Supplement to
"Large Risks, Limited Liability and Dynamic Moral Hazard"

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Abstract

In this document, we give complete proofs for the results exposed in "Large Risks, Limited Liability and Dynamic Moral Hazard." A precise description of the stochastic environment is provided in Appendix A. In Appendix B, we use martingale techniques to formulate the agent's incentive compatibility constraint. Appendix C is devoted to the free boundary problem that characterizes the principal's value function. The verification theorem is established in Appendix D. In Appendix E, we analyze the asymptotic properties of firm size dynamics. Finally, a heuristic approach to small perturbations of the constant returns to scale model is offered in Appendix F.

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A The Stochastic Environment

In this Appendix, we provide a precise description of the stochastic environment. Let be given a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ over which is defined a Poisson process $N = \{N_t\}_{t\geq 0}$ of intensity λ. Denote by $\mathcal{F}^N = \{\mathcal{F}^N_t\}_{t\geq 0}$ the filtration generated by N and augmented by the $\mathbf{P}$–null sets. This filtration satisfies the usual conditions (Dellacherie and Meyer (1978, Chapter IV, Definition 48)). The process $M = \{M_t\}_{t\geq 0}$ defined by

$$M_t = N_t - \lambda t$$

for all $t \geq 0$ is an $\mathcal{F}^N$–martingale under $\mathbf{P}$. For any $\mathcal{F}^N$–predictable process $\Lambda = \{\Lambda_t\}_{t\geq 0}$ with values in $\{\lambda, \lambda + \Delta\lambda\}$, denote by $Z^\Lambda = \{Z^\Lambda_t\}_{t\geq 0}$ the unique solution to the stochastic differential equation

$$Z^\Lambda_t = 1 + \int_0^t Z^\Lambda_{s^-} \left(\frac{\Lambda_s}{\lambda} - 1 \right) dM_s$$

for all $t \geq 0$. By the exponential formula for Lebesgue–Stieltjes calculus (Brémaud (1981, Appendix A4, Theorem T4)), one has

$$Z^\Lambda_t = \prod_{s\in(0,t]} \left[1 + \left(\frac{\Lambda_s}{\lambda} - 1 \right) \Delta N_s \right] \exp\left(\int_0^t (\lambda - \Lambda_s)\, ds \right)$$

for all $t \geq 0$, where $\Delta N_s = N_s - N_{s^-}$ for all $s \in [0, t]$, with $N_{0^-} = 0$ and $\prod_\emptyset = 1$ by convention. From Brémaud (1981, Chapter VI, Theorem T2), Z^Λ is a positive $\mathcal{F}^N$–local martingale under $\mathbf{P}$. Moreover $\mathbf{E}\,[Z^\Lambda_t] = 1$ for all $t \geq 0$. A standard extension argument implies that there exists a unique probability measure $\mathbf{P}^\Lambda$ over $(\Omega, \mathcal{F})$ defined by the family of Radon–Nikodym derivatives

$$\left. \frac{d\mathbf{P}^\Lambda}{d\mathbf{P}} \right|_{\mathcal{F}^N_t} = Z^\Lambda_t$$

for all $t \geq 0$. It then follows from Brémaud (1981, Chapter VI, Theorem T3) that the process M^Λ defined by (11) is an $\mathcal{F}^N$–martingale under $\mathbf{P}^\Lambda$.

B The Incentive Compatibility Constraint

Proof of Lemma 1. Since $U_\tau(\Gamma, \Lambda)$ is integrable by (8), one can define a positive $\mathcal{F}^N$–martingale $U(\Gamma, \Lambda)$ under $\mathbf{P}^\Lambda$ by choosing for each $t \geq 0$ a random variable $U_t(\Gamma, \Lambda)$ in the equivalence class of the conditional expectation in (10). Moreover, since the filtration $\mathcal{F}^N$ satisfies the usual conditions, one can for each $t \geq 0$ choose $U_t(\Gamma, \Lambda)$ in such a way that the martingale $U(\Gamma,\Lambda)$ is right-continuous with left-hand limits (Dellacherie and Meyer (1982, Chapter VI, Theorem 4)). The predictable representation (12) then follows directly from Brémaud (1981, Chapter III, Theorems T9 and T17). ■

Proof of Proposition 1. Let U'_t denote the agent's lifetime expected payoff, given the information available at date t, when she acts according to $\Lambda' = \{\Lambda'_t\}_{t\geq 0}$ until date t and then reverts to $\Lambda = \{\Lambda_t\}_{t\geq 0}$:

$$U'_t = \int_0^{t\wedge\tau^-} e^{-\rho s}(dL_s + 1_{\{\Lambda'_s=\lambda+\Delta\lambda\}} X_s B ds) + e^{-\rho t} W_t(\Gamma,\Lambda). \tag{B.1}$$

Following Sannikov (2008, Proposition 2), the proof now proceeds as follows. First, one shows that if $U' = \{U'_t\}_{t\geq 0}$ is an $\mathcal{F}^N$–submartingale under $\mathbf{P}^{\Lambda'}$ that is not a martingale, then Λ is suboptimal for the agent. Indeed, in that case there exists some $t > 0$ such that

$$U_{0^-}(\Gamma,\Lambda) = U'_{0^-} < \mathbf{E}^{\Lambda'}[U'_t],$$

where $U_{0^-}(\Gamma,\Lambda)$ and U'_{0^-} correspond to unconditional expected payoffs at date 0. By (B.1), the agent is then strictly better off acting according to Λ' until date t and then reverting to Λ. The claim follows. Next, one shows that if U' is a $\mathcal{F}^N$–supermartingale under $\mathbf{P}^{\Lambda'}$, then Λ is at least as good as Λ' for the agent. From (10) and (B.1), one has

$$U'_t = U_t(\Gamma,\Lambda) + \int_0^{t\wedge\tau} e^{-\rho s}(1_{\{\Lambda'_s=\lambda+\Delta\lambda\}} - 1_{\{\Lambda_s=\lambda+\Delta\lambda\}}) X_s B\, ds \tag{B.2}$$

for all $t \geq 0$. Hence, since $U(\Gamma,\Lambda)$ as given by (12) is right-continuous with left-hand limits, so is U'. Moreover, since U' is positive, it has a last element. Hence, by the optional sampling theorem (Dellacherie and Meyer (1982, Chapter VI, Theorem 10)),

$$U'_{0^-} \geq \mathbf{E}^{\Lambda'}[U'_\tau] = U_{0^-}(\Gamma,\Lambda'),$$

where again $U_{0^-}(\Gamma,\Lambda')$ is an unconditional expected payoff at date 0. Since $U'_{0^-} = U_{0^-}(\Gamma,\Lambda)$ by (B.1), the claim follows. Now, for each $t \geq 0$,

$$\begin{aligned} U'_t &= U_t(\Gamma,\Lambda) + \int_0^{t\wedge\tau} e^{-\rho s}(1_{\{\Lambda'_s=\lambda+\Delta\lambda\}} - 1_{\{\Lambda_s=\lambda+\Delta\lambda\}}) X_s B\, ds \\ &= U_0(\Gamma,\Lambda) - \int_0^{t\wedge\tau} e^{-\rho s} H_s(\Gamma,\Lambda)\, dM_s^{\Lambda} + \int_0^{t\wedge\tau} e^{-\rho s}(1_{\{\Lambda'_s=\lambda+\Delta\lambda\}} - 1_{\{\Lambda_s=\lambda+\Delta\lambda\}}) X_s B\, ds \\ &= U_0(\Gamma,\Lambda) - \int_0^{t\wedge\tau} e^{-\rho s} H_s(\Gamma,\Lambda)\, dM_s^{\Lambda'} - \int_0^{t\wedge\tau} e^{-\rho s} H_s(\Gamma,\Lambda)(\Lambda'_s - \Lambda_s)\, ds \\ &\quad + \int_0^{t\wedge\tau} e^{-\rho s}(1_{\{\Lambda'_s=\lambda+\Delta\lambda\}} - 1_{\{\Lambda_s=\lambda+\Delta\lambda\}}) X_s B\, ds \end{aligned}$$

$$= U_0(\Gamma, \Lambda) - \int_0^{t\wedge\tau} e^{-\rho s} H_s(\Gamma, \Lambda)\, dM_s^{\Lambda'}$$

$$+ \int_0^{t\wedge\tau} e^{-\rho s} \Delta\lambda (1_{\{\Lambda'_s = \lambda + \Delta\lambda\}} - 1_{\{\Lambda_s = \lambda + \Delta\lambda\}})[X_s b - H_s(\Gamma, \Lambda)]\, ds,$$

where the first equality follows from (B.2), the second from (12), the third from (11), and the fourth from a straightforward computation. Since $H(\Gamma, \Lambda)$ is $\mathcal{F}^N$–predictable and $M^{\Lambda'}$ is an $\mathcal{F}^N$–martingale under $\mathbf{P}^{\Lambda'}$, the drift of U' has the same sign as

$$(1_{\{\Lambda'_t = \lambda + \Delta\lambda\}} - 1_{\{\Lambda_t = \lambda + \Delta\lambda\}})[X_t b - H_t(\Gamma, \Lambda)]$$

for all $t \in [0, \tau)$. If (14) holds for the effort process Λ, then this drift remains negative for all $t \in [0, \tau)$ and all choices of $\Lambda'_t \in \{\lambda, \lambda + \Delta\lambda\}$. This implies that for any effort process Λ', U' is an $\mathcal{F}^N$–supermartingale under $\mathbf{P}^{\Lambda'}$, and thus that Λ is at least as good as Λ' for the agent. If (14) does not hold for the effort process Λ, then choose Λ' such that for each $t \in [0, \tau)$, $\Lambda'_t = \lambda$ if and only if $H_t(\Gamma, \Lambda) \geq X_t b$. The drift of U' is then everywhere positive, and strictly positive over a set of $\mathbf{P}^{\Lambda'}$–strictly positive measure. As a result of this, U' is an $\mathcal{F}^N$–submartingale under $\mathbf{P}^{\Lambda'}$ that is not a martingale, and thus Λ is suboptimal for the agent. This concludes the proof. ■

C The Value Function

To simplify the exposition, we shall work in this appendix with the size-adjusted social value function, v, rather than with the size-adjusted value function of the principal, f. These two functions are related by $v(w) = f(w) + w$ for all $w \geq 0$, so that (41) can be rewritten as:

$$\begin{cases} v(w) = \frac{v(b)}{b}\, w & \text{if } w \in [0, b], \\ rv(w) = \mu - \lambda C - (\rho - r)w + \mathcal{L}v(w) & \text{if } w \in (b, w^i], \\ (r-\gamma)v(w) = \mu - \lambda C - \gamma c - (\rho - r)w + \mathcal{L}_\gamma v(w) & \text{if } w \in (w^i, w^p], \\ v(w) = v(w^p) & \text{if } w \in (w^p, \infty), \end{cases} \tag{C.1}$$

where $\mathcal{L}$ and $\mathcal{L}_\gamma$ are linear first-order delay differential operators defined by

$$\mathcal{L}u(w) = (\rho w + \lambda b)u'(w) - \lambda[u(w) - u(w-b)] \tag{C.2}$$

and

$$\mathcal{L}_\gamma u(w) = \mathcal{L}u(w) - \gamma w u'(w) \tag{C.3}$$

for all $w > b$ and any continuous function u of class $C^1(\mathbb{R}_+\backslash\{b\})$. We assume that

$$\mu - \lambda C > (\rho - r)b\Big(2 + \frac{r}{\lambda}\Big) \tag{C.4}$$

throughout this appendix.

C.1 The No Investment Case

As a preliminary, we deal with the case in which investment is not feasible, that is $\gamma = 0$. For each $\beta \geq 0$, consider the delay differential equation

$$\begin{cases} v_\beta(w) = \beta w & \text{if } w \in [0, b], \\ r v_\beta(w) = \mu - \lambda C - (\rho - r)w + \mathcal{L} v_\beta(w) & \text{if } w \in (b, \infty). \end{cases} \tag{C.5}$$

Given the initial condition over the interval $[0, b]$, which is fixed by the slope parameter β, (C.5) reduces to a sequence of initial value problems over the intervals $(kb, (k+1)b]$, $k \in \mathbb{N}\backslash\{0\}$, that satisfy the assumptions of the Cauchy–Lipschitz theorem. This ensures that there exists a unique continuous solution v_β to (C.5), which can be recursively constructed. One can check from (C.4) and (C.5) that v_β is not differentiable at b:

$$v'_{\beta+}(b) = \frac{(\rho - r)b - \mu + \lambda C}{(\rho + \lambda)b} + \beta\,\frac{r + \lambda}{\rho + \lambda} < \beta = v'_{\beta-}(b). \tag{C.6}$$

Since v_β is continuous, however, it follows from (C.5) that it is of class $C^1(\mathbb{R}_+\backslash\{b\})$. As a result, one can differentiate (C.5) over $\mathbb{R}_+\backslash\{b, 2b\}$, which in turn implies that v_β is of class $C^2(\mathbb{R}_+\backslash\{b, 2b\})$. By iterating this procedure, one can easily verify that v_β is of class $C^k(\mathbb{R}_+\backslash\{b, \ldots, kb\})$ for all $k \in \mathbb{N}\backslash\{0\}$.

For each $\beta \geq 0$, it is convenient to decompose v_β as follows:

$$v_\beta = u_1 + \beta u_2, \tag{C.7}$$

where the auxiliary functions u_1 and u_2 are the continuous solutions to the delay differential equations

$$\begin{cases} u_1(w) = 0 & \text{if } w \in [0, b], \\ r u_1(w) = \mu - \lambda C - (\rho - r)w + \mathcal{L} u_1(w) & \text{if } w \in (b, \infty) \end{cases} \tag{C.8}$$

and

$$\begin{cases} u_2(w) = w & \text{if } w \in [0, b], \\ r u_2(w) = \mathcal{L} u_2(w) & \text{if } w \in (b, \infty), \end{cases} \tag{C.9}$$

respectively. Just as v_β, u_1 and u_2 are of class $C^k(\mathbb{R}_+\backslash\{b,\dots,kb\})$ for all $k\in\mathbb{N}\backslash\{0\}$. The decomposition (C.7) allows us to strictly order the derivatives of the functions $(v_\beta)_{\beta\geq 0}$.

Proposition C.1.1 *If $\beta>\beta'\geq 0$, then $v'_\beta>v'_{\beta'}$ over $\mathbb{R}_+\backslash\{b\}$.*

Given the decomposition (C.7), Proposition C.1.1 is an immediate consequence of the following result.

Lemma C.1.1 *$u'_2>0$ over $\mathbb{R}_+\backslash\{b\}$.*

Proof. From (C.9), $u'_2=1$ over the interval $[0,b)$. Consider now the interval (b,∞). From (C.9) again, it is easy to check that

$$u'_{2+}(b)=\frac{r+\lambda}{\rho+\lambda}>0. \tag{C.10}$$

Thus, since u_2 is of class $C^1(\mathbb{R}_+\backslash\{b\})$, one only needs to check that u'_2 has no zero in (b,∞). Arguing by contradiction, let $\tilde{w}>b$ be the first point at which u'_2 vanishes. Note that $u'_2>0$ over $[0,\tilde{w})\backslash\{b\}$. Then, using (C.9) yet again, one obtains that

$$-\lambda[u_2(\tilde{w})-u_2(\tilde{w}-b)]-ru_2(\tilde{w})=0,$$

which is impossible since u_2 is strictly increasing and strictly positive over $(0,\tilde{w}]$. This contradiction establishes the result. ■

Proposition C.1.1 shows that the derivatives of the functions $(v_\beta)_{\beta\geq 0}$ are strictly ordered by their slopes β over $[0,b)$. We now show that the subfamily of $(v_\beta)_{\beta\geq 0}$ composed of those functions whose derivatives have at least a zero in (b,∞) has a maximal element.

Proposition C.1.2 *There exists a maximum value β_0 of β such that the equation $v'_\beta=0$ has a solution over (b,∞). The function v_{β_0} is increasing over $\mathbb{R}_+$.*

The proof of Proposition C.1.2 proceeds as follows. For each $w\in[b,\infty)$, the ratio $-\frac{u'_{1+}(w)}{u'_{2+}(w)}$ is well defined since $u'_{2+}>0$ over $[b,\infty)$ by Lemma C.1.1. In the first step of the proof, we show that this ratio attains a maximum $\beta_0>0$ over $[b,\infty)$. Using Proposition C.1.1 along with the decomposition (C.7), we then obtain that

$$v'_\beta>v'_{\beta_0}=u'_1+\beta_0u'_2\geq 0$$

over (b,∞) for all $\beta>\beta_0$. Hence, for any such β, v'_β has no zero in (b,∞). By contrast, let $w^p_{\beta_0}$ be the smallest point at which the function $-\frac{u'_{1+}}{u'_{2+}}$ attains its maximum β_0 over $[b,\infty)$.

In the second step of the proof, we show that $w^p_{\beta_0} > b$, so that v'_{β_0} is differentiable at $w^p_{\beta_0}$. By construction,

$$v'_{\beta_0}(w^p_{\beta_0}) = u'_1(w^p_{\beta_0}) + \beta_0 u'_2(w^p_{\beta_0}) = 0,$$

and v_{β_0} is increasing over $\mathbb{R}_+$, and strictly so over $[0, w^p_{\beta_0}]$. We now provide a detailed exposition of each step of the proof.

Step 1 Because u_1 and u_2 are of class $C^1(\mathbb{R}_+\backslash\{b\})$, the function $-\frac{u'_{1+}}{u'_{2+}}$ is continuous over $[b,\infty)$. Moreover, since $u'_{2+}(b) > 0$ by (C.10) and

$$u'_{1+}(b) = \frac{(\rho - r)b - \mu + \lambda C}{(\rho+\lambda)b} < 0 \tag{C.11}$$

by (C.4) and (C.8), $-\frac{u'_{1+}(b)}{u'_{2+}(b)} > 0$. Hence, to show that the function $-\frac{u'_{1+}}{u'_{2+}}$ attains its maximum over $[b,\infty)$, one only needs to check that it takes strictly negative values beyond some point. Given Lemma C.1.1, this is an immediate consequence of the following result.

Lemma C.1.2 $\liminf_{w\to\infty} u'_1(w) \geq 1$.

