z)
\]

By assumption this will be positive when \( r \) is large enough. Applying Bend-and-Break we obtain a rational curve \( Y_z \) through every point of \( C_z \).
Bend-and-Break

**Sketch of proof:**

**Step 3: deforming back**

For every closed point $z \in \text{Spec}(Z)$ and every point $p_z \in C_z$ we have found a rational curve $Y_z$ through $p_z$. We would now like to “deform” these rational curves back to the generic fiber to find a rational curve on our original variety $X$.

If we knew that the rational curves $Y_z$ were bounded – that is, if they were contained in a finite-type subscheme of the relative Hilbert scheme – then there would have to be a single component of the Hilbert scheme that parametrized the curves for a dense open subset of $\text{Spec}(Z)$. By Chevalley’s Theorem, the image of this component in $\text{Spec}(Z)$ would also contain the generic point. Since the geometric genus is constant in the family, we would obtain the desired rational curve through the point $p \in C$. 
Sketch of proof:

Step 4: breaking down

Unfortunately Bend-and-Break gives us essentially no control over the rational curves $Y_z$ we constructed in Step 2. In particular, there is no reason to expect that as we vary the closed point $z \in \text{Spec}(\mathbb{Z})$ the rational curves $Y_z$ form a bounded family. In other words, if we fix an ample divisor $A$ on $X$ then the degrees of the $Y_z$ against $A$ could be unbounded.

Fortunately, Bend-and-Break comes to our rescue again. If the $A$-degree of $Y_z$ is large enough then the deformation space of $Y_z$ through the point $p_z$ has dimension $> \dim(X) + 1$. In particular, we can find a curve in the moduli space parametrizing deformations of $Y_z$ through $p_z$ and through some other fixed point. Applying the rational curve version of Bend-and-Break, we find a different rational curve $Y'_z$ through $p_z$ of smaller $A$-degree. Arguing inductively, we eventually find a bounded family of rational curves $Y_z$ which allow us to conclude by the previous step.