Proof. Suppose first by way of contradiction that $\liminf_{w\to\infty} u'_1(w) = -\infty$. Then there exists an increasing divergent sequence $(w_n)_{n\geq 1}$ in $(2b,\infty)$ such that $\lim_{n\to\infty} u'_1(w_n) = -\infty$ and $w_n = \arg\min_{w\in[0,w_n]}\{u'_{1+}(w)\}$. For each $n \geq 1$, one can find some $\tilde{w}_n \in (w_n - b, w_n)$ such that

$$\begin{aligned}(\rho w_n + \lambda b)u'_1(w_n) &= \lambda[u_1(w_n) - u_1(w_n - b)] + r u_1(w_n) + (\rho - r)w_n - \mu + \lambda C \\ &= \lambda b u'_1(\tilde{w}_n) + r u_1(w_n) + (\rho - r)w_n - \mu + \lambda C,\end{aligned}$$

where the first equality follows from (C.8) and the second from the mean value theorem. This can conveniently be rewritten as:

$$u'_1(\tilde{w}_n) = \frac{w_n}{\lambda b}\left[\rho u'_1(w_n) - \frac{r}{w_n}\, u_1(w_n)\right] + u'_1(w_n) + \frac{\mu - \lambda C - (\rho - r)w_n}{\lambda b}.$$

Since $u_1(0) = 0$, one has $u_1(w_n) \geq w_n u'_1(w_n)$ by construction of the sequence $(w_n)_{n\geq 1}$. Moreover, $u'_1(w_n) < 0$ for n large enough. It then follows that for any such n,

$$u'_1(\tilde{w}_n) \leq \frac{(\rho - r)w_n u'_1(w_n) + \mu - \lambda C}{\lambda b}.$$

Therefore, since $u'_1(w_n) < 0$,

$$\frac{u'_1(\tilde{w}_n)}{u'_1(w_n)} \geq \frac{(\rho - r)w_n}{\lambda b} + \frac{\mu - \lambda C}{\lambda b u'_1(w_n)},$$

so that the ratio $\frac{u_1'(\tilde{w}_n)}{u_1'(w_n)}$ goes to ∞ as n goes to ∞. As $u_1'(w_n) < 0$ for n large enough, one obtains that eventually $u_1'(\tilde{w}_n) < u_1'(w_n)$, which, since $\tilde{w}_n < w_n$, contradicts the fact that $w_n = \arg\min_{w \in [0,w_n]}\{u_{1+}'(w)\}$. Thus $\liminf_{w\to\infty} u_1'(w) > -\infty$. Assume without loss of generality that $\liminf_{w\to\infty} u_1'(w)$ is a finite number l. It remains to prove that $l \geq 1$. Consider an increasing divergent sequence $(w_n)_{n\geq 1}$ in $(2b, \infty)$ such that $\lim_{n\to\infty} u_1'(w_n) = l$. Then there exists a constant U such that $u_1(w_n) \geq lw_n + U$ for all $n \geq 1$. Constructing $\tilde{w}_n \in (w_n - b, w_n)$ as above and rearranging, it follows that

$$\rho[u_1'(w_n) - 1] - r(l-1) \geq \frac{\lambda b[u_1'(\tilde{w}_n) - u_1'(w_n)] + rU - \mu + \lambda C}{w_n}$$

for all $n \geq 1$. Letting n go to ∞, one obtains that

$$(\rho - r)(l-1) \geq \lambda b \limsup_{n\to\infty} \frac{u_1'(\tilde{w}_n)}{w_n}.$$

If $l < 1$, this implies that $\limsup_{n\to\infty} u_1'(\tilde{w}_n) = -\infty$, which in turn contradicts the finiteness of $l = \liminf_{w\to\infty} u_1'(w)$. Hence $l \geq 1$, and the result follows. ■

Step 2 A sufficient condition for $w^p_{\beta_0} > b$ is that the right derivative at b of the function $-\frac{u_{1+}'}{u_{2+}'}$ be strictly positive. Differentiating (C.8) and (C.9) at the right of b leads to

$$u_{1+}''(b) = \frac{(\lambda - \rho + r)u_{1+}'(b) + \rho - r}{(\rho + \lambda)b}$$

and

$$u_{2+}''(b) = \frac{(\lambda - \rho + r)u_{2+}'(b) - \lambda}{(\rho + \lambda)b}.$$

Combining these expressions with (C.10) and (C.11), one obtains that

$$\begin{aligned} -u_{1+}''(b)u_{2+}'(b) + u_{2+}''(b)u_{1+}'(b) &= -\frac{(\rho - r)u_{2+}'(b) + \lambda u_{1+}'(b)}{(\rho + \lambda)b} \\ &= \frac{\lambda}{b^2(\rho+\lambda)^2}\left[\mu - \lambda C - (\rho - r)b\Big(2 + \frac{r}{\lambda}\Big)\right], \end{aligned}$$

which is strictly positive by (C.4). The result follows. By construction, one has[1]

$$w^p_{\beta_0} = \inf\big\{(v_{\beta_0}')^{-1}(0)\big\} > b. \tag{C.12}$$

This concludes the proof of Proposition C.1.2.

[1]One can show along the lines of the proof of Lemma C.1.6 that v_{β_0}' vanishes at $w^p_{\beta_0}$ only, so that v_{β_0} is actually strictly increasing over $\mathbb{R}_+$. This refined statement is however not required for our purposes.

In the remainder of this section, we study the concavity of the function v_{β_0}. The following proposition summarizes our findings.

Proposition C.1.3 *v_{β_0} is concave over $[0, w^p_{\beta_0}]$, and strictly so over $[b, w^p_{\beta_0}]$.*

The proof of Proposition C.1.3 proceeds through a sequence of lemmas.

Lemma C.1.3 *$v''_{\beta_0+}(b) < 0$.*

Proof. By (C.6) and (C.12), one has

$$v'_{\beta_0+}(b) = \frac{(\rho - r)b - \mu + \lambda C}{(\rho + \lambda)b} + \beta_0 \frac{r + \lambda}{\rho + \lambda} > 0. \tag{C.13}$$

As a result of this,

$$\beta_0 > \frac{\mu - \lambda C - (\rho - r)b}{(r + \lambda)b}. \tag{C.14}$$

Now, differentiating (C.5) at the right of any $w \geq b$ leads to

$$(\rho w + \lambda b)v''_{\beta_0+}(w) = \lambda[v'_{\beta_0+}(w) - v'_{\beta_0+}(w - b)] - (\rho - r)[v'_{\beta_0+}(w) - 1].$$

Applying this formula at b and using (C.13) and (C.14), one then obtains that

$$\begin{aligned}(\rho + \lambda)bv''_{\beta_0+}(b) &= \lambda[v'_{\beta_0+}(b) - \beta_0] - (\rho - r)[v'_{\beta_0+}(b) - 1] \\ &= \frac{(\lambda - \rho + r)[(\rho - r)b - \mu + \lambda C]}{(\rho + \lambda)b} + \beta_0 \frac{(r - \rho)(r + 2\lambda)}{\rho + \lambda} + \rho - r \\ &< \frac{\lambda[(\rho - r)b - \mu + \lambda C]}{(r + \lambda)b} + \rho - r,\end{aligned}$$

which is strictly negative under (C.4). Hence the result. ■

Lemma C.1.4 *v''_{β_0+} is upper semicontinuous over $[b, \infty)$.*

Proof. As v_{β_0} is of class $C^2(\mathbb{R}_+ \backslash \{b, 2b\})$, one only needs to check that $v''_{\beta_0+}(2b) > v''_{\beta_0-}(2b)$. Differentiating (C.5) both at the left and at the right of any $w > b$ and using the fact that v_{β_0} is of class $C^1(\mathbb{R}_+ \backslash \{b\})$ leads to

$$(\rho w + \lambda b)[v''_{\beta_0+}(w) - v''_{\beta_0-}(w)] = \lambda[v'_{\beta_0-}(w - b) - v'_{\beta_0+}(w - b)].$$

Applying this formula at $2b$ and using (C.6) yields the result. ■

It follows from Lemma C.1.4 that the set $\{w \geq b \,|\, v''_{\beta_0+}(w) \geq 0\}$ is closed. Denote by $w^c_{\beta_0}$ its smallest element. By Lemma C.1.3, $w^c_{\beta_0} > b$ and $v''_{\beta_0+} < 0$ over $[b, w^c_{\beta_0})$. Thus v_{β_0} is strictly concave over $[b, w^c_{\beta_0}]$. Moreover, v_{β_0} is linear over $[0, b]$ and $v'_{\beta_0+}(b) < v'_{\beta_0-}(b)$ by (C.6). Thus v_{β_0} is concave over $[0, w^c_{\beta_0}]$. To complete the proof of Proposition C.1.3, we now show that $w^p_{\beta_0}$ coincides with $w^c_{\beta_0}$. We shall need the following result.

Lemma C.1.5 $w^c_{\beta_0} \geq 2b$.

Proof. Suppose by way of contradiction that $w^c_{\beta_0} < 2b$. Then, as $w^c_{\beta_0} > b$ and v_{β_0} is of class $C^2(\mathbb{R}_+ \backslash \{b, 2b\})$, $v''_{\beta_0}(w^c_{\beta_0}) = 0$ and $v''_{\beta_0} < 0$ over $(b, w^c_{\beta_0})$. There are three cases to consider.

Case 1 Suppose first that $\lambda \leq \rho - r$. Since $w^c_{\beta_0} - b < b$ and $v''_{\beta_0}(w^c_{\beta_0}) = 0$, differentiating (C.5) at $w^c_{\beta_0}$ yields

$$\lambda[v'_{\beta_0}(w^c_{\beta_0}) - \beta_0] - (\rho - r)[v'_{\beta_0}(w^c_{\beta_0}) - 1] = 0.$$

Using the fact that $\lambda \leq \rho - r$ and that $v'_{\beta_0} \geq 0$ over (b, ∞), one obtains that

$$\beta_0 = \frac{(\lambda - \rho + r)v'_{\beta_0}(w^c_{\beta_0}) + \rho - r}{\lambda} \leq \frac{\rho - r}{\lambda}.$$

By (C.14), it follows that

$$\frac{\mu - \lambda C - (\rho - r)b}{(r + \lambda)b} \leq \frac{\rho - r}{\lambda}$$

which contradicts (C.4).

Case 2 Suppose next that $\lambda \geq 2\rho - r$. Differentiating (C.5) twice over $(b, 2b)$ and using the fact that v_{β_0} is linear over $(0, b)$ yields

$$(\rho w + \lambda b)v'''_{\beta_0}(w) = \lambda[v''_{\beta_0}(w) - v''_{\beta_0}(w - b)] - (2\rho - r)v''_{\beta_0}(w) = (\lambda - 2\rho + r)v''_{\beta_0}(w)$$

for all $w \in (b, 2b)$. Since $\lambda \geq 2\rho - r$ and $v''_{\beta_0} < 0$ over $(b, w^c_{\beta_0})$, this implies that $v'''_{\beta_0} \leq 0$ over this interval, and hence $v''_{\beta_0}(w^c_{\beta_0}) \leq v''_{\beta_0+}(b)$. This leads to a contradiction since $v''_{\beta_0}(w^c_{\beta_0}) = 0$ and $v''_{\beta_0+}(b) < 0$ by Lemma C.1.3.

Case 3 Suppose finally that $\rho - r < \lambda < 2\rho - r$. Differentiating (C.5) twice as in Case 2 shows that v''_{β_0} and v'''_{β_0} have opposite signs over $(b, 2b)$. It follows that $v'''_{\beta_0} > 0$ and hence $v''_{\beta_0} > v''_{\beta_0+}(b)$ over $(b, w^c_{\beta_0}]$. Since $\lambda - 2\rho + r < 0$, one obtains that

$$v'''_{\beta_0}(w) = \frac{(\lambda - 2\rho + r)v''_{\beta_0}(w)}{\rho w + \lambda b} < \frac{(\lambda - 2\rho + r)v''_{\beta_0+}(b)}{\rho w + \lambda b}$$

for all $w \in (b, w^c_{\beta_0})$. One then has

$$v''_{\beta_0}(w^c_{\beta_0}) = v''_{\beta_0+}(b) + \int_b^{w^c_{\beta_0}} \frac{(\lambda - 2\rho + r)v''_{\beta_0}(w)}{\rho w + \lambda b}\, dw < \left(1 + \int_b^{w^c_{\beta_0}} \frac{\lambda - 2\rho + r}{\rho w + \lambda b}\, dw\right) v''_{\beta_0+}(b).$$

Since $v''_{\beta_0}(w^c_{\beta_0}) = 0$ and $v''_{\beta_0+}(b) < 0$ by Lemma C.1.3, one obtains a contradiction if

$$1 + \int_b^{w^c_{\beta_0}} \frac{\lambda - 2\rho + r}{\rho w + \lambda b}\, dw > 0.$$

To see that this actually holds whenever $w^c_{\beta_0} \in (b, 2b)$, observe that

$$\int_b^{w^c_{\beta_0}} \frac{1}{\rho w + \lambda b}\, dw < \int_b^{2b} \frac{1}{\rho w + \lambda b}\, dw < \frac{1}{\rho + \lambda}.$$

Since $\rho - r < \lambda < 2\rho - r$, this implies that

$$1 + \int_b^{w^c_{\beta_0}} \frac{\lambda - 2\rho + r}{\rho w + \lambda b}\, dw > \frac{2\lambda - \rho + r}{\rho + \lambda} > 0,$$

and the result follows. ■

Proposition C.1.3 is then an immediate consequence of the following result.

Lemma C.1.6 $w^p_{\beta_0} = w^c_{\beta_0}$.

Proof. Since v_{β_0} is increasing and $v'_{\beta_0}(w^p_{\beta_0}) = 0$ by Proposition C.1.2, one must have $v''_{\beta_0+}(w^p_{\beta_0}) \geq 0$, and thus $w^p_{\beta_0} \geq w^c_{\beta_0}$. It remains therefore to prove that $w^p_{\beta_0} \leq w^c_{\beta_0}$. One first shows that $v''_{\beta_0} > 0$ over an interval $(w^c_{\beta_0}, w^c_{\beta_0} + \varepsilon)$ for some $\varepsilon > 0$. Whenever $w^c_{\beta_0} = 2b$ and $v''_{\beta_0+}(2b) > 0$, this is immediate since v_{β_0} is of class $C^2(\mathbb{R}_+ \backslash \{b, 2b\})$. In all the other cases, $v''_{\beta_0+}(w^c_{\beta_0}) = 0$. Differentiating (C.5) twice at the right of $w^c_{\beta_0}$ then yields

$$(\rho w^c_{\beta_0} + \lambda b) v'''_{\beta_0+}(w^c_{\beta_0}) = \lambda[v''_{\beta_0+}(w^c_{\beta_0}) - v''_{\beta_0+}(w^c_{\beta_0} - b)] - (2\rho - r) v''_{\beta_0+}(w^c_{\beta_0}) = -\lambda v''_{\beta_0+}(w^c_{\beta_0} - b) > 0,$$

where the strict inequality follows from the fact that $w^c_{\beta_0} - b \in [b, w^c_{\beta_0})$ by Lemma C.1.5, and that $v''_{\beta_0+} < 0$ over $[b, w^c_{\beta_0})$. Since $v''_{\beta_0+}(w^c_{\beta_0}) = 0$ and $v'''_{\beta_0+}(w^c_{\beta_0}) > 0$, one has $v''_{\beta_0} > 0$ over an interval $(w^c_{\beta_0}, w^c_{\beta_0} + \varepsilon)$ for some $\varepsilon > 0$, as claimed. Suppose by way of contradiction that $w^p_{\beta_0} > w^c_{\beta_0}$. Then $v'_{\beta_0}(w^c_{\beta_0}) > 0$ by (C.12), so that v''_{β_0} cannot be positive everywhere over $(w^c_{\beta_0}, w^p_{\beta_0})$. Let $\tilde{w} = \inf\{w > w^c_{\beta_0} \,|\, v''_{\beta_0}(w) < 0\} \in (w^c_{\beta_0}, w^p_{\beta_0})$. One has $v''_{\beta_0} > 0$ over $(w^c_{\beta_0}, \tilde{w})$ and $v''_{\beta_0}(\tilde{w}) = 0$ since v_{β_0} is of class $C^2(\mathbb{R}_+ \backslash \{b, 2b\})$ and $\tilde{w} > w^c_{\beta_0} \geq 2b$ by Lemma C.1.5. One now shows that $\tilde{w} - b \geq w^c_{\beta_0}$. Note that one must have $v'''_{\beta_0+}(\tilde{w}) \leq 0$, because v''_{β_0} would otherwise be strictly positive over an interval $(\tilde{w}, \tilde{w} + \eta)$ for some $\eta > 0$. Differentiating

(C.5) twice at the right of $\tilde{w}$ then yields

$$0 \geq (\rho\tilde{w} + \lambda b)v'''_{\beta_0+}(\tilde{w}) = \lambda[v''_{\beta_0}(\tilde{w}) - v''_{\beta_0+}(\tilde{w} - b)] - (2\rho - r)v''_{\beta_0}(\tilde{w}) = -\lambda v''_{\beta_0+}(\tilde{w} - b),$$

and thus $v''_{\beta_0+}(\tilde{w} - b) \geq 0$. Now, $v''_{\beta_0+} < 0$ over $(b, w^c_{\beta_0})$. Since $\tilde{w} > 2b$ and thus $\tilde{w} - b > b$, it follows that $\tilde{w} - b \geq w^c_{\beta_0}$, as claimed. Because $v''_{\beta_0} > 0$ over $(w^c_{\beta_0}, \tilde{w})$, this implies that v_{β_0} is convex over $[\tilde{w} - b, \tilde{w}]$. Then, since

$$0 = (\rho\tilde{w} + \lambda b)v''_{\beta_0}(\tilde{w}) = \lambda[v'_{\beta_0}(\tilde{w}) - v'_{\beta_0}(\tilde{w} - b)] - (\rho - r)[v'_{\beta_0}(\tilde{w}) - 1]$$

by differentiating (C.5) at $\tilde{w}$, one obtains that $v'_{\beta_0}(\tilde{w}) \geq 1$. One then has

$$\begin{aligned} \rho\tilde{w} + \lambda b v'_{\beta_0}(\tilde{w}) &\leq (\rho\tilde{w} + \lambda b)v'_{\beta_0}(\tilde{w}) \\ &= \lambda[v_{\beta_0}(\tilde{w}) - v_{\beta_0}(\tilde{w} - b)] + rv_{\beta_0}(\tilde{w}) + (\rho - r)\tilde{w} - \mu + \lambda C \qquad \text{(C.15)} \\ &\leq \lambda b v'_{\beta_0}(\tilde{w}) + rv_{\beta_0}(\tilde{w}) + (\rho - r)\tilde{w} - \mu + \lambda C, \end{aligned}$$

where the first inequality reflects the fact that $v'_{\beta_0}(\tilde{w}) \geq 1$, while the second follows from (C.5) and the third from the convexity of v_{β_0} over $[\tilde{w} - b, \tilde{w}]$. As a result of (C.15), one has $v_{\beta_0}(\tilde{w}) > \frac{\mu - \lambda C}{r}$. Since $w^p_{\beta_0} > \tilde{w}$ and v_{β_0} is increasing, one must have $v_{\beta_0}(w^p_{\beta_0}) > \frac{\mu - \lambda C}{r}$ as well. However, writing (C.5) at $w^p_{\beta_0}$ yields

$$0 = (\rho w^p_{\beta_0} + \lambda b)v'_{\beta_0}(w^p_{\beta_0}) = \lambda[v_{\beta_0}(w^p_{\beta_0}) - v_{\beta_0}(w^p_{\beta_0} - b)] + rv_{\beta_0}(w^p_{\beta_0}) + (\rho - r)w^p_{\beta_0} - \mu + \lambda C,$$

which, since v_{β_0} is increasing, implies that $v_{\beta_0}(w^p_{\beta_0}) < \frac{\mu - \lambda C}{r}$, a contradiction. The result follows. ■

C.2 The Investment Case

Proof of Proposition 2. Suppose now that investment is feasible, that is $\gamma > 0$. Our goal is to construct a solution to (C.1) that satisfies the following three requirements:

(i) The first-order condition for investment holds at the investment threshold w^i:

$$w^i = \inf\{w \geq b \,|\, v(w) - wv'_+(w) > c\}. \qquad \text{(C.16)}$$

(ii) The first-order condition for transfers holds at the payment threshold w^p:

$$v'(w^p) = 0. \qquad \text{(C.17)}$$

(iii) The solution is maximal among the solutions to (C.1) for which there exist thresholds w^i and w^p that satisfy (C.16) and (C.17).

We shall proceed as in Section C.1. For each $\beta \geq \beta_0$, consider the delay differential equation

$$
\begin{cases}
v_{\beta,\gamma}(w) = \beta w & \text{if } w \in [0,b], \\
r v_{\beta,\gamma}(w) = \mu - \lambda C - (\rho - r)w + \mathcal{L} v_{\beta,\gamma}(w) & \text{if } w \in (b, w^i_\beta], \\
(r-\gamma) v_{\beta,\gamma}(w) = \mu - \lambda C - \gamma c - (\rho - r)w + \mathcal{L}_\gamma v_{\beta,\gamma}(w) & \text{if } w \in (w^i_\beta, \infty),
\end{cases}
\tag{C.18}
$$

where the operators $\mathcal{L}$ and $\mathcal{L}_\gamma$ are defined by (C.2) and (C.3) and the threshold w^i_β satisfies

$$
w^i_\beta = \inf\{w \geq b \,|\, v_{\beta,\gamma}(w) - w v'_{\beta,\gamma+}(w) > c\}. \tag{C.19}
$$

It should be noted that one may have $w^i_\beta = b$, in which case the intermediary region $(b, w^i_\beta]$ is empty. We assume that

$$
\bar{c} = v_{\beta_0}(w^p_{\beta_0}) > c \tag{C.20}
$$

throughout this section. As we will see in Appendix D, (C.20) is a necessary and sufficient condition for investment to ever be strictly profitable. The existence of a solution to (C.18)–(C.19) is guaranteed by the following result.

Lemma C.2.1 *For each $\beta \geq \beta_0$, there exists a unique continuous solution $v_{\beta,\gamma}$ to (C.18) with w^i_β given by (C.19). Moreover, $v_{\beta,\gamma}$ is of class $C^1(\mathbb{R}_+ \backslash \{b\})$, $w^i_\beta \in [b, w^p_{\beta_0})$, and w^i_β is decreasing and continuous with respect to β.*

Proof. The proof consists of three steps.

Step 1 One first shows that if $\beta > \beta' \geq \beta_0$, then

$$
v_\beta(w) - w v'_{\beta+}(w) > v_{\beta'}(w) - w v'_{\beta'+}(w)
$$

for all $w \geq b$. Since

$$
v_\beta(w) - w v'_\beta(w) = u_1(w) - w u'_{1+}(w) + \beta[u_2(w) - w u'_{2+}(w)]
$$

by (C.7), one must prove that $u_2(w) - w u'_{2+}(w) > 0$ for all $w \geq b$. This holds at b since, by (C.9) and (C.10), $u_2(b) - b u'_{2+}(b) = \frac{(\rho - r)b}{\rho + \lambda} > 0$. The claim then follows if $u''_{2+} < 0$ over $[b, \infty)$. Proceeding as for v''_{β_0+} in the proof of Lemma C.1.4, it is easy to check that u''_{2+} is upper semicontinuous. Therefore, the set $\{w \geq b \,|\, u''_{2+}(w) \geq 0\}$ is closed. Suppose by way of contradiction that this set is nonempty, and denote by $\tilde{w}$ its smallest element. Observe that $\tilde{w} > b$, since $u''_{2+}(b) = \frac{(\lambda - \rho + r)u'_{2+}(b) - \lambda}{(\rho + \lambda)b}$ as shown in Step 2 of the proof of Proposition C.1.1 and $u'_{2+}(b) < 1 = u'_{2-}(b)$ by (C.10), which implies that $u''_{2+}(b) < 0$. As a result, $u''_{2+} < 0$

over $[b, \tilde{w})$, and in particular $u_2'(\tilde{w}) < u_{2+}'(\tilde{w} - b)$. Differentiating (C.9) at the right of $\tilde{w}$, one therefore obtains that

$$(r - \rho)u_2'(\tilde{w}) = (\rho\tilde{w} + \lambda b)u_{2+}''(\tilde{w}) - \lambda[u_2'(\tilde{w}) - u_{2+}'(\tilde{w} - b)] > 0$$

which, since $r < \rho$, contradicts the fact that $u_2'(\tilde{w}) > 0$ by Lemma C.1.1. The claim follows. Note that u_2 is concave over $\mathbb{R}_+$, and strictly so over $[b, \infty)$.

Step 2 One next shows that, for each $\beta \geq \beta_0$, $v_\beta(w) - wv_{\beta+}'(w)$ is a strictly increasing function of w over $[b, w_{\beta_0}^p]$. To this end, one only needs to check that $v_{\beta+}'' < 0$ over $[b, w_{\beta_0}^p)$. For each $\beta \geq \beta_0$, it follows from (C.7) and Step 1 that

$$v_{\beta+}'' = u_{1+}'' + \beta u_{2+}'' < u_{1+}'' + \beta_0 u_{2+}'' = v_{\beta_0+}'',$$

which is strictly negative over $[b, w_{\beta_0}^p)$ as shown in the proof of Proposition C.1.3. This implies the claim.

Step 3 There are now two cases to consider.

Case 1 First, fix some $\beta \geq \beta_0$, and suppose that $v_\beta(b) - bv_{\beta+}'(b) < c$. From Step 1, this is the case whenever

$$\beta < \hat{\beta} = \frac{c - u_1(b) + bu_{1+}'(b)}{u_2(b) - bu_{2+}'(b)}. \tag{C.21}$$

From Step 1 again, $v_\beta(w) - wv_{\beta+}'(w) \geq v_{\beta_0}(w) - wv_{\beta_0+}'(w)$ for all $w \in [b, w_{\beta_0}^p]$. Hence, by (C.12) and (C.20),

$$v_\beta(w_{\beta_0}^p) - w_{\beta_0}^p v_\beta'(w_{\beta_0}^p) \geq v_{\beta_0}(w_{\beta_0}^p) - w_{\beta_0}^p v_{\beta_0}'(w_{\beta_0}^p) = v_{\beta_0}(w_{\beta_0}^p) > c.$$

Since $v_\beta(w) - wv_{\beta+}'(w)$ is continuous and strictly increasing with respect to w over $[b, w_{\beta_0}^p]$ by Step 2, there exists a unique $w_\beta^i \in (b, w_{\beta_0}^p)$ such that $v_\beta(w_\beta^i) - w_\beta^i v_\beta'(w_\beta^i) = c$. It follows from Step 1 that, as long as $v_\beta(b) - bv_{\beta+}'(b) < c$, w_β^i is strictly decreasing and continuous with respect to β. One can then construct $v_{\beta,\gamma}$ by setting it equal to v_β over $[0, w_\beta^i]$ and extending it to (w_β^i, ∞) as stipulated in (C.18). Using the fact that $v_{\beta,\gamma}(w_\beta^i) - w_\beta^i v_{\beta,\gamma-}'(w_\beta^i) = c$, it is easy to check from (C.18) that $v_{\beta,\gamma-}'(w_\beta^i) = v_{\beta,\gamma+}'(w_\beta^i) = v_\beta'(w_\beta^i)$. This, along with (C.18), implies that $v_{\beta,\gamma}$ is of class $C^1(\mathbb{R}_+ \backslash \{b\})$. One can further show that $v_{\beta,\gamma}$ is of class $C^k(\mathbb{R}_+ \backslash \{b, ..., kb, w_\beta^i, ..., w_\beta^i + (k-2)b\})$ for all $k \in \mathbb{N} \backslash \{0, 1\}$. To conclude, one must verify that w_β^i satisfies (C.19). A sufficient condition for this is that $v_{\beta,\gamma+}''(w_\beta^i) < 0$. Differentiating (C.5) and (C.18) at the right of w_β^i and using the fact that $v_{\beta,\gamma} = v_\beta$ over $[b, w_\beta^i]$ yields

$$[(\rho - \gamma)w_\beta^i + \lambda b]v_{\beta,\gamma+}''(w_\beta^i) = \lambda[v_\beta'(w_\beta^i) - v_{\beta+}'(w_\beta^i - b)] - (\rho - r)[v_\beta'(w_\beta^i) - 1]$$

$$= (\rho w_\beta^i + \lambda b) v''_{\beta+}(w_\beta^i)$$

which implies that $v''_{\beta,\gamma+}(w_\beta^i) < 0$ since $w_\beta^i \in (b, w_{\beta_0}^p)$ and, as shown in Step 2, $v''_{\beta+} < 0$ over $[b, w_{\beta_0}^p)$ whenever $\beta \geq \beta_0$.

Case 2 Next, fix some $\beta \geq \beta_0$, and suppose that $\beta \geq \hat{\beta}$ with $\hat{\beta}$ given by (C.21), so that $v_\beta(b) - bv'_{\beta+}(b) \geq c$. Define $v_{\beta,\gamma}$ as the continuous solution to the delay differential equation

$$\begin{cases} v_{\beta,\gamma}(w) = \beta w & \text{if } w \in [0, b], \\ (r-\gamma) v_{\beta,\gamma}(w) = \mu - \lambda C - \gamma c - (\rho - r) w + \mathcal{L}_\gamma v_{\beta,\gamma}(w) & \text{if } w \in (b, \infty), \end{cases} \tag{C.22}$$

reflecting that the intermediary region $(b, w_\beta^i]$ is empty. To show that this is consistent with (C.19), one must verify that $w_\beta^i = b$ for all $\beta \geq \max\{\beta_0, \hat{\beta}\}$. In analogy with (C.7), for each $\beta \geq \hat{\beta}$, it is convenient to decompose $v_{\beta,\gamma}$ as follows:

$$v_{\beta,\gamma} = u_{1,\gamma} + \beta u_{2,\gamma}, \tag{C.23}$$

where $u_{1,\gamma}$ and $u_{2,\gamma}$ are the continuous solutions to the delay differential equations

$$\begin{cases} u_{1,\gamma}(w) = 0 & \text{if } w \in [0, b], \\ (r-\gamma) u_{1,\gamma}(w) = \mu - \lambda C - \gamma c - (\rho - r) w + \mathcal{L}_\gamma u_{1,\gamma}(w) & \text{if } w \in (b, \infty) \end{cases} \tag{C.24}$$

and

$$\begin{cases} u_{2,\gamma}(w) = w & \text{if } w \in [0, b], \\ (r-\gamma) u_{2,\gamma}(w) = \mathcal{L}_\gamma u_{2,\gamma}(w) & \text{if } w \in (b, \infty), \end{cases} \tag{C.25}$$

respectively. Proceeding as in Step 1, one can show that $u''_{2,\gamma+} < 0$ over $[b, \infty)$, which implies that if $\beta > \beta' \geq \hat{\beta}$, then

$$v_{\beta,\gamma}(w) - wv'_{\beta,\gamma+}(w) > v_{\beta',\gamma}(w) - wv'_{\beta',\gamma+}(w)$$

for all $w \geq b$. As $v_{\hat{\beta},\gamma}(b) = v_{\hat{\beta}}(b) = \beta b$ and $v'_{\hat{\beta},\gamma+}(b) = v'_{\hat{\beta}+}(b)$, which follows from (C.5) and (C.22) along with the fact that $v_{\hat{\beta}}(b) - bv'_{\hat{\beta}+}(b) = c$, one has $v_{\hat{\beta},\gamma}(b) - bv'_{\hat{\beta},\gamma+}(b) = c$. If $\beta_0 > \hat{\beta}$, one immediately obtains that $v_{\beta,\gamma}(b) - bv'_{\beta,\gamma+}(b) > c$ for all $\beta > \beta_0$, which implies that $w_\beta^i = b$, as claimed. If $\hat{\beta} \geq \beta_0$, one must in addition check that $v''_{\hat{\beta},\gamma+}(b) < 0$. Arguing as in Case 1 yields

$$(\rho - \gamma + \lambda) v''_{\hat{\beta},\gamma+}(b) = (\rho + \lambda) v''_{\hat{\beta}+}(b)$$

which implies that $v''_{\hat{\beta},\gamma+}(b) < 0$ since, as shown in Step 2, $v''_{\hat{\beta}+} < 0$ over $[b, w^p_{\beta_0})$ whenever $\hat{\beta} \geq \beta_0$. The result follows. ■

As for the functions $(v_\beta)_{\beta \geq 0}$, a key result is that one can strictly order the derivatives of the functions $(v_{\beta,\gamma})_{\beta \geq \beta_0}$.

Proposition C.2.1 *If $\beta > \beta' \geq \beta_0$, then $v'_{\beta,\gamma} > v'_{\beta',\gamma}$ over $\mathbb{R}_+ \backslash \{b\}$.*

Proof. If $\beta > \beta' \geq \hat{\beta}$, with $\hat{\beta}$ given by (C.21), the proof proceeds along the lines of that of Proposition C.1.1, replacing the decomposition (C.7) into the auxiliary functions (C.8) and (C.9) by the decomposition (C.23) into the auxiliary functions (C.24) and (C.25), and showing similarly to Lemma C.1.1 that $u'_{2,\gamma+} > 0$ over $\mathbb{R}_+ \backslash \{b\}$. From now on, suppose instead that $\hat{\beta} \geq \beta > \beta'$. By Case 1 of Step 3 of the proof of Lemma C.2.1, $w^i_{\beta'} > w^i_\beta > b$. It immediately follows from (C.18) and Proposition C.1.1 that $v'_{\beta,\gamma} > v'_{\beta',\gamma}$ over $[0, w^i_\beta] \backslash \{b\}$. The remainder of the proof consists of two steps.

Step 1 Consider first the interval $[w^i_\beta, w^i_{\beta'}]$. Since $v_{\beta,\gamma}$ is of class $C^1(\mathbb{R}_+ \backslash \{b\})$, one has

$$v'_{\beta,\gamma}(w^i_\beta) = v'_\beta(w^i_\beta) > v'_{\beta'}(w^i_\beta) = v'_{\beta',\gamma}(w^i_\beta),$$

where the inequality follows from Proposition C.1.1. Therefore, since $v_{\beta,\gamma} - v_{\beta',\gamma}$ is of class $C^1(\mathbb{R}_+ \backslash \{b\})$, one only needs to check that $v'_{\beta,\gamma} - v'_{\beta',\gamma}$ has no zero in $(w^i_\beta, w^i_{\beta'}]$. Arguing by contradiction, let $\tilde{w} > w^i_\beta$ be the first point at which $v'_{\beta,\gamma} - v'_{\beta',\gamma}$ vanishes. Note that $v'_{\beta,\gamma} > v'_{\beta',\gamma}$ over $[0, \tilde{w}) \backslash \{b\}$. Then, writing (C.18) for $v_{\beta,\gamma}$ and $v_{\beta',\gamma}$ at $\tilde{w}$ and rearranging yields

$$\begin{aligned}(r-\gamma)[v_{\beta,\gamma}(\tilde{w}) - v_{\beta',\gamma}(\tilde{w})] &= \gamma[v_{\beta',\gamma}(\tilde{w}) - \tilde{w} v'_{\beta',\gamma}(\tilde{w}) - c] \\ &\quad - \lambda[v_{\beta,\gamma}(\tilde{w}) - v_{\beta,\gamma}(\tilde{w} - b) - v_{\beta',\gamma}(\tilde{w}) + v_{\beta',\gamma}(\tilde{w} - b)].\end{aligned} \tag{C.26}$$

Now, since $\tilde{w} \leq w^i_{\beta'}$,

$$v_{\beta',\gamma}(\tilde{w}) - \tilde{w} v'_{\beta',\gamma}(\tilde{w}) \leq c.$$

Moreover, since $v'_{\beta,\gamma} > v'_{\beta',\gamma}$ over $[0, \tilde{w}) \backslash \{b\}$,

$$v_{\beta,\gamma}(\tilde{w}) - v_{\beta,\gamma}(\tilde{w} - b) > v_{\beta',\gamma}(\tilde{w}) - v_{\beta',\gamma}(\tilde{w} - b).$$

Substituting these two inequalities into (C.26), one obtains that $v_{\beta,\gamma}(\tilde{w}) < v_{\beta',\gamma}(\tilde{w})$, which is impossible since $v_{\beta,\gamma}(0) = v_{\beta\gamma}(0) = 0$ and $v'_{\beta,\gamma} > v'_{\beta',\gamma}$ over $[0, \tilde{w}) \backslash \{b\}$. This contradiction establishes that $v'_{\beta,\gamma} > v'_{\beta',\gamma}$ over $[w^i_\beta, w^i_{\beta'}]$.

Step 2 Consider next the interval $[w^i_{\beta'}, \infty)$. By Step 1, $v'_{\beta,\gamma}(w^i_{\beta'}) > v'_{\beta',\gamma}(w^i_{\beta'})$, and thus one only needs to check that $v'_{\beta,\gamma} - v'_{\beta',\gamma}$ has no zero in $[w^i_{\beta'}, \infty)$. Arguing by contradiction, let $\tilde{w} > w^i_{\beta'}$ be the first point at which $v'_{\beta,\gamma} - v'_{\beta',\gamma}$ vanishes. Observe that $v'_{\beta,\gamma} > v'_{\beta',\gamma}$ over $[0, \tilde{w})\backslash\{b\}$. Then, writing (C.18) for $v_{\beta,\gamma}$ and $v_{\beta',\gamma}$ at $\tilde{w}$ and rearranging yields

$$(r-\gamma)[v_{\beta,\gamma}(\tilde{w}) - v_{\beta',\gamma}(\tilde{w})] = -\lambda[v_{\beta,\gamma}(\tilde{w}) - v_{\beta,\gamma}(\tilde{w}-b) - v_{\beta',\gamma}(\tilde{w}) + v_{\beta',\gamma}(\tilde{w}-b)].$$

As in Step 1, one obtains that $v_{\beta,\gamma}(\tilde{w}) < v_{\beta',\gamma}(\tilde{w})$, which is impossible. This contradiction establishes that $v'_{\beta,\gamma} > v'_{\beta',\gamma}$ over $[w^i_{\beta'}, \infty)$. The result follows. ■

Proposition C.2.1 shows that the derivatives of the functions $(v_{\beta,\gamma})_{\beta\geq\beta_0}$ are strictly ordered by their slopes β over $[0, b)$. As in the no investment case of Section C.1, we now show that the subfamily of $(v_{\beta,\gamma})_{\beta\geq\beta_0}$ composed of those functions whose derivatives have at least a zero in (b, ∞) has a maximal element.

Proposition C.2.2 *There exists a maximum value β_γ of β such that the equation $v'_{\beta,\gamma} = 0$ has a solution over (b, ∞). The function $v_{\beta_\gamma,\gamma}$ is increasing over $\mathbb{R}_+$ and $\beta_\gamma > \beta_0$.*

The proof of Proposition C.2.2 proceeds as follows. We first show that the set of $\beta \geq \beta_0$ such that $v'_{\beta,\gamma+}(b) > 0$ and $v'_{\beta,\gamma}$ has at least a zero in (b, ∞) is a nonempty interval. Next, we show that this interval is bounded. Then, we show that it is closed, so that it contains its upper bound β_γ. Finally, we show that the function $v_{\beta_\gamma,\gamma}$ is increasing over $\mathbb{R}_+$ and that I is not reduced to a point, so that in particular $\beta_\gamma > \beta_0$. We now provide a detailed exposition of each step of the proof.

Step 1 Let $I = \{\beta \geq \beta_0 \,|\, v'_{\beta,\gamma+}(b) > 0 \text{ and } (v'_{\beta,\gamma})^{-1}(0) \neq \emptyset\}$. One has the following result.

Lemma C.2.2 *I is a nonempty interval.*

Proof. That I is an interval is an immediate consequence of Proposition C.2.1. It remains to show that I is nonempty. There are three cases to consider.

Case 1 Suppose first that $\beta_0 < \hat{\beta}$, with $\hat{\beta}$ given by (C.21), which corresponds to Case 1 of Step 3 of the proof of Lemma C.2.1. One shows that in this case $\beta_0 \in I$. One has $w^i_{\beta_0} \in (b, w^p_{\beta_0})$ and $v_{\beta_0,\gamma} = v_{\beta_0}$ over $[0, w^i_{\beta_0}]$ so clearly $v'_{\beta_0,\gamma+}(b) > 0$. Moreover, since $v_{\beta_0,\gamma}$ is of class $C^1(\mathbb{R}_+\backslash\{b\})$, $v'_{\beta_0,\gamma}(w^i_{\beta_0}) = v'_{\beta_0}(w^i_{\beta_0})$. Finally,

$$\frac{v''_{\beta_0,\gamma+}(w^i_{\beta_0})}{v''_{\beta_0+}(w^i_{\beta})} = \frac{\rho w^i_{\beta_0} + \lambda b}{(\rho-\gamma)w^i_{\beta_0} + \lambda b} > 1,$$

which implies that $v''_{\beta_0,\gamma+}(w^i_{\beta_0}) < v''_{\beta_0+}(w^i_{\beta_0})$ since $w^i_{\beta_0} \in (b, w^p_{\beta_0})$ and $v''_{\beta_0+} < 0$ over $[b, w^p_{\beta_0})$ as shown in the proof of Proposition C.1.3. It follows that $v'_{\beta_0,\gamma} < v'_{\beta_0}$ over an interval

$(w^i_{\beta_0}, w^i_{\beta_0} + \varepsilon)$ for some $\varepsilon > 0$. One now shows that actually $v'_{\beta_0,\gamma} < v'_{\beta_0}$ over $(w^i_{\beta_0}, w^p_{\beta_0}]$. Since $v'_{\beta_0,\gamma}(w^i_{\beta_0}) = v'_{\beta_0}(w^i_{\beta_0})$, one only needs to check that $v'_{\beta_0,\gamma} - v'_{\beta_0}$ does not have a zero in $(w^i_{\beta_0}, w^p_{\beta_0}]$. Arguing by contradiction, let $\tilde{w} > w^i_{\beta_0}$ be the first point at which $v'_{\beta_0,\gamma} - v'_{\beta_0}$ vanishes. Observe that $v'_{\beta_0,\gamma} \leq v'_{\beta_0}$ over $[0, \tilde{w})\backslash\{b\}$, this inequality being strict over $(w^i_{\beta_0}, \tilde{w})$. Then, writing (C.5) and (C.18) for v_{β_0} and $v_{\beta_0,\gamma}$ at $\tilde{w}$ and rearranging yields

$$\begin{aligned}(r-\gamma)[v_{\beta_0,\gamma}(\tilde{w}) - v_{\beta_0}(\tilde{w})] = \gamma[v_{\beta_0}(\tilde{w}) - \tilde{w}v'_{\beta_0}(\tilde{w}) - c]& \\ - \lambda[v_{\beta_0,\gamma}(\tilde{w}) - v_{\beta_0,\gamma}(\tilde{w} - b) - v_{\beta_0}(\tilde{w}) + v_{\beta_0}(\tilde{w} - b)]&.\end{aligned} \tag{C.27}$$

Now, since $\tilde{w} \in (w^i_{\beta_0}, w^p_{\beta_0}]$ and $v''_{\beta_0+} < 0$ over $[w^i_{\beta_0}, w^p_{\beta_0})$,

$$v_{\beta_0}(\tilde{w}) - \tilde{w}v'_{\beta_0}(\tilde{w}) > c.$$

Moreover, since $v'_{\beta_0,\gamma} \leq v'_{\beta_0}$ over $[0, \tilde{w})\backslash\{b\}$,

$$v_{\beta_0,\gamma}(\tilde{w}) - v_{\beta_0,\gamma}(\tilde{w} - b) \leq v_{\beta_0}(\tilde{w}) - v_{\beta_0}(\tilde{w} - b).$$

Substituting these two inequalities into (C.27), one obtains that $v_{\beta_0,\gamma}(\tilde{w}) > v_{\beta_0}(\tilde{w})$, which is impossible since $v_{\beta_0,\gamma}(w^i_{\beta_0}) = v_{\beta_0}(w^i_{\beta_0})$ and $v'_{\beta_0,\gamma} < v'_{\beta_0}$ over $(w^i_{\beta_0}, \tilde{w})$. This contradiction establishes that $v'_{\beta_0,\gamma} < v'_{\beta_0}$ over $(w^i_{\beta_0}, w^p_{\beta_0}]$. As $v'_{\beta_0}(w^p_{\beta_0}) = 0$ and $v_{\beta_0,\gamma}$ is of class $C^1(\mathbb{R}_+\backslash\{b\})$ and has a strictly positive derivative at $w^i_{\beta,\gamma}$, this implies that $v'_{\beta_0,\gamma}$ has at least a zero in $(w^i_{\beta_0}, w^p_{\beta_0})$. Thus $\beta_0 \in I$, as claimed.

Case 2 Suppose next that $\beta_0 \geq \hat{\beta}$, so that $w^i_{\beta_0} = b$, which corresponds to Case 2 of Step 3 of the proof of Lemma C.2.1, and that $v'_{\beta_0,\gamma+}(b) > 0$. One shows that in this case also $\beta_0 \in I$. Writing (C.5) and (C.18) for v_{β_0} and $v_{\beta_0,\gamma}$ at the right of b and rearranging yields

$$(\rho - \gamma + \lambda)b[v'_{\beta_0+}(b) - v'_{\beta_0,\gamma+}(b)] = \gamma[v_{\beta_0}(b) - bv'_{\beta_0+}(b) - c],$$

which is positive if $\beta_0 \geq \hat{\beta}$, and strictly positive if $\beta_0 > \hat{\beta}$. Whenever $\beta_0 = \hat{\beta}$, one has $v'_{\beta_0+}(b) = v'_{\beta_0,\gamma+}(b)$ but $v''_{\beta_0+}(b) > v''_{\beta_0,\gamma+}(b)$ since $v''_{\beta_0+}(b) < 0$ by Lemma C.1.3 and

$$\frac{v''_{\beta_0,\gamma+}(b)}{v''_{\beta_0+}(b)} = \frac{\rho + \lambda}{\rho - \gamma + \lambda} > 1.$$

Hence, in any case, $v'_{\beta_0,\gamma} < v'_{\beta_0}$ over an interval $(b, b + \varepsilon)$ for some $\varepsilon > 0$. One can then show as in Case 1 that actually $v'_{\beta_0,\gamma} < v'_{\beta_0}$ over $(b, w^p_{\beta_0}]$. As $v'_{\beta_0}(w^p_{\beta_0}) = 0$ and $v_{\beta_0,\gamma}$ is of class $C^1(\mathbb{R}_+\backslash\{b\})$ and has a strictly positive right derivative at b, this implies that $v'_{\beta_0,\gamma}$ has at least a zero in $(b, w^p_{\beta_0})$. Thus $\beta_0 \in I$, as claimed.

Case 3 Suppose finally that $\beta_0 \geq \hat{\beta}$, so that $w^i_{\beta_0} = b$, and that $v'_{\beta_0,\gamma+}(b) \leq 0$, that is, by (C.22), and in analogy with (C.6):

$$v'_{\beta_0,\gamma+}(b) = \frac{(\rho - r)b - \mu + \lambda C + \gamma c}{(\rho - \gamma + \lambda)b} + \beta_0 \frac{r - \gamma + \lambda}{\rho - \gamma + \lambda} \leq 0.$$

Define then $\beta'_0 > \beta_0$ as the unique solution to the equation $v'_{\beta'_0,\gamma+}(b) = 0$,

$$\beta'_0 = \frac{\mu - \lambda C - \gamma c - (\rho - r)b}{(r - \gamma + \lambda)b}.$$

Arguing by contradiction, suppose that $v'_{\beta,\gamma} > 0$ over (b, ∞) for all $\beta > \beta'_0$. Given the decomposition (C.23), which is valid for all $\beta \geq \hat{\beta}$, it follows by taking limits as β decreases to β'_0 that $v'_{\beta'_0,\gamma} \geq 0$ over (b, ∞). Yet, differentiating (C.22) at the right of b and using the fact that $v'_{\beta'_0,\gamma+}(b) = 0$ along with (C.14) leads to

$$(\rho - \gamma + \lambda)bv''_{\beta'_0,\gamma+}(b) = -\lambda\beta'_0 + \rho - r < -\lambda\beta_0 + \rho - r < \frac{\lambda[(\rho - r)b - \mu + \lambda C]}{(r + \lambda)b} + \rho - r,$$

which is strictly negative under (C.4). Since $v'_{\beta'_0,\gamma+}(b) = 0$, this implies that $v'_{\beta'_0,\gamma+} < 0$ in an interval $(b, b + \varepsilon)$ for some $\varepsilon > 0$, a contradiction. It follows that there exists some $\beta''_0 > \beta'_0$ such that $v'_{\beta''_0,\gamma}$ has at least a zero in (b, ∞). Since $v'_{\beta''_0,\gamma+}(b) > v'_{\beta'_0,\gamma+}(b) = 0$ as $\beta''_0 > \beta'_0$, it follows that $\beta''_0 \in I$. Note that, unlike in Cases 1 and 2, this argument establishes that I has a nonempty interior since any $\beta \in (\beta'_0, \beta''_0)$ also belongs to I. The result follows. ■

Step 2 The following result shows that the interval I is bounded.

Lemma C.2.3 *For β large enough, the equation $v'_{\beta,\gamma} = 0$ has no solution over (b, ∞).*

Proof. Consider the functions $u_{1,\gamma}$ and $u_{2,\gamma}$ defined by (C.24) and (C.25). As observed in the proof of Proposition C.2.1, it is easy to check along the lines of the proof of Lemma C.1.1 that $u'_{2,\gamma} > 0$ over $\mathbb{R}_+ \backslash \{b\}$. Similarly, it is easy to check along the lines of the proof of Lemma C.1.2 that $\limsup_{w \to \infty} u'_{1,\gamma}(w) \geq 1$. Combining these observations with the fact that the function $-\frac{u'_{1,\gamma+}}{u'_{2,\gamma+}}$ is continuous over $[b, \infty)$ as $u_{1,\gamma}$ and $u_{2,\gamma}$ are of class $C^1(\mathbb{R}_+ \backslash \{b\})$, one obtains that

$$\sup_{w \in [b,\infty)} \left\{ -\frac{u'_{1,\gamma+}(w)}{u'_{2,\gamma+}(w)} \right\} < \infty. \tag{C.28}$$

Defining $\hat{\beta}$ as in (C.21), the decomposition (C.23) then implies that whenever

$$\beta > \max \left\{ \hat{\beta}, \sup_{w \in [b,\infty)} \left\{ -\frac{u'_{1,\gamma+}(w)}{u'_{2,\gamma+}(w)} \right\} \right\},$$

$v'_{\beta,\gamma}$ has no zero in (b, ∞). The result follows. ■

Remark The supremum in (C.28) is actually a maximum. As shown in Lemma C.2.7, the conditions (C.4) and (C.20) imply that $\mu - \lambda C - \gamma c > (\rho - r)b$, so that by (C.24)

$$u'_{1,\gamma+}(b) = \frac{(\rho - r)b - \mu + \lambda C + \gamma c}{(\rho - \gamma + \lambda)b} < 0. \tag{C.29}$$

Since by (C.25)

$$u'_{2,\gamma+}(b) = \frac{r - \gamma + \lambda}{\rho - \gamma + \lambda} > 0, \tag{C.30}$$

it follows that $-\frac{u'_{1,\gamma+}(b)}{u'_{2,\gamma+}(b)} > 0$. As the function $-\frac{u'_{1,\gamma+}}{u'_{2,\gamma+}}$ is continuous and takes strictly negative values beyond some point, it must therefore attain its maximum over $[b, \infty)$.

Step 3 Denote by β_γ the upper bound of the interval I, which is finite by Lemma C.2.3. We now show that $\beta_\gamma \in I$. For each $\beta \in I$, let $w^p_{\beta,\gamma} = \inf\{(v'_{\beta,\gamma})^{-1}(0)\} > b$. Observe that since $v'_{\beta,\gamma+}(b) > 0$ whenever $\beta \in I$, for any such β the function $v'_{\beta,\gamma}$ remains strictly positive over the interval $(b, w^p_{\beta,\gamma})$. As the derivatives of the functions $(v_{\beta,\gamma})_{\beta \in I}$ are strictly ordered by their slopes β over $[0, b)$, it follows that $w^p_{\beta,\gamma}$ is strictly increasing with respect to β over I. The following result implies that the family $(w^p_{\beta,\gamma})_{\beta \in I}$ is uniformly bounded above, so that $w^p_{\beta,\gamma}$ converges to a finite limit when β converges to β_γ from below.

Lemma C.2.4 *For each $\varepsilon > 0$, there exists $w_\varepsilon > b$ such that $v'_{\beta,\gamma}(w) > 1 - \varepsilon$ for all $\beta \geq \beta_0$ and $w \geq w_\varepsilon$.*

Proof. One shows that $\liminf_{w \to \infty} v'_{\beta_0,\gamma}(w) \geq 1$, which implies the result by Proposition C.2.1. It is convenient to decompose $v_{\beta_0,\gamma}$ as follows:

$$v_{\beta_0,\gamma} = u_{1,\gamma,0} + \beta u_{2,\gamma,0}, \tag{C.31}$$

where $u_{1,\gamma,0}$ and $u_{2,\gamma,0}$ are the continuous solutions to the delay differential equations

$$\begin{cases} u_{1,\gamma,0}(w) = u_1(w) & \text{if } w \in [0, w^i_{\beta_0}], \\ (r - \gamma)u_{1,\gamma,0}(w) = \mu - \lambda C - \gamma c - (\rho - r)w + \mathcal{L}_\gamma u_{1,\gamma,0}(w) & \text{if } w \in (w^i_{\beta_0}, \infty) \end{cases} \tag{C.32}$$

and

$$\begin{cases} u_{2,\gamma,0}(w) = u_2(w) & \text{if } w \in [0, w^i_{\beta_0}], \\ (r - \gamma)u_{2,\gamma,0}(w) = \mathcal{L}_\gamma u_{2,\gamma,0}(w) & \text{if } w \in (w^i_{\beta_0}, \infty), \end{cases} \tag{C.33}$$

respectively. Note that whenever $\beta_0 \geq \hat{\beta}$, with $\hat{\beta}$ given by (C.21), one has $w^i_{\beta_0} = b$, in which case $u_{1,\gamma,0} = u_{1,\gamma}$ and $u_{2,\gamma,0} = u_{2,\gamma}$, where $u_{1,\gamma}$ and $u_{2,\gamma}$ are defined by (C.24) and (C.25). One can easily show that $u_{1,\gamma,0}$ and $u_{2,\gamma,0}$ are of class $C^1(\mathbb{R}_+\backslash\{b, w^i_{\beta_0}\})$. The proof then proceeds along the lines of Lemmas C.1.1 and C.1.2.

First, one shows that $u'_{2,\gamma,0} > 0$ over $\mathbb{R}_+\backslash\{b, w^i_{\beta_0}\}$. From (C.33) and Lemma C.1.1, $u'_{2,\gamma,0} = u'_2 > 0$ over the set $[0, w^i_{\beta_0})\backslash\{b\}$. Consider now the interval $(w^i_{\beta_0}, \infty)$. From (C.33), it is easy to check that

$$u'_{2,\gamma,0+}(w^i_{\beta_0}) = \frac{(r-\gamma)u_2(w^i_{\beta_0}) + \lambda[u_2(w^i_{\beta_0}) - u_2(w^i_{\beta_0} - b)]}{(\rho-\gamma)w^i_{\beta_0} + \lambda b} > 0.$$

Thus, since $u_{2,\gamma,0}$ is of class $C^1(\mathbb{R}_+\backslash\{b, w^i_{\beta_0}\})$, one only needs to check that $u'_{2,\gamma,0}$ has no zero in $(w^i_{\beta_0}, \infty)$. The proof mimics that of the similar claim about u'_2 in Lemma C.1.1, and is therefore omitted.

Second, one shows that $\liminf_{w\to\infty} u'_{1,\gamma,0}(w) \geq 1$, which completes the proof given (C.31). Suppose first by way of contradiction that $\liminf_{w\to\infty} u'_{1,\gamma,0}(w) = -\infty$. Then there exists an increasing divergent sequence $(w_n)_{n\geq 1}$ in $(w^i_{\beta_0} + b, \infty)$ such that $\lim_{n\to\infty} u'_{1,\gamma,0}(w_n) = -\infty$ and $w_n = \arg\min_{w\in[0,w_n]}\{u'_{1,\gamma,0+}(w)\}$. For each $n \geq 1$, one can find some $\tilde{w}_n \in (w_n - b, w_n)$ such that

$$\begin{aligned}
[(\rho-\gamma)w_n + \lambda b]u'_{1,\gamma,0}(w_n) &= \lambda[u_{1,\gamma,0}(w_n) - u_{1,\gamma,0}(w_n - b)] + (r-\gamma)u_{1,\gamma,0}(w_n) \\
&\quad + (\rho - r)w_n - \mu + \lambda C + \gamma c \\
&= \lambda b u'_{1,\gamma,0}(\tilde{w}_n) + (r-\gamma)u_{1,\gamma,0}(w_n) + (\rho-r)w_n - \mu + \lambda C + \gamma c,
\end{aligned}$$

where the first equality follows from (C.32) and the second from the mean value theorem. Since $u_{1,\gamma,0}(w_n) \geq u_1(w^i_{\beta_0}) + u'_{1,\gamma,0}(w_n)(w_n - w^i_{\beta_0})$ by construction of the sequence $(w_n)_{n\geq 1}$, it is easy to verify as in the proof of Lemma C.1.2 that, for n large enough,

$$\frac{u'_{1,\gamma,0}(\tilde{w}_n)}{u'_{1,\gamma,0}(w_n)} \geq \frac{(\rho-r)w_n}{\lambda b} + \frac{\mu - \lambda C - \gamma c - (r-\gamma)u_1(w^i_{\beta_0})}{\lambda b u'_{1,\gamma,0}(w_n)},$$

so that the ratio $\frac{u'_{1,\gamma,0}(\tilde{w}_n)}{u'_{1,\gamma,0}(w_n)}$ goes to ∞ as n goes to ∞, which in turn contradicts the fact that $w_n = \arg\min_{w\in[0,w_n]}\{u'_{1,\gamma,0+}(w)\}$. Thus $\liminf_{w\to\infty} u'_{1,\gamma,0}(w) > -\infty$. Assume without loss of generality that $\liminf_{w\to\infty} u'_{1,\gamma,0}(w)$ is a finite number l_γ. Proceeding as in the proof of Lemma C.1.2, one obtains that there exists a divergent sequence $(\tilde{w}_n)_{n\geq 1}$ such that

$$(\rho - r)(l_\gamma - 1) \geq \lambda b \limsup_{n\to\infty} \frac{u'_{1,\gamma,0}(\tilde{w}_n)}{w_n}.$$

If $l_\gamma < 1$, this implies that $\limsup_{n\to\infty} u'_{1,\gamma,0}(\tilde{w}_n) = -\infty$, which in turn contradicts the finiteness of $l_\gamma = \liminf_{w\to\infty} u'_{1,\gamma,0}(w)$. Hence $l_\gamma \geq 1$, and the result follows. ■

Let $w^p_{\beta_\gamma,\gamma} > b$ be the limit of $w^p_{\beta,\gamma}$ when β converges to β_γ from below. For each $\beta \in I$, $v'_{\beta,\gamma}(w^p_{\beta,\gamma}) = 0$. To establish that I contains its upper bound β_γ, we need to show that this equality also holds at β_γ. This immediately follows from the following result, which states that the derivatives of the functions $(v_{\beta,\gamma})_{\beta\geq\beta_0}$ vary continuously with β.

Lemma C.2.5 *Let $(\beta_n)_{n\geq 1}$ be a sequence in $[\beta_0,\infty)$ that converges to β_∞. Then the sequence $(v'_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to $v'_{\beta_\infty,\gamma}$ over $\mathbb{R}_+\setminus\{b\}$.*

Proof. One will repeatedly use the following simple technical fact.

Fact 1 *Let $(g_n)_{n\geq 1}$ be a sequence of real valued continuous functions that converges uniformly to a function g_∞ over a compact subset K of $\mathbb{R}$, and let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences in $\mathbb{R}$ converging to a_∞ and b_∞. Then, if J is a compact subset of $\mathbb{R}$ for which there exists $n_0 \geq 1$ such that $b_n J \subset K$ for all $n \geq n_0$, the sequence $(a_n g_n \circ (b_n \mathrm{Id}))_{n\geq 1}$ converges uniformly to $a_\infty g_\infty \circ (b_\infty \mathrm{Id})$ over J.*

Proof. Note first that g is continuous over K, being the uniform limit of the sequence of continuous functions $(g_n)_{n\geq 1}$. By assumption, $b_n x \in K$ for all $n \geq n_0$ and $x \in J$, and thus $b_\infty x \in K$ for all $x \in J$ since the sequence $(b_n)_{n\geq 1}$ converges to b_∞ and K is compact. For each $n \geq n_0$ and $x \in J$,

$$\begin{aligned} |a_n g_n(b_n x) - a_\infty g_\infty(b_\infty x)| \leq\ & |a_n||g_n(b_n x) - g_\infty(b_n x)| \\ & + |a_n - a_\infty||g_\infty(b_n x)| \\ & + |a_\infty||g_\infty(b_n x) - g_\infty(b_\infty x)|. \end{aligned} \tag{C.34}$$

Consider now each term on the right-hand side of (C.34). For each $n \geq n_0$ and $x \in J$,

$$|a_n||g_n(b_n x) - g_\infty(b_n x)| \leq \sup_{n\geq n_0}\{|a_n|\}\,\|g_n - g_\infty\|_K,$$

which converges to 0 when n goes to ∞ because the sequence $(g_n)_{n\geq 1}$ converges uniformly to g_∞ over K. Next, for each $n \geq n_0$ and $x \in J$,

$$|a_n - a_\infty||g_\infty(b_n x)| \leq |a_n - a_\infty|\|g_\infty\|_K,$$

which converges to 0 when n goes to ∞ because the sequence $(a_n)_{n\geq 1}$ converges to a_∞. Finally, for each $n \geq n_0$ and $x \in J$,

$$|g_\infty(b_n x) - g_\infty(b_\infty x)| \leq \sup_{\{(y,y')\in K^2\,|\,|y-y'|\leq|b_n - b_\infty|\sup J\}}\{|g_\infty(y) - g_\infty(y')|\},$$

which converges to 0 when n goes to ∞ because the sequence $(b_n)_{n\geq 1}$ converges to b_∞ and because, by the Heine–Cantor theorem, the function g_∞ is uniformly continuous over K as it is continuous over K and K is compact. Substituting these three uniform bounds into (C.34) yields the result. ■

One can now proceed with the proof of Lemma C.2.5. It is sufficient to prove the result for monotone sequences $(\beta_n)_{n\geq 1}$ that converge to β_∞ from below or from above. Focus without loss of generality on the first case. According to Proposition C.2.1, the derivatives of the functions $(v_{\beta_n,\gamma})_{n\geq 1}$ over $\mathbb{R}_+\backslash\{b\}$ are ordered by their slopes $(\beta_n)_{n\geq 1}$ over $[0,b)$. As a result, the sequence $(v_{\beta_n,\gamma})_{n\geq 1}$ is increasing and bounded above by $v_{\beta_\infty,\gamma}$ over $\mathbb{R}_+$, and thus it has a pointwise limit over $\mathbb{R}_+$, hereafter denoted by $\tilde{v}_{\beta_\infty,\gamma}$. Now, fix some compact interval $[\underline{w},\overline{w}]$ of $\mathbb{R}_+$. By Proposition C.2.1 again, the following holds for each $n\geq 1$ and $w\in[\underline{w},\overline{w}]$:

$$\min_{y\in[\underline{w},\overline{w}]}\{v'_{\beta_1,\gamma+}(y)\}\leq v'_{\beta_n,\gamma+}(w)\leq \max_{y\in[\underline{w},\overline{w}]}\{v'_{\beta_\infty,\gamma+}(y)\},$$

hence the sequence $(v_{\beta_n,\gamma})_{n\geq 1}$ is equicontinuous over $[\underline{w},\overline{w}]$. Since $[\underline{w},\overline{w}]$ is an arbitrary compact interval of $\mathbb{R}_+$, the sequence $(v_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to its pointwise limit $\tilde{v}_{\beta_\infty,\gamma}$ by the Arzelà–Ascoli theorem. To translate this into a uniform convergence result for the sequence $(v'_{\beta_n,\gamma})_{n\geq 1}$, it is convenient to change variables as follows. For each $(\beta,z)\in[\beta_0,\infty)\times\mathbb{R}_+$, define $v^i_{\beta,\gamma}(z)=v_{\beta,\gamma}(w^i_\beta z)$, and similarly let $\tilde{v}^i_{\beta_\infty,\gamma}(z)=\tilde{v}_{\beta_\infty,\gamma}(w^i_{\beta_\infty}z)$. Observe also for further reference that for each $\beta\geq\beta_0$, $v^i_{\beta,\gamma}$ satisfies the following delay differential equation:

$$\begin{cases} v^i_{\beta,\gamma}(z)=v_\beta(w^i_\beta z) & \text{if } z\in[0,1],\\ (r-\gamma)v^i_{\beta,\gamma}(z)=\mu-\lambda C-\gamma c-(\rho-r)w^i_{\beta,\gamma}z+\mathcal{L}_{\beta,\gamma}v^i_{\beta,\gamma}(z) & \text{if } z\in(1,\infty),\end{cases}\tag{C.35}$$

where $\mathcal{L}_{\beta,\gamma}$ is a linear first-order delay differential operator defined by

$$\mathcal{L}_{\beta,\gamma}u(z)=\left[(\rho-\gamma)z+\frac{\lambda b}{w^i_\beta}\right]u'(z)-\lambda\left[u(z)-u\left(z-\frac{b}{w^i_\beta}\right)\right]\tag{C.36}$$

for all $z>1$ and any continuous function u of class $C^1\big(\mathbb{R}_+\backslash\{\frac{b}{w^i_\beta}\}\big)$. From Lemma C.2.1, the sequence $(w^i_{\beta_n})_{n\geq 1}$ is decreasing and converges to $w^i_{\beta_\infty}$. Now, fix some interval $J=[\underline{z},\overline{z}]$ of $\mathbb{R}_+$, and apply Fact 1 to the sequence $(g_n)_{n\geq 1}=(v_{\beta_n,\gamma})_{n\geq 1}$ that converges uniformly to $g_\infty=\tilde{v}_{\beta_\infty,\gamma}$ over the interval $K=[w^i_{\beta_\infty}\underline{z},w^i_{\beta_1}\overline{z}]$ and to the sequences $(a_n)_{n\geq 1}=(1)_{n\geq 1}$ and $(b_n)_{n\geq 1}=(w^i_{\beta_n})_{n\geq 1}$ with limits $a_\infty=1$ and $b_\infty=w^i_{\beta_\infty}$. Since the interval J is arbitrary, it follows that the sequence $(a_n g_n\circ(b_n\mathrm{Id}))_{n\geq 1}=(v^i_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to $a_\infty g_\infty\circ(b_\infty\mathrm{Id})=\tilde{v}^i_{\beta_\infty,\gamma}$ over $\mathbb{R}_+$. One now shows that $\tilde{v}^i_{\beta_\infty,\gamma}=v^i_{\beta_\infty,\gamma}$ or equivalently, letting

$\delta_{\beta_n} = v^i_{\beta_\infty,\gamma} - v^i_{\beta_n,\gamma}$ for all $n \geq 1$, that δ, the locally uniform limit of the sequence $(\delta_{\beta_n})_{n\geq 1}$, is identically equal to 0. Consider first the interval $[0,1]$. For each $n \geq 1$ and $z \in [0,1]$, one has by (C.35)

$$\delta_{\beta_n}(z) = v_{\beta_\infty}(w^i_{\beta_\infty} z) - v_{\beta_n}(w^i_{\beta_n} z). \tag{C.37}$$

The decomposition (C.7) implies that the sequence $(v_{\beta_n})_{n\geq 1}$ converges locally uniformly to v_{β_∞}. Therefore, since the sequence $(w^i_{\beta_n})_{n\geq 1}$ converges to $w^i_{\beta_\infty}$, it follows from (C.37) that the sequence $(\delta_{\beta_n}(z))_{n\geq 1}$ converges to 0 for all $z \in [0,1]$ and thus that $\delta = 0$ over $[0,1]$. Consider next the interval $(1,\overline{z}]$, for some given $\overline{z} > 1$. For each $n \geq 1$ and $z \in (1,\overline{z}]$, one has by (C.35) and (C.36)

$$\begin{aligned}(r-\gamma)\delta_{\beta_n}(z) = {} & \left[(\rho-\gamma)z + \frac{\lambda b}{w^i_{\beta_\infty}}\right]\delta'_{\beta_n}(z) + \lambda b\left(\frac{1}{w^i_{\beta_\infty}} - \frac{1}{w^i_{\beta_n}}\right) v^{i\prime}_{\beta_n,\gamma}(z) \\ & - (\rho - r)(w^i_{\beta_\infty} - w^i_{\beta_n})z \\ & - \lambda\left[\delta_{\beta_n}(z) - \delta_{\beta_n}\left(z - \frac{b}{w^i_{\beta_\infty}}\right)\right] + \lambda\left[v^i_{\beta_n,\gamma}\left(z - \frac{b}{w^i_{\beta_\infty}}\right) - v^i_{\beta_n,\gamma}\left(z - \frac{b}{w^i_{\beta_n}}\right)\right].\end{aligned} \tag{C.38}$$

Now, the sequence $(w^i_{\beta_n})_{n\geq 1}$ converges to $w^i_{\beta_\infty}$. Moreover, the sequence $(\delta_{\beta_n})_{n\geq 1}$ converges uniformly over $(1,\overline{z}]$. Finally the sequence $(v^{i\prime}_{\beta_n,\gamma+})_{n\geq 0}$ is uniformly bounded over $[0,\overline{z}]$ since, by Proposition C.2.1 and the definition of the functions $(v^i_{\beta,\gamma})_{\beta\geq 0}$,

$$\left|v^{i\prime}_{\beta_n,\gamma+}(z)\right| \leq w^i_{\beta_1} \max\left\{\left|\inf_{w\in[0,w^i_{\beta_1}\overline{z}]}\{v'_{\beta_1,\gamma+}(w)\}\right|, \left|\sup_{w\in[0,w^i_{\beta_1}\overline{z}]}\{v'_{\beta_\infty,\gamma+}(w)\}\right|\right\}$$

for all $n \geq 1$ and $z \in [0,\overline{z}]$. Using these three observations along with (C.38), one then obtains that the sequence $(\delta'_{\beta_n})_{n\geq 1}$ converges uniformly over $(1,\overline{z}]$. Since δ_{β_n} is of class $C^1\big(\mathbb{R}_+ \backslash \big\{\frac{b}{w^i_{\beta_n}}\big\}\big)$ and $\frac{b}{w^i_{\beta_n}} \leq 1$ for all $n \geq 1$, it follows from the fundamental theorem of calculus that the uniform limit over $(1,\overline{z}]$ of the sequence $(\delta'_{\beta_n})_{n\geq 1}$ must be equal to the derivative δ' of δ. Taking limits in (C.38) as n goes to ∞ then reveals that δ is the unique continuous solution over $[0,\overline{z}]$ to the delay differential equation

$$\begin{cases} \delta(z) = 0 & \text{if } z \in [0,1], \\ (r-\gamma)\delta(z) = \mathcal{L}_{\beta_\infty,\gamma}\delta(z) & \text{if } z \in (1,\overline{z}]. \end{cases} \tag{C.39}$$

But the constant function everywhere equal to 0 is clearly a continuous solution to (C.39) over

$[0, \overline{z}]$. Since $\overline{z}$ is arbitrary, one obtains that $\delta = 0$ over $\mathbb{R}_+$, as claimed. Thus the sequence $(v^i_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to $v^i_{\beta_\infty,\gamma}$. Now consider the derivatives of the functions $(v^i_{\beta_n,\gamma})_{n\geq 1}$. It has already been established that the sequence $(v^{i\prime}_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to $v^{i\prime}_{\beta_\infty,\gamma}$ over $(1, \infty)$. If $w^i_{\beta_\infty} = b$, this is all what is needed in what follows. If $w^i_{\beta_\infty} > b$, one must in addition prove that the sequence $(v^{i\prime}_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to $v^{i\prime}_{\beta_\infty,\gamma}$ over $\big(\frac{b}{w^i_{\beta_\infty}}, 1\big]$. For each $n \geq 1$ and $z \in \big(\frac{b}{w^i_{\beta_\infty}}, 1\big]$, one has by (C.7) and (C.35)

$$v^{i\prime}_{\beta_n,\gamma}(z) = w^i_{\beta_n} v'_{\beta_n}(w^i_{\beta_n} z) = w^i_{\beta_n}[u'_1(w^i_{\beta_n} z) + \beta_n u'_2(w^i_{\beta_n} z)].$$

Given this decomposition, fix some interval $J = [\underline{z}, \overline{z}]$ of $\big(\frac{b}{w^i_{\beta_\infty}}, 1\big]$, and apply Fact 1 to the sequence $(g_n)_{n\geq 1} = (u'_1 + \beta_n u'_2)_{n\geq 1}$ that converges uniformly to $g_\infty = u'_1 + \beta_\infty u'_2$ over the interval $K = [w^i_{\beta_\infty}\underline{z}, w^i_{\beta_1}\overline{z}]$ and to the sequences $(a_n)_{n\geq 1} = (b_n)_{n\geq 1} = (w^i_{\beta_n})_{n\geq 1}$ with limits $a_\infty = b_\infty = w^i_{\beta_\infty}$. Since the interval J is arbitrary, it follows that the sequence $(a_n g_n \circ (b_n \mathrm{Id}))_{n\geq 1} = (v^{i\prime}_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to $a_\infty g_\infty \circ (b_\infty \mathrm{Id}) = v^{i\prime}_{\beta_\infty,\gamma}$ over $\big(\frac{b}{w^i_{\beta_\infty}}, 1\big]$. Combining this with the previous result, one thus obtains that the sequence $(v^{i\prime}_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to $v^{i\prime}_{\beta_\infty,\gamma}$ over $\big(\frac{b}{w^i_{\beta_\infty}}, \infty\big)$. It remains to show that this implies that the sequence $(v'_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to $v'_{\beta_\infty,\gamma}$ over (b, ∞). Note that since the sequence $(w^i_{\beta_n})_{n\geq 1}$ converges to $w^i_{\beta_\infty}$, for any interval $J = [\underline{w}, \overline{w}]$ of (b, ∞) and for each $\varepsilon > 0$, there exists some $n_0(J, \varepsilon) \geq 1$ such that $\frac{w}{w^i_{\beta_n}} \geq \frac{w-\varepsilon}{w^i_{\beta_\infty}}$ for all $n \geq n_0(J, \varepsilon)$ and $w \in J$, so that, letting $K = \big[\frac{\underline{w}-\varepsilon}{w^i_{\beta_\infty}}, \frac{\overline{w}}{w^i_{\beta_\infty}}\big]$, $\frac{1}{w^i_{\beta_n}} J \subset K$ for all $n \geq n_0(J, \varepsilon)$. Now, choose $\varepsilon > 0$ such that $\underline{w} - \varepsilon > b$, and apply Fact 1 to the sequence $(g_n)_{n\geq 1} = (v^{i\prime}_{\beta_n,\gamma})_{n\geq 1}$ that converges uniformly to $g_\infty = v^{i\prime}_{\beta_\infty,\gamma}$ over K and to the sequences $(a_n)_{n\geq 1} = (b_n)_{n\geq 1} = \big(\frac{1}{w^i_{\beta_n}}\big)_{n\geq 1}$ with limits $a_\infty = b_\infty = \frac{1}{w^i_{\beta_\infty}}$. Since the interval J is arbitrary, it follows that the sequence $(a_n g_n \circ (b_n \mathrm{Id}))_{n\geq 1} = (v'_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to $a_\infty g_\infty \circ (b_\infty \mathrm{Id}) = v'_{\beta_\infty,\gamma}$ over (b, ∞). Finally, since the sequence $(\beta_n)_{n\geq 1}$ converges to β_∞, the uniform convergence of $(v'_{\beta_n,\gamma})_{n\geq 1}$ to $(v'_{\beta_\infty,\gamma})_{n\geq 1}$ over $[0, b)$ follows immediately from (C.18). Hence the result. ■

To complete the proof of Proposition C.2.2, we only need to check that $v_{\beta_\gamma,\gamma}$ is increasing over $\mathbb{R}_+$ and that $\beta_\gamma > \beta_0$. The first of these claims follows from considering a strictly decreasing sequence $(\beta_n)_{n\geq 1}$ converging to β_γ. By construction of β_γ, the derivatives of the functions $(v_{\beta_n,\gamma})_{n\geq 1}$ are strictly positive over $\mathbb{R}_+ \backslash \{b\}$, and according to Lemma C.2.5 the sequence $(v'_{\beta_n,\gamma})_{n\geq 1}$ converges locally uniformly to $v'_{\beta_\gamma,\gamma}$ over $\mathbb{R}_+ \backslash \{b\}$. Hence $v'_{\beta_\gamma,\gamma} \geq 0$ over $\mathbb{R}_+ \backslash \{b\}$, which implies the first claim as $v_{\beta_\gamma,\gamma}$ is continuous over $\mathbb{R}_+$. To prove the second claim, we have to go back to the proof of Lemma C.2.2, where three cases were distinguished. In Case 3, we already observed that $\beta_\gamma > \beta_0$. In Cases 1 and 2, we established that $v'_{\beta_0,\gamma}(w^p_{\beta_0}) < 0$. Hence, since $v'_{\beta_\gamma,\gamma}(w^p_{\beta_0}) \geq 0$ by the above argument, it follows from Proposition C.2.1 that $\beta_\gamma > \beta_0$. This concludes the proof of Proposition C.2.2.

In the remainder of this section, we study the concavity of the function $v_{\beta_\gamma,\gamma}$. The following proposition summarizes our findings.

Proposition C.2.3 *$v_{\beta_\gamma,\gamma}$ is concave over $[0, w^p_{\beta_\gamma,\gamma}]$, and strictly so over $[b, w^p_{\beta_\gamma,\gamma}]$.*

The proof of Proposition C.2.3 is very similar to that of Proposition C.1.3. It proceeds through a sequence of lemmas.

Lemma C.2.6 *$v''_{\beta_\gamma,\gamma+}(w^i_{\beta_\gamma}) < 0$.*

Proof. There are two cases to consider.

Case 1 Suppose first that $\beta_\gamma < \hat{\beta}$, with $\hat{\beta}$ given by (C.21). This corresponds to Case 1 of Step 3 of the proof of Lemma C.2.1. Since $\beta_\gamma > \beta_0$, the result follows along the same lines.

Case 2 Suppose next that $\beta_\gamma \geq \hat{\beta}$. Then $w^i_{\beta_\gamma} = b$. This corresponds to Case 2 of Step 3 of the proof of Lemma C.2.1. The function $v_{\beta_\gamma,\gamma}$ can then be decomposed as in (C.23). Since $u''_{2,\gamma+} < 0$ over $[b, \infty)$ and $v''_{\hat{\beta},\gamma+}(b) < 0$, the result follows. ∎

Lemma C.2.7 *$v''_{\beta_\gamma,\gamma+}$ is upper semicontinuous over $[w^i_{\beta_\gamma}, \infty)$.*

Proof. By construction, $w^i_{\beta_\gamma} \geq b$. If $w^i_{\beta_\gamma} \geq 2b$, the result is immediate since $v_{\beta_\gamma,\gamma}$ is of class $C^2(\mathbb{R}_+ \backslash \{b, 2b, w^i_{\beta_\gamma}\})$. If $w^i_{\beta_\gamma} < 2b$, one only needs to check that $v''_{\beta_\gamma,\gamma+}(2b) > v''_{\beta_\gamma,\gamma-}(2b)$. Differentiating (C.18) both at the left and at the right of any $w > b$ and using the fact that $v_{\beta_\gamma,\gamma}$ is of class $C^1(\mathbb{R}_+ \backslash \{b\})$ leads to

$$[(\rho - \gamma)w + \lambda b][v''_{\beta_\gamma,\gamma+}(w) - v''_{\beta_\gamma,\gamma-}(w)] = \lambda[v'_{\beta_\gamma,\gamma-}(w - b) - v'_{\beta_\gamma,\gamma+}(w - b)]. \tag{C.40}$$

There are now two cases to consider.

Case 1 Suppose first that $\beta_\gamma < \hat{\beta}$, with $\hat{\beta}$ given by (C.21). This corresponds to Case 1 of Step 3 of the proof of Lemma C.2.1. Then $v'_{\beta,\gamma+}(b) = v'_{\beta_\gamma+}(b)$, and applying formula (C.40) at $2b$ and using (C.6) yields that $v''_{\beta_\gamma,\gamma+}(2b) > v''_{\beta_\gamma,\gamma-}(2b)$, as claimed.

Case 2 Suppose next that $\beta_\gamma \geq \hat{\beta}$. Then $w^i_{\beta_\gamma} = b$. This corresponds to Case 2 of Step 3 of the proof of Lemma C.2.1. Applying formula (C.40) at $2b$ and using (C.29) and (C.30) yields that $v''_{\beta_\gamma,\gamma+}(2b) > v''_{\beta_\gamma,\gamma-}(2b)$ if and only if

$$v'_{\beta_\gamma,\gamma+}(b) = \frac{(\rho-r)b-\mu+\lambda C+\gamma c}{(\rho-\gamma+\lambda)b} + \beta_\gamma\,\frac{r-\gamma+\lambda}{\rho-\gamma+\lambda} < \beta_\gamma = v'_{\beta_\gamma,\gamma-}(b).$$

A sufficient condition for this to be true is that $\mu-\lambda C-\gamma c > (\rho-r)b$. Now, since $w^p_{\beta_\gamma,\gamma} > b$ and $v'_{\beta_\gamma,\gamma}(w^p_{\beta_\gamma,\gamma}) = 0$, one has by (C.18)

$$\begin{aligned}\mu-\lambda C-\gamma c-(\rho-r)b &> \mu-\lambda C-\gamma c-(\rho-r)w^p_{\beta_\gamma,\gamma}\\ &= (r-\gamma)v_{\beta_\gamma,\gamma}(w^p_{\beta_\gamma,\gamma}) + \lambda[v_{\beta_\gamma,\gamma}(w^p_{\beta_\gamma,\gamma}) - v_{\beta_\gamma,\gamma}(w^p_{\beta_\gamma,\gamma}-b)],\end{aligned}$$

which is strictly positive since $v_{\beta_\gamma,\gamma}$ is strictly increasing and strictly positive over $(0, w^p_{\beta_\gamma,\gamma}]$. Hence the result. ■

Remark It should be noted that the inequality $\mu-\lambda C-\gamma c > (\rho-r)b$ derived in the proof of Lemma C.2.7 is a consequence of our standing assumptions (C.4) and (C.20), from which the whole analysis conducted so far follows. It may at first seem a bit odd that a parameter restriction involving γ can in this way be obtained from two conditions from which γ is absent. This apparent paradox results from the assumption of constant returns to scale, which implies that the desirability of investment depends in a bang-bang way on the level of the agent's size-adjusted payoff. It follows that size growth when it takes place does so at a constant rate, which essentially amounts to an equal reduction in the principal's and in the agent's discount rates. The only restriction to which γ is subjected to is thus that it be strictly lower than the least of these discount rates, that is $\gamma < r$.

It follows from Lemma C.2.7 that the set $\{w \geq w^i_{\beta_\gamma} \,|\, v''_{\beta_\gamma,\gamma+}(w) \geq 0\}$ is closed. Denote by $w^c_{\beta_\gamma,\gamma}$ its smallest element. By Lemma C.2.6, $w^c_{\beta_\gamma,\gamma} > w^i_{\beta_\gamma}$ and $v''_{\beta_\gamma,\gamma+} < 0$ over $[w^i_{\beta_\gamma}, w^c_{\beta_\gamma,\gamma})$. Thus $v_{\beta_\gamma,\gamma}$ is strictly concave over $[w^i_{\beta_\gamma}, w^c_{\beta_\gamma,\gamma}]$. Moreover, $v_{\beta_\gamma,\gamma}$ coincides with v_{β_γ} over $[0, w^i_{\beta_\gamma}]$. Since $\beta_\gamma > \beta_0$ and u_2 is concave over $\mathbb{R}_+$ as shown in Step 1 of the proof of Lemma C.2.1, the decomposition (C.7) implies that $v''_{\beta_\gamma,\gamma+} \leq v''_{\beta_0+}$ over $[0, w^i_{\beta_\gamma})$. As $w^i_{\beta_\gamma} < w^p_{\beta_0}$ by Lemma C.2.1, and v_{β_0} is concave over $[0, w^p_{\beta_0}]$, and strictly so over $[b, w^p_{\beta_0}]$ by Proposition C.1.3, it follows that $v_{\beta_\gamma,\gamma}$ is concave over $[0, w^i_{\beta_\gamma}]$, and strictly so over $[b, w^i_{\beta_\gamma}]$. Finally, observe that either $v'_{\beta_\gamma,\gamma+}(w^i_{\beta_\gamma}) = v'_{\beta_\gamma,\gamma-}(w^i_{\beta_\gamma})$ if $w^i_{\beta_\gamma} > b$ as shown in Case 1 of Step 3 of the proof of Lemma C.2.1, or $v'_{\beta_\gamma,\gamma+}(w^i_{\beta_\gamma}) < v'_{\beta_\gamma,\gamma-}(w^i_{\beta_\gamma})$ if $w^i_{\beta_\gamma} = b$ as shown in Case 2 of the proof of Lemma C.2.7. Thus $v_{\beta_\gamma,\gamma}$ is concave over $[0, w^c_{\beta_\gamma,\gamma}]$. To complete the proof of Proposition C.2.3, we now show that $w^p_{\beta_\gamma,\gamma}$ coincides with $w^c_{\beta_\gamma,\gamma}$. We shall need the following result.

Lemma C.2.8 $w^c_{\beta_\gamma,\gamma} \geq 2b$.

Proof. Suppose by way of contradiction that $w^c_{\beta_\gamma} < 2b$. Then, as $w^c_{\beta_\gamma,\gamma} > b$ and $v_{\beta_\gamma,\gamma}$ is of class $C^2(\mathbb{R}_+\backslash\{b, 2b, w^i_{\beta_\gamma}\})$, $v''_{\beta_\gamma,\gamma}(w^c_{\beta_\gamma,\gamma}) = 0$ and $v''_{\beta_\gamma,\gamma} < 0$ over $(w^i_{\beta_\gamma}, w^c_{\beta_\gamma,\gamma})$. There are three cases to consider.

Case 1 Suppose first that $\lambda \leq \rho - r$. Proceeding as in Case 1 of the proof of Lemma C.1.5, one obtains that $\beta_\gamma \leq \frac{\rho-r}{\lambda}$. Using (C.14) in combination with $\beta_\gamma > \beta_0$ then shows that this is in contradiction with (C.4).

Case 2 Suppose next that $\lambda \geq 2\rho - r - \gamma$. One closely follows Case 2 of the proof of Lemma C.1.5. Differentiating (C.18) twice over $(w^i_{\beta_\gamma}, 2b)$, which is feasible as $w^i_{\beta_\gamma} + b \geq 2b$, one obtains that $v'''_{\beta_\gamma,\gamma} \leq 0$ over this interval, and hence $v''_{\beta_\gamma,\gamma}(w^c_{\beta_\gamma,\gamma}) \leq v''_{\beta_\gamma,\gamma+}(w^i_{\beta_\gamma})$. This leads to a contradiction since $v''_{\beta_\gamma,\gamma}(w^c_{\beta_\gamma,\gamma}) = 0$ and $v''_{\beta_\gamma,\gamma+}(w^i_{\beta_\gamma}) < 0$ by Lemma C.2.6.

Case 3 Suppose finally that $\rho - r < \lambda < 2\rho - r - \gamma$. Arguing as in Case 3 of the proof of Lemma C.1.5, one obtains that $v'''_{\beta_\gamma,\gamma} > 0$ and hence $v''_{\beta_\gamma,\gamma} > v''_{\beta_\gamma,\gamma+}(w^i_{\beta_\gamma})$ over $(w^i_{\beta_\gamma}, w^c_{\beta_\gamma,\gamma}]$, which in turn implies that

$$v''_{\beta_\gamma,\gamma}(w^c_{\beta_\gamma,\gamma}) < \left[1 + \int_{w^i_{\beta_\gamma}}^{w^c_{\beta_\gamma,\gamma}} \frac{\lambda - 2\rho + r + \gamma}{(\rho - \gamma)w + \lambda b}\, dw\right] v''_{\beta_\gamma,\gamma+}(w^i_{\beta_\gamma}).$$

Since $v''_{\beta_\gamma,\gamma}(w^c_{\beta_\gamma,\gamma}) = 0$ and $v''_{\beta_\gamma,\gamma+}(w^i_{\beta_\gamma}) < 0$ by Lemma C.2.6, this yields a contradiction as

$$1 + \int_{w^i_{\beta_\gamma}}^{w^c_{\beta_\gamma,\gamma}} \frac{\lambda - 2\rho + r + \gamma}{(\rho - \gamma)w + \lambda b}\, dw > 1 + \int_b^{2b} \frac{\lambda - 2\rho + r + \gamma}{(\rho - \gamma)w + \lambda b}\, dw > \frac{2\lambda - \rho + r}{\rho - \gamma + \lambda} > 0.$$

The result follows. ■

Proposition C.2.3 is then an immediate consequence of the following result.

Lemma C.2.9 $w^p_{\beta_\gamma,\gamma} = w^c_{\beta_\gamma,\gamma}$.

Proof. Since $v_{\beta_\gamma,\gamma}$ is increasing and $v'_{\beta_\gamma,\gamma}(w^p_{\beta_\gamma,\gamma}) = 0$ by Proposition C.2.2, one must have $v''_{\beta_\gamma,\gamma+}(w^p_{\beta_\gamma,\gamma}) \geq 0$, and thus $w^p_{\beta_\gamma,\gamma} \geq w^c_{\beta_\gamma,\gamma}$. It remains therefore to prove that $w^p_{\beta_\gamma,\gamma} \leq w^c_{\beta_\gamma,\gamma}$. The proof closely follows that of Lemma C.1.6. One first shows that $v''_{\beta_\gamma,\gamma} > 0$ over an interval $(w^c_{\beta_\gamma,\gamma}, w^c_{\beta_\gamma,\gamma} + \varepsilon)$ for some $\varepsilon > 0$. One then shows that, if $w^p_{\beta_\gamma,\gamma} > w^c_{\beta_\gamma,\gamma}$, then $v''_{\beta_\gamma,\gamma}$ must have a zero in $(w^c_{\beta_\gamma,\gamma}, w^p_{\beta_\gamma,\gamma})$. Letting $\tilde{w}$ be the least of the points at which $v''_{\beta_\gamma,\gamma}$ vanishes, one next shows by differentiating (C.18) twice at the right of $\tilde{w}$ that $\tilde{w} - b \geq w^c_{\beta_\gamma,\gamma}$, which in turn implies that $v_{\beta_\gamma,\gamma}$ is convex over $[\tilde{w} - b, \tilde{w}]$. Using this information along with the fact that $v''_{\beta_\gamma,\gamma}(\tilde{w}) = 0$, one can establish by differentiating (C.18) at $\tilde{w}$ that $v'_{\beta_\gamma,\gamma}(\tilde{w}) \geq 1$. Finally, using inequalities similar to (C.15) reveals that this implies that $v_{\beta_\gamma,\gamma}(\tilde{w}) > \frac{\mu - \lambda C - \gamma c}{r - \gamma}$. This leads to a contradiction since $\tilde{w} < w^p_{\beta_\gamma,\gamma}$ and, as is easily checked from (C.18), $v_{\beta_\gamma,\gamma}(w^p_{\beta_\gamma,\gamma}) < \frac{\mu-\lambda C-\gamma c}{r-\gamma}$. The result follows. ■

To simplify notation, we shall hereafter write w^i and w^p instead of $w^i_{\beta_\gamma}$ and $w^p_{\beta_\gamma,\gamma}$. The function v defined by

$$v(w) = v_{\beta_\gamma,\gamma}(w) \wedge v_{\beta_\gamma,\gamma}(w^p)$$

for all $w \geq 0$ is the unique solution to (C.1) that satisfies the requirements (i) to (iii) laid down at the beginning of this section. Our candidate for the optimal value function of the principal is the function f defined by $f(w) = v(w) - w$ for all $w \geq 0$. This function is linear over $[0, b]$, and affine with slope -1 over $[w^p, \infty)$. Moreover, it is concave over $\mathbb{R}_+$, and strictly so over $[b, w^p]$. Finally, $f(w) - wf'(w) > c$ if and only if $w > w^i$. This completes the proof of Proposition 2. ■

D The Verification Theorem

In this appendix, we establish that, under conditions (C.4) and (C.20), the function F defined by $F(X, W) = Xf\left(\frac{W}{X}\right)$ for all $(X, W) \in \mathbb{R}_{++} \times \mathbb{R}_+$ is the principal's optimal value function.

D.1 An Upper Bound for the Principal's Expected Payoff

In this section, we show that the function F provides an upper bound for the expected payoff that the principal obtains from any incentive compatible contract that incites the agent to always exert effort. The following lemma is crucial in establishing this result. Observe that f is of class $C^1(\mathbb{R}_+\backslash\{b\})$, just as v, so that $f'_+ = f'$ over (b, ∞).

Lemma D.1.1 *Whenever* $0 \leq g \leq \gamma$ *and* $w \geq b$,

$$[(\rho - g)w + \lambda b]f'_+(w) - \lambda[f(w) - f(w - b)] - (r - g)f(w) \leq -\mu + \lambda C + gc. \tag{D.1}$$

Proof. There are three cases to consider.

Case 1 Suppose first that $w \in [b, w^i)$. Then

$$(\rho w + \lambda b)f'_+(w) - \lambda[f(w) - f(w - b)] - rf(w) = -\mu + \lambda C$$

and

$$f(w) - wf'_+(w) < c,$$

from which (D.1) follows as $g \geq 0$.

Case 2 Suppose next that $w \in [w^i, w^p)$. Then

$$[(\rho-\gamma)w+\lambda b]f'_+(w)-\lambda[f(w)-f(w-b)]-(r-\gamma)f(w)=-\mu+\lambda C+\gamma c$$

and

$$f(w)-wf'_+(w)\geq c,$$

from which (D.1) follows as $g \leq \gamma$.

Case 3 Suppose finally that $w \in [w^p, \infty)$. Then

$$\begin{aligned}
&\mathcal{L}_\gamma v(w)-(r-\gamma)v(w)-(\rho-r)w+\mu-\lambda C-\gamma c\\
&\quad=-\lambda[v(w^p)-v(w-b)]-(r-\gamma)v(w^p)-(\rho-r)w+\mu-\lambda C-\gamma c,\\
&\quad=\lambda[v(w-b)-v(w^p-b)]-(\rho-r)(w-w^p)\\
&\quad\leq[\lambda v'_+(w^p-b)-\rho+r](w-w^p)\\
&\quad=-[(\rho-\gamma)w^p+\lambda b]v''_{\beta_\gamma,\gamma+}(w^p)(w-w^p)\\
&\quad\leq 0,
\end{aligned}$$

where the first equality follows from the fact that v is constant above w^p, the second equality from substituting $\mathcal{L}_\gamma v(w^p)-(r-\gamma)v(w^p)=(\rho-r)w^p-\mu+\lambda C+\gamma c$ into the second line and from observing that $v'(w^p)=0$, the first inequality from the concavity of v, the third equality from the fact that $v'_+(w^p-b)=v'_{\beta_\gamma,\gamma+}(w^p-b)$, from differentiating (C.18) at the right of w^p and from observing that $v'_{\beta_\gamma,\gamma}(w^p)=0$, and the last inequality from the fact that $v_{\beta_\gamma,\gamma}$ is increasing and that $v'_{\beta_\gamma,\gamma}(w^p)=0$. One thus has

$$[(\rho-\gamma)w+\lambda b]f'(w)-\lambda[f(w)-f(w-b)]-(r-\gamma)f(w)\leq-\mu+\lambda C+\gamma c,$$

and the result follows as in Case 2. ■

Then the following holds.

Proposition D.1.1 *Suppose that conditions (C.4) and (C.20) hold. Then, for any contract* $\Gamma=(X,L,\tau)$ *that induces maximal risk prevention* $\Lambda_t=\lambda$ *for all* $t\in[0,\tau)$, *and yields the* *agent an initial expected payoff W_{0^-} given an initial project size X_0, one has*

$$F(X_0, W_{0^-}) \geq \mathbf{E}\left[\int_0^\tau e^{-rt}\{X_t[(\mu - g_t c)dt - C dN_t] - dL_t\}\right], \tag{D.2}$$

so that the principal's initial expected payoff is at most $F(X_0, W_{0^-})$.

Proof. Fix an arbitrary contract $\Gamma = (X, L, \tau)$ that has the required properties. Since $\Lambda_t = \lambda$ for all $t \in [0, \tau)$, one has $\mathbf{P}^\Lambda = \mathbf{P}$, see Appendix A. For simplicity, one shall drop the mention of the contract Γ and of the effort process Λ in the remainder of the proof. The agent's continuation payoff follows a process W whose dynamics is described by (13) with $\Lambda_t = \lambda$. In line with the assumption that X is $\mathcal{F}^N$–predictable while W is $\mathcal{F}^N$–adapted, there is no loss of generality in assuming that X has left-continuous paths, while W has right-continuous paths. The limited liability and incentive compatibility constraints imply that $\frac{W_{t^-}}{X_t} \geq b$ for all $t \in [0, \tau)$. Now, observe that because f is of class $C^1((b, \infty))$, F is of class $C^1(\{(X, W) \in \mathbb{R}_{++} \times \mathbb{R}_+ \mid \frac{W}{X} > b\})$. Moreover, since f is continuous at b and f' has a finite right-hand limit $f'_+(b)$ at b, one can continuously extend the derivative of F to the set $\{(X, W) \in \mathbb{R}_{++} \times \mathbb{R}_+ \mid \frac{W}{X} = b\}$. This in turn ensures that one can apply the change of variable formula for processes of locally bounded variation (Dellacherie and Meyer (1982, Chapter VI, Section 92)) to the pair $(X, W_{\cdot^-}) = \{(X_t, W_{t^-})\}_{t \geq 0}$, yielding

$$\begin{aligned} e^{-rT}F(X_{T^+}, W_T) = F(X_0, W_{0^-}) &+ \int_0^T e^{-rt}[(\rho W_{t^-} + \lambda H_t)F_W(X_t, W_{t^-}) - rF(X_t, W_{t^-})]\, dt \\ &+ \int_0^T e^{-rt} F_X(X_t, W_{t^-})\,(dX_t^{d,c} + g_t X_t\, dt) \\ &- \int_0^T e^{-rt} F_W(X_t, W_{t^-})\, dL_t^c \\ &+ \sum_{t \in [0,T]} e^{-rt}[F(X_{t^+}, W_t) - F(X_t, W_{t^-})] \end{aligned} \tag{D.3}$$

for all $T \in [0, \tau)$, where $X^{d,c}$ and L^c stand for the pure continuous parts of X^d and L. For each $t \in [0, T]$, one has the following decomposition of the jump in $F(X_t, W_{t^-})$ at time t:

$$\begin{aligned} F(X_{t^+}, W_t) - F(X_t, W_{t^-}) = {} & F(X_{t^+}, W_t) - F(X_t, W_t) \\ &+ F(X_t, W_{t^-} - H_t \Delta N_t - \Delta L_t) - F(X_t, W_{t^-} - H_t \Delta N_t) \\ &+ F(X_t, W_{t^-} - H_t \Delta N_t) - F(X_t, W_{t^-}), \end{aligned}$$

reflecting that $W_t = W_{t^-} - H_t \Delta N_t - \Delta L_t$, where $\Delta N_t = N_t - N_{t^-}$ and $\Delta L_t = L_t - L_{t^-}$ for all $t \in [0, T]$, with $N_{0^-} = L_{0^-} = 0$ by convention. Now fix $T \in [0, \tau)$ and, as in Appendix A, let $M_t = N_t - \lambda t$ for all $t \geq 0$. Using the above decomposition along with

$$\sum_{t \in [0,T]} e^{-rt}[F(X_t, W_{t^-} - H_t \Delta N_t) - F(X_t, W_{t^-})] = \int_0^T e^{-rt}[F(X_t, W_{t^-} - H_t) - F(X_t, W_{t^-})] \, dN_t,$$

one can then rewrite (D.3) as:

$$\begin{aligned} e^{-rT} F(X_{T^+}, W_T) = F(X_0, W_{0^-}) + \int_0^T e^{-rt}[F(X_t, W_{t^-} - H_t) - F(X_t, W_{t^-})] \, dM_t \\ + A_1 + A_2 + A_3, \end{aligned} \tag{D.4}$$

where A_1 is a standard integral with respect to time,

$$\begin{aligned} A_1 = \int_0^T e^{-rt} \{ (\rho W_{t^-} + \lambda H_t) F_W(X_t, W_{t^-}) - \lambda [F(X_t, W_{t^-}) - F(X_t, W_{t^-} - H_t)] \\ + F_X(X_t, W_{t^-}) g_t X_t - r F(X_t, W_{t^-}) \} \, dt, \end{aligned} \tag{D.5}$$

A_2 accounts for downsizing, that is, negative changes in the size of the project,

$$A_2 = \int_0^T e^{-rt} F_X(X_t, W_{t^-}) \, dX_t^{d,c} + \sum_{t \in [0,T]} e^{-rt}[F(X_{t^+}, W_t) - F(X_t, W_t)], \tag{D.6}$$

and A_3 accounts for changes in cumulative transfers,

$$\begin{aligned} A_3 = - \int_0^T e^{-rt} F_W(X_t, W_{t^-}) \, dL_t^c \\ + \sum_{t \in [0,T]} e^{-rt}[F(X_t, W_{t^-} - H_t \Delta N_t - \Delta L_t) - F(X_t, W_{t^-} - H_t \Delta N_t)]. \end{aligned} \tag{D.7}$$

One now treats each of these terms in turn.

Consider first A_1. For each $t \in [0, T]$, let $w_t = \frac{W_{t-}}{X_t}$ and $h_t = \frac{H_t}{X_t}$. Since F is homogenous of degree 1, one has $F_W(X_t, W_{t^-}) = f'_+(w_t)$ and $F_X(X_t, W_{t^-}) = f(w_t) - w_t f'_+(w_t)$ for all $t \in [0, T]$. Thus

$$\begin{aligned} A_1 &= \int_0^T e^{-rt} X_t \{ [(\rho - g_t) w_t + \lambda h_t] f'_+(w_t) - \lambda [f(w_t) - f(w_t - h_t)] - (r - g_t) f(w_t) \} \, dt \\ &\leq \int_0^T e^{-rt} X_t \{ [(\rho - g_t) w_t + \lambda b] f'_+(w_t) - \lambda [f(w_t) - f(w_t - b)] - (r - g_t) f(w_t) \} \, dt \end{aligned} \tag{D.8}$$

$$\leq \int_0^T e^{-rt} X_t(-\mu + \lambda C + g_t c)\, dt$$

where the first and second inequalities respectively follow from the concavity of f and from Lemma D.1.1, along with the fact that $w_t \geq h_t \geq b$ for all $t \in [0, T]$ by limited liability and incentive compatibility.

Consider next A_2. Since F is homogenous of degree 1, one has

$$A_2 = \int_0^T e^{-rt}[f(w_t) - w_t f'_+(w_t)]\, dX_t^{d,c} + \sum_{t\in[0,T]} e^{-rt} W_t \left[\frac{X_{t^+}}{W_t} f\left(\frac{W_t}{X_{t^+}}\right) - \frac{X_t}{W_t} f\left(\frac{W_t}{X_t}\right)\right] \leq 0, \tag{D.9}$$

where the inequality can be justified as follows. Since f is concave and vanishes at 0, $f(w) - w f'_+(w) \geq 0$ for all $w \geq 0$. Because the process $X^{d,c}$ is decreasing, this implies that the first term on the right-hand side of (D.9) is negative. The properties of f stated above also imply that $\frac{f(w)}{w}$ is a decreasing function of w. Since $\frac{W_t}{X_{t^+}} \geq \frac{W_t}{X_t}$ for all $t \in [0, T]$, this implies that the second term on the right-hand side of (D.9) is negative. As a result of this, one has $A_2 \leq 0$.

Consider finally A_3. Since F is homogenous of degree 1 and f is concave, one has

$$\begin{aligned} F(X_t, W_{t^-} - H_t\Delta N_t - \Delta L_t) - F(X_t, W_{t^-} - H_t\Delta N_t) &= X_t\left[f\left(\frac{W_{t^-} - H_t\Delta N_t - \Delta L_t}{X_t}\right) - f\left(\frac{W_{t^-} - H_t\Delta N_t}{X_t}\right)\right] \\ &= -f'_+\left(\frac{W_{t^-} - H_t\Delta N_t}{X_t}\right)\Delta L_t \\ &\leq \Delta L_t, \end{aligned}$$

for all $t \in [0, T]$, where the last inequality reflects that $f'_+ \geq -1$. Using again the fact that $-F_W(X_t, W_{t^-}) = -f'_+(w_t) \leq 1$ for all $t \in [0, T]$ along with the definition of A_3, one therefore obtains that

$$A_3 \leq \int_0^T e^{-rt}\, dL_t^c + \sum_{t\in[0,T]} e^{-rt}\Delta L_t = \int_0^T e^{-rt}\, dL_t. \tag{D.10}$$

Substituting the upper bounds (D.8), (D.9) and (D.10) for A_1, A_2 and A_3 into (D.4) and rearranging then yields

$$F(X_0, W_{0^-}) \geq e^{-rT} F(X_{T^+}, W_T) + \int_0^T e^{-rt}\{X_t[(\mu - g_t c)dt - C dN_t] - dL_t\} + \tilde{M}_T \quad \text{(D.11)}$$

for all $T \in [0, \tau)$, where the process $\tilde{M} = \{\tilde{M}_t\}_{t \geq 0}$ is defined by

$$\tilde{M}_t = \int_0^{t \wedge \tau} e^{-rs}[F(X_s, W_{s^-}) - F(X_s, W_{s^-} - H_s) + X_s C]\, dM_{s \wedge \tau} \quad \text{(D.12)}$$

for all $t \geq 0$. For each $t \geq 0$,

$$\begin{aligned}
&\mathbf{E}\left[\int_0^{t \wedge \tau} e^{-rs}|F(X_s, W_{s^-}) - F(X_s, W_{s^-} - H_s) + X_s C|\, ds\right] \\
&\qquad = \mathbf{E}\left[\int_0^{t \wedge \tau} e^{-rs} X_s \left| f\left(\frac{W_{s^-}}{X_s}\right) - f\left(\frac{W_{s^-} - H_s}{X_s}\right) + C\right| ds\right] \\
&\qquad \leq \mathbf{E}\left[\int_0^{t \wedge \tau} e^{-rs}\left(W_{s^-} \sup_{w \in (b, w^p]}\{|f'(w)|\} + X_s C\right) ds\right] \\
&\qquad \leq \mathbf{E}\left[\int_0^{t \wedge \tau} e^{-rs}\left(W_{0^-} e^{(\rho + \lambda)s} \sup_{w \in (b, w^p]}\{|f'(w)|\} + X_0 e^{\gamma s} C\right) ds\right] \\
&\qquad < \infty,
\end{aligned}$$

where the first inequality follows from the limited liability constraint (16) and the second inequality is an immediate consequence of (13) and of the fact that X grows at most at rate γ. Since the integrand in (D.12) is $\mathcal{F}^N_{\cdot \wedge \tau}$–predictable, where by definition $\mathcal{F}^N_{\cdot \wedge \tau} = \{\mathcal{F}^N_{t \wedge \tau}\}_{t \geq 0}$, a straightforward adaptation of Brémaud (1981, Chapter II, Lemma L3) shows that $\tilde{M}$ is an $\mathcal{F}^N_{\cdot \wedge \tau}$–martingale under $\mathbf{P}$. In particular, $\mathbf{E}[\tilde{M}_T] = \tilde{M}_0 = 0$. Taking expectations in (D.11) then leads to

$$\begin{aligned}
&F(X_0, W_{0^-}) \\
&\quad \geq \mathbf{E}\left[e^{-rT \wedge \tau} F(X_{T \wedge \tau^+}, W_{T \wedge \tau}) + \int_0^{T \wedge \tau} e^{-rt}\{X_t[(\mu - g_t c)dt - C dN_t] - dL_t\}\right] \\
&\quad = \mathbf{E}\left[\int_0^{\tau} e^{-rt}\{X_t[(\mu - g_t c)dt - C dN_t] - dL_t\}\right] \\
&\qquad - \mathbf{E}\left[1_{\{T < \tau\}}\left(\int_T^{\tau} e^{-rt}\{X_t[(\mu - g_t c)dt - C dN_t] - dL_t\} - e^{-rT} F(X_{T^+}, W_T)\right)\right]
\end{aligned}$$

$$= \mathbf{E}\Big[\int_0^\tau e^{-rt}\{X_t[(\mu - g_t c)dt - C dN_t] - dL_t\}\Big] \tag{D.13}$$

$$- e^{-rT}\,\mathbf{E}\Big[1_{\{T<\tau\}}\Big(\mathbf{E}\Big[\int_T^\tau e^{-r(t-T)}\{X_t[(\mu - g_t c)dt - C dN_t] - dL_t\}\,|\,\mathcal{F}_T^N\Big] - F(X_{T^+}, W_T)\Big)\Big]$$

$$\geq \mathbf{E}\Big[\int_0^\tau e^{-rt}\{X_t[(\mu - g_t c)dt - C dN_t] - dL_t\}\Big]$$

$$- e^{-rT}\,\mathbf{E}\Big[1_{\{T<\tau\}}\Big[\frac{X_T(\mu - \lambda C)}{r-\gamma} - W_T - F(X_{T^+}, W_T)\Big]\Big]$$

for all $T \geq 0$, where the first equality reflects that $W_\tau = 0$ by (9), while the second inequality follows from the fact that X grows at most at rate $\gamma < r$ and from the definition (9) of W_T, bearing in mind that $\rho > r$. Now, for each $T \geq 0$,

$$e^{-rT}\left|\frac{X_T(\mu - \lambda C)}{r-\gamma} - W_T - F(X_{T^+}, W_T)\right| = e^{-rT}\left|\frac{X_T(\mu - \lambda C)}{r-\gamma} - X_{T^+} v\Big(\frac{W_T}{X_{T^+}}\Big)\right|$$

$$\leq e^{-(r-\gamma)T} X_0 \Big[\frac{\mu - \lambda C}{r-\gamma} + v(w^p)\Big].$$

Since $r > \gamma$, taking limits as T goes to ∞ in (D.13) yields (D.2). Hence the result. ■

D.2 Attaining the Upper Bound: The Optimal Contract

We now show that the upper bound (D.2) for the principal's expected payoff derived in Proposition D.1.1 can actually be attained by an incentive compatible contract, which is therefore optimal in the class of contracts that induce maximal risk prevention. We assume as in Proposition D.1.1 that conditions (C.4) and (C.20) hold.

Proof of Proposition 3. Since $\Lambda_t = \lambda$ for all $t \geq 0$ under maximal risk prevention, one has $\mathbf{P}^\Lambda = \mathbf{P}$, see Appendix A. It follows from (42) and (43) that $w_t > b$ for all $t \geq 0$. This ensures that the size process $X = \{X_t\}_{t\geq 0}$ defined by (44) always remains strictly positive. The proof then consists of four steps.

Step 1 One first justifies equation (44) for X. The proposed downsizing policy stipulates that the project be downsized by a factor $\frac{w_t - b}{b} \wedge 1$ at any time t at which the process N jumps. Hence the cumulative downsizing process X^d satisfies

$$X_t^d = \int_0^{t^-} X_s\Big(\frac{w_s - b}{b} \wedge 1 - 1\Big)\,dN_s$$

for all $t \geq 0$. Next, the proposed investment policy stipulates that the size of the project grow at rate γ as long as $w_t > w^i$ and at rate 0 otherwise. Hence the cumulative investment process X^i satisfies

$$X_t^i = \int_0^t X_s \gamma 1_{\{w_s > w^i\}} \, ds$$

for all $t \geq 0$. As $X = X_0 + X^d + X^i$, X solves the stochastic differential equation

$$X_t = X_0 + \int_0^{t^-} X_s \left[\left(\frac{w_s - b}{b} \wedge 1 - 1 \right) dN_s + \gamma 1_{\{w_s > w^i\}} \, ds \right] \tag{D.14}$$

for all $t \geq 0$. Since X has left-continuous paths, it follows from the exponential formula for Lebesgue–Stieltjes calculus (Brémaud (1981, Appendix A4, Theorem T4)) that

$$X_t = X_0 \prod_{s \in (0,t)} \left[1 + \left(\frac{w_s - b}{b} \wedge 1 - 1 \right) \Delta N_s \right] \exp\left(\int_0^t \gamma 1_{\{w_s > w^i\}} \, ds \right)$$

for all $t \geq 0$, where $\Delta N_s = N_s - N_{s^-}$ for all $s \in [0, t]$, with $N_{0^-} = 0$ and $\prod_\emptyset = 1$ by convention. This in turn yields (44) by definition of the stopping times $(T_k)_{k \geq 1}$.

Step 2 One now shows that

$$X_t w_t = X_0 w_0 + \int_0^{t^-} \{ X_s [(\rho w_s + \lambda b) \, ds - b \, dN_s] - dL_s \} \tag{D.15}$$

for all $t \geq 0$. Adapting the integration by parts formula for functions of locally bounded variation (Dellacherie and Meyer (1982, Chapter VI, Theorem 90)) to the case of the product of two processes with left-continuous paths, one obtains that

$$X_t w_t = X_0 w_0 + \int_0^{t^-} X_s \, dw_s + \int_0^{t^-} w_s \, dX_s + \sum_{s \in [0,t)} \Delta X_s \Delta w_s \tag{D.16}$$

for all $t \geq 0$, where $\Delta X_s = X_{s^+} - X_s$ and $\Delta w_s = w_{s^+} - w_s$ for all $s \in [0, t)$, with $\sum_\emptyset = 0$ by convention. Substituting (D.14) and (42) into (D.16) and using (45) yields

$$X_t w_t = X_0 w_0 + \int_0^{t^-} [X_s (\rho w_s + \lambda b) \, ds - dL_s] + \int_0^{t^-} X_s \left[(w_s - b) \left(\frac{w_s - b}{b} \wedge 1 \right) - w_s \right] dN_s$$

$$+ \sum_{s \in [0,t)} X_s b \left(\frac{w_s - b}{b} \wedge 1 \right) \left(\frac{w_s - b}{b} \wedge 1 - 1 \right) (\Delta N_s)^2,$$

from which (D.15) follows after a straightforward computation.

Step 3 One then shows that, for each $t \geq 0$, the proposed contract delivers the agent a continuation payoff $W_t = \lim_{s \downarrow t} X_s w_s$ after the realization of uncertainty at time t. From Step 2 one has

$$W_t = W_{0^-} + \int_0^t [(\rho W_s + X_s \lambda b)\, ds - X_s b\, dN_s - dL_s]$$

for all $t \geq 0$. Applying the change of variable formula for processes of locally bounded variation (Dellacherie and Meyer (1982, Chapter VI, Section 92)) to $W = \{W_t\}_{t \geq 0}$ yields, after simplifications,

$$e^{-\rho T} W_T = e^{-\rho t} W_t - \int_t^T e^{-\rho s} (X_s b\, dM_s + dL_s)$$

for all $T \geq t$, where M is defined as in Appendix A. Since X is $\mathcal{F}^N$–predictable and grows at most at rate $\gamma < \rho$, it then follows from Brémaud (1981, Chapter II, Lemma L3) that

$$W_t = \mathbf{E}\,[e^{-\rho(T-t)} W_T \,|\, \mathcal{F}_t^N] + \mathbf{E}\left[\int_t^T e^{-\rho(s-t)}\, dL_s \,|\, \mathcal{F}_t^N\right] \tag{D.17}$$

for all $T \geq t$. Now, observe from (42) and (43) that $w_t \in (b, w^p]$ for all $t \geq 0$, so that

$$0 < e^{-\rho(T-t)} W_T \leq e^{\rho t} e^{-(\rho - \gamma) T} w^p \tag{D.18}$$

for all $T \geq t$. Besides, an immediate consequence of (43) and (45) is that

$$\begin{aligned} \int_t^T e^{-\rho(s-t)}\, dL_s &= \int_t^T e^{-\rho(s-t)} X_s [(\rho - \gamma) w^p + \lambda b] 1_{\{w_{s^+} = w^p\}}\, ds \\ &\leq \frac{X_t[(\rho - \gamma) w^p + \lambda b]}{\rho - \gamma}. \end{aligned} \tag{D.19}$$

for all $T \geq t$. Note that both (D.18) and (D.19) reflect the fact that X grows at most at rate $\gamma < \rho$. Since L is increasing, the family of functions $\{\int_t^T e^{-\rho(s-t)}\, dL_s\}_{T \geq t}$ is increasing and by (D.19) it is uniformly bounded. Hence, by the monotone convergence theorem, taking limits as T goes to ∞ in (D.17) yields

$$W_t = \mathbf{E}\left[\int_t^\infty e^{-\rho(s-t)}\, dL_s \,|\, \mathcal{F}_t^N\right],$$

from which the claim follows.

Step 4 From Step 3, the proposed contract generates a continuation utility process that satisfies (13) with $\Lambda_t = \lambda$ and $H_t = X_t b$ for all $t \geq 0$. Thus, by Proposition 1, this contract induces maximal risk prevention. It remains to show that it is optimal in the class of contracts that induce maximal risk prevention and yield the agent an initial expected payoff W_{0^-} given an initial project size X_0. By Proposition D.1.1, one only needs to show that this contract yields the principal an initial expected payoff $F(X_0, W_{0^-})$. Fix some $T > 0$. Proceeding as for the derivation of (D.4), one obtains that

$$e^{-rT}F(X_{T^+}, W_T) = F(X_0, W_{0^-}) + \int_0^T e^{-rt}[F(X_t, W_{t^-} - X_t b) - F(X_t, W_{t^-})]\, dM_t \\ + A_1 + A_2 + A_3, \tag{D.20}$$

where A_1, A_2 and A_3 are defined as in (D.5), (D.6) and (D.7), with $g_t = \gamma 1_{\{w_t > w^i\}}$ and $H_t = X_t b$ for all $t \geq 0$. One now treats each of these terms in turn.

Consider first A_1. By (D.8) one has

$$\begin{aligned} A_1 &= \int_0^T e^{-rt} X_t \{[(\rho - \gamma 1_{\{w_t > w^i\}}) w_t + \lambda b) f'(w_t) \\ &\qquad - \lambda[f(w_t) - f(w_t - b)] - (r - \gamma 1_{\{w_t > w^i\}}) f(w_t)\}\, dt \\ &= \int_0^T e^{-rt} X_t(-\mu + \lambda C + g_t c)\, dt, \end{aligned} \tag{D.21}$$

where the second equality follows from (41) and from the fact that $g_t = \gamma 1_{\{w_t > w^i\}}$ and $w_t \in (b, w^p]$ for all $t \in [0, T]$.

Consider next A_2. Since the process X^d is purely discontinuous,

$$\begin{aligned} A_2 &= \sum_{t \in [0,T]} e^{-rt}[F(X_{t^+}, W_t) - F(X_t, W_t)] \\ &= \sum_{t \in [0,T]} e^{-rt}\left[X_{t^+} f\left(\frac{W_t}{X_{t^+}}\right) - X_t f\left(\frac{W_t}{X_t}\right)\right] \\ &= \sum_{t \in [0,T]} e^{-rt} X_t \left[\frac{w_t - b}{b} f(b) - f(w_t - b)\right] 1_{\{\Delta X_t < 0\}} \\ &= 0, \end{aligned} \tag{D.22}$$

where the second equality follows from the homogeneity of degree 1 of F, the third from the fact that $X_{t^+} = \frac{w_t - b}{b} X_t$ and $W_t = W_{t^-} - X_t b = X_t(w_t - b)$ whenever $\Delta X_t < 0$, and the fourth from the linearity of f over $[0, b]$ along with the fact that $\Delta X_t < 0$ implies $w_t - b < b$.

Consider finally A_3. Since the process L is continuous except perhaps at time 0,

$$\begin{aligned} A_3 &= -\int_0^T e^{-rt} F_W(X_t, W_{t-})\, dL_t^c + F(X_0, W_{0^-} - L_0) - F(X_0, W_{0^-}) \\ &= -\int_0^T e^{-rt} f'(w_t) X_t[(\rho-\gamma)w^p + \lambda b] 1_{\{w_{t^+}=w^p\}}\, dt + (W_{0^-} - X_0 w^p) \vee 0 \qquad \text{(D.23)} \\ &= \int_0^T e^{-rt}\, dL_t, \end{aligned}$$

where the second equality follows from the homogeneity of degree 1 of F together with (43) and (45), and the third from (45) along with the fact that $w_{t^+} = w^p$ implies $w_t = w^p$ and thus $f'(w_t) = -1$.

The end of the proof proceeds along the lines of that of Proposition D.1.1. First, taking expectations in (D.20) and using (D.21), (D.22) and (D.23) leads to

$$F(X_0, W_{0^-}) = \mathbf{E}\left[e^{-rT} F(X_{T^+}, W_T) + \int_0^T e^{-rt}\{X_t[(\mu - g_t c)dt - C dN_t] - dL_t\}\right] \qquad \text{(D.24)}$$

for all $T \geq 0$. By construction, $\frac{W_t}{X_{t^+}} = \lim_{s \downarrow t} w_s \in [b, w^p]$ for all $t \geq 0$. Thus, for each $T \geq 0$,

$$\left|e^{-rT} F(X_{T^+}, W_T)\right| = \left|e^{-rT} X_{T^+} f\left(\frac{W_T}{X_{T^+}}\right)\right| \leq e^{-(r-\gamma)T} X_0 \max_{w \in [b, w^p]} \{|f(w)|\},$$

reflecting that X grows at most at rate $\gamma < r$. Then, as in Step 3, one can take limits as T goes to ∞ in (D.24), which yields

$$\mathbf{E}\left[\int_0^\infty e^{-rt}\{X_t[(\mu - g_t c)dt - C dN_t] - dL_t\}\right] = F(X_0, W_{0^-}),$$

and the result follows. ■

Remark An implication of our analysis is that, given (C.4), (C.20) is a sufficient condition for the optimal contract to entail investment. One can actually show that (C.20) is also necessary for investment to ever be strictly profitable. Indeed, suppose that (C.20) fails to hold and define an alternative value function for the principal by

$$f_{\beta_0}(w) = v_{\beta_0}(w) \wedge v_{\beta_0}(w^p_{\beta_0}) - w$$

for all $w \geq 0$. Observe that since v_{β_0} is concave over $[0, w^p_{\beta_0}]$ and $v'_{\beta_0} = 0$ over $[w^p_{\beta_0}, \infty)$, f_{β_0} is concave over $\mathbb{R}_+$ and $f'_{\beta_0} = -1$ over $[w^p_{\beta_0}, \infty)$. Hence

$$f_{\beta_0}(w) - wf'_{\beta_0+}(w) \leq f_{\beta_0}(w^p_{\beta_0}) - w^p_{\beta_0} f'_{\beta_0}(w^p_{\beta_0}) = v_{\beta_0}(w^p_{\beta_0}) \leq c \tag{D.25}$$

for all $w \geq 0$. Now, proceeding as in the proof of Lemma D.1.1, it is easy to check that

$$(\rho w + \lambda b) f'_{\beta_0+}(w) - \lambda[f_{\beta_0}(w) - f_{\beta_0}(w-b)] - r f_{\beta_0}(w) \leq -\mu + \lambda C \tag{D.26}$$

for all $w \geq b$. From (D.25) and (D.26), one obtains that whenever $0 \leq g \leq \gamma$ and $w \geq b$,

$$[(\rho - g)w + \lambda b] f'_{\beta_0+}(w) - \lambda[f_{\beta_0}(w) - f_{\beta_0}(w-b)] - (r-g) f_{\beta_0}(w) \leq -\mu + \lambda C + gc, \tag{D.27}$$

in analogy with (D.1). Arguing as in the proof of Proposition D.1.1, the inequality (D.27) can then be used to show that any contract that induces maximal risk prevention and yields the agent an initial expected payoff W_{0-} given an initial project size X_0 yields the principal an initial expected payoff at most equal to $F_{\beta_0}(X_0, W_{0-}) = X_0 f_{\beta_0}\left(\frac{W_{0-}}{X_0}\right)$. Finally, an incentive compatible contract that attains this upper bound can easily be constructed along the lines of Proposition 3, replacing w^p by $w^p_{\beta_0}$ throughout and requiring that no investment ever take place, $g_t = 0$ for all $t \geq 0$.

D.3 Initialization

Proposition 3 describes the optimal contract for a given initial project size X_0 and a given initial promised utility W_{0-} for the agent. In this section, we briefly examine how X_0 and W_{0-} are optimally determined at time 0. Consider for simplicity the case in which the principal is competitive. We then look for a pair (X_0, W_{0-}) that maximizes utilitarian welfare under the constraint that the principal breaks even on average. Letting $w_0 = \frac{W_{0-}}{X_0}$, the corresponding maximization problem is

$$\max_{(X_0, w_0)} \{X_0[f(w_0) + w_0]\}, \tag{D.28}$$

subject to the principal's participation constraint

$$X_0 f(w_0) \geq 0, \tag{D.29}$$

the agent's limited liability contraint

$$w_0 \geq 0, \tag{D.30}$$

and the feasibility constraint

$$1 \geq X_0, \tag{D.31}$$

reflecting that the initial size of the project is at most 1. Let η be the Lagrange multiplier for constraint (D.29), and focus on the interesting case where $(1+\eta)f(w_0)+w_0>0$ at the optimum.[2] It immediately follows that it is optimal to start operating the project at full scale, $X_0=1$. This result hinges on the homogeneity of the principal's value function F. As shown in (D.28), this enables one to separate at time 0 the determination of the project's size from that of the manager's size-adjusted utility. The initial size-adjusted utility of the agent is given by the first-order condition $f'(w_0)=-\frac{1}{1+\eta}$. Two cases arise, depending on whether constraint (D.29) is slack or binding at the optimum. If $f(w^p)\geq 0$, this constraint is slack, so that $\eta=0$ and $w_0=w^p$, which from the point of view of utilitarian welfare is optimal. If $f(w^p)<0$, this constraint is binding, so that $\eta>0$ and $w_0<w^p$, reflecting that an initial size-adjusted utility for the agent equal to w^p is inconsistent with the participation constraint of the principal.

E Firm Size Dynamics

Proof of Proposition 4. One will repeatedly use the following simple technical fact.

Fact 2 *Let $(Y_n)_{n\geq 1}$ be a sequence of real valued random variables that converges $\mathbf{P}$–almost surely to a constant y, and let $(n_t)_{t\geq 0}$ be a family of integer valued random variables that diverges $\mathbf{P}$–almost surely to ∞ as t goes to ∞. Then the family $(Y_{n_t})_{t\geq 0}$ converges $\mathbf{P}$–almost surely to y as t goes to ∞.*

Proof. Since $(Y_n)_{n\geq 1}$ converges $\mathbf{P}$–almost surely to y, there exists a measurable set Ω_0 with $\mathbf{P}[\Omega_0]=1$ such that for each $\omega\in\Omega_0$ and $\varepsilon>0$, there exists $m_0(\omega,\varepsilon)\geq 1$ such that $|Y_n(\omega)-y|\leq\varepsilon$ for all $n\geq m_0(\omega,\varepsilon)$. Next, since $(n_t)_{t\geq 0}$ diverges $\mathbf{P}$–almost surely to ∞ as t goes to ∞, there exists a measurable set Ω_1 with $\mathbf{P}[\Omega_1]=1$ such that for each $\omega\in\Omega_1$ and $m_0\geq 1$, there exists $t_0(\omega,m_0)\geq 0$ such that $n_t(\omega)\geq m_0$ for all $t\geq t_0(\omega,m_0)$. Hence, for each $\omega\in\Omega_0\cap\Omega_1$ and $\varepsilon>0$, one has $n_t(\omega)\geq m_0(\omega,\varepsilon)$ and thus $|Y_{n_t(\omega)}(\omega)-y|\leq\varepsilon$ for all $t\geq t_0(\omega,m_0(\omega,\varepsilon))$. This implies the result as $\mathbf{P}[\Omega_0\cap\Omega_1]=1$. ■

Now, from (47), one has

$$\frac{\ln(X_t)}{t}=\frac{1}{t}\left[\sum_{k=1}^{N_{t-}}\ln\left(\frac{w_{T_k}-b}{b}\wedge 1\right)+\int_0^{T_{N_{t-}}}\gamma 1_{\{w_s>w^i\}}\,ds+\int_{T_{N_{t-}}}^{t}\gamma 1_{\{w_s>w^i\}}\,ds\right] \tag{E.1}$$

[2]This is the case whenever f takes strictly positive values. Otherwise the solution to problem (D.28) to (D.31) is $X_0=w_0=0$ and the project is not operated.

for all $t \geq 0$. One now treats each of the terms on the right-hand side of (E.1) in turn.

Claim 1 *Let $\boldsymbol{\mu}^{\boldsymbol{w}}$ be the unique invariant measure associated to the process $\{w_{T_k}\}_{k\geq 1}$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{N_{t-}} \ln\left(\frac{w_{T_k} - b}{b} \wedge 1\right) = \lambda \int_{[b,2b)} \ln\left(\frac{w-b}{b}\right) \boldsymbol{\mu}^{\boldsymbol{w}}(dw),$$

P–*almost surely.*

Proof. The proof goes through a sequence of steps.

Step 1 A straightforward implication of (42) is that $\{w_{T_k}\}_{k\geq 1}$ is a Markov process. Let $P : [b, w^p] \times \mathcal{B}([b, w^p]) \to [0, 1]$ denote the associated transition function, where $\mathcal{B}([b, w^p])$ is the Borel σ–field over $[b, w^p]$. Let $A \in \mathcal{B}([b, w^p])$ be Markov invariant for $\{w_{T_k}\}_{k\geq 1}$, that is $P(w, A) = 1$ for all $w \in A$. Then a further implication of (42) is that for all $w \in A$, A must contain a subset of full Lebesgue measure in $[(w - b) \vee b, w^p]$. Hence there are no disjoint Markov invariant sets, and $\{w_{T_k}\}_{k\geq 1}$ is Markov ergodic (Stout (1974, Definition 3.6.8)). One now shows that $\{w_{T_k}\}_{k\geq 1}$ has a stationary initial distribution. Define t_{b,w^p} to be the minimum amount of time it takes for the process $\{w_t\}_{t\geq 0}$ to transit from b to w^p, that is, from (42):

$$t_{b,w^p} = \frac{1}{\rho} \ln\left(\frac{\rho w^i + \lambda b}{\rho b + \lambda b}\right) + \frac{1}{\rho - \gamma} \ln\left(\frac{(\rho - \gamma)w^p + \lambda b}{(\rho - \gamma)w^i + \lambda b}\right). \tag{E.2}$$

Then clearly $P(w, \{w^p\}) \geq e^{-\lambda t_{b,w^p}}$ for all $w \in [b, w^p]$. Hence the transition function P satisfies Condition M in Stokey and Lucas (1989, Chapter 11, Section 4). Specifically, for each $A \in \mathcal{B}([b, w^p])$ the following holds. Either $w^p \in A$ and $P(w, A) \geq e^{-\lambda t_{b,w^p}}$ for all $w \in [b, w^p]$, or $w^p \notin A$ and $P(w, [b, w^p]\backslash A) \geq e^{-\lambda t_{b,w^p}}$ for all $w \in [b, w^p]$. Let $\Delta([b, w^p])$ be the space of Borel probability measures over $[b, w^p]$, and let $T^* : \Delta([b, w^p]) \to \Delta([b, w^p])$ be the adjoint operator associated with P, defined by

$$(T^*\boldsymbol{\mu})(A) = \int_{[b,w^p]} P(w, A)\, \boldsymbol{\mu}(dw)$$

for all $(\boldsymbol{\mu}, A) \in \Delta([b, w^p]) \times \mathcal{B}([b, w^p])$. Condition M as stated above implies that T^* is a contraction of modulus $1 - e^{-\lambda t_{b,w^p}}$ over the space $\Delta([b, w^p])$ endowed with the total variation norm $\|\cdot\|_{TV}$ (Stokey and Lucas (1989, Lemma 11.11)). Because this is a complete metric space, it follows from the contraction mapping theorem that T^* has an unique invariant measure $\boldsymbol{\mu}^{\boldsymbol{w}}$, which corresponds to the unique stationary initial distribution of $\{w_{T_k}\}_{k\geq 1}$.

Step 2 One next shows that

$$\int_{[b,w^p]} \left| \ln\left(\frac{w-b}{b} \wedge 1 \right) \right| \boldsymbol{\mu}^{\boldsymbol{w}}(dw) < \infty. \tag{E.3}$$

To do so, define an auxiliary process $\{\hat{w}_t\}_{t\geq 0}$ by

$$\hat{w}_t = [1 + (\rho - \gamma + \lambda)(t - T_{N_{t-}})]b \wedge 2b$$

for all $t \geq 0$. It is easy to check from (42) that $\hat{w}_t \leq w_t$ for all $t \geq 0$. Now, for each $k \geq 1$,

$$\hat{w}_{T_k} = [1 + (\rho - \gamma + \lambda)(T_k - T_{k-1})]b \wedge 2b,$$

where $T_0 = 0$ by convention. Thus, by the properties of the Poisson process, $(\hat{w}_{T_k})_{k\geq 1}$ is a sequence of independently and identically distributed random variables, with

$$\begin{aligned} \mathbf{P}\left[\hat{w}_{T_k} \leq w\right] &= 1 - e^{-\frac{\lambda(w-b)}{(\rho-\gamma+\lambda)b}} \quad \text{if } w \in [b, 2b), \\ \mathbf{P}\left[\hat{w}_{T_k} = 2b\right] &= e^{-\frac{\lambda}{\rho-\gamma+\lambda}} \end{aligned} \tag{E.4}$$

for all $k \geq 1$. Denote by $\boldsymbol{\mu}^{\hat{\boldsymbol{w}}}$ the corresponding measure over $[b, 2b]$. For each $j \geq 1$ and $w \in [b, w^p]$, define $g_j(w) = \ln\left(\frac{w-b}{b} \wedge 1\right) \vee (-j)$, and observe that $-j \leq g_j \leq 0$ over $[b, w^p]$ and $g_j = 0$ over $[2b, w^p]$. Since $\hat{w}_{T_k} \leq w_{T_k}$ for all $k \geq 1$,

$$\frac{1}{n} \sum_{k=1}^{n} g_j(\hat{w}_{T_k}) \leq \frac{1}{n} \sum_{k=1}^{n} g_j(w_{T_k}) \tag{E.5}$$

for all $n \geq 1$, $\mathbf{P}$–almost surely. Since the random variables $(\hat{w}_{T_k})_{k\geq 1}$ are independently and identically distributed over $[b, 2b]$ according to $\boldsymbol{\mu}^{\hat{\boldsymbol{w}}}$, and since the function g_j is measurable and bounded, and hence $\boldsymbol{\mu}^{\hat{\boldsymbol{w}}}$–integrable, it follows from the strong law of large numbers that the sequence $\left(\frac{1}{n}\sum_{k=1}^{n} g_j(\hat{w}_{T_k})\right)_{n\geq 1}$ converges $\mathbf{P}$–almost surely to

$$\int_{[b,2b]} g_j(w)\, \boldsymbol{\mu}^{\hat{\boldsymbol{w}}}(dw) = \int_b^{2b} g_j(w)\, \frac{\lambda}{(\rho - \gamma + \lambda)b}\, e^{-\frac{\lambda(x-b)}{(\rho-\gamma+\lambda)b}}\, dw,$$

where the equality follows from (E.4) and from the fact that $g_j(2b) = 0$. Similarly, since the process $\{w_{T_k}\}_{k\geq 1}$ is Markov ergodic by Step 1, with invariant measure $\boldsymbol{\mu}^{\boldsymbol{w}}$ over $[b, w^p]$, and since the function g_j is measurable and bounded, and hence $\boldsymbol{\mu}^{\boldsymbol{w}}$–integrable, it follows from the strong law of large numbers for Markov ergodic processes (Stout (1974, Theorem 3.6.7)) that the sequence $\left(\frac{1}{n}\sum_{k=1}^{n} g_j(w_{T_k})\right)_{n\geq 1}$ converges $\mathbf{P}$–almost surely to

$$\int_{[b,w^p]} g_j(w)\, \boldsymbol{\mu}^{\boldsymbol{w}}(dw).$$

Combining these observations with (E.5), and using the fact that $g_j \leq 0$, one obtains that

$$\int_{[b,w^p]} |g_j(w)|\, \boldsymbol{\mu}^{\boldsymbol{w}}(dw) \leq \int_b^{2b} |g_j(w)| \frac{\lambda}{(\rho-\gamma+\lambda)b}\, e^{-\frac{\lambda(w-b)}{(\rho-\gamma+\lambda)b}}\, dw. \tag{E.6}$$

By construction, the sequence of functions $(|g_j|)_{j\geq 1}$ is increasing and converges pointwise to the function $|g_\infty| : [b, w^p] \to \mathbb{R} \cup \{\infty\}$ defined by $g_\infty(w) = \ln\left(\frac{w-b}{b} \wedge 1\right) \in \mathbb{R} \cup \{-\infty\}$ for all $w \in [b, w^p]$. Applying the monotone convergence theorem to both sides of (E.6) and using the fact that $g_\infty(w) = \ln\left(\frac{w-b}{b}\right)$ if $w \in [b, 2b]$ then yields

$$\begin{aligned}
\int_{[b,w^p]} \left|\ln\left(\frac{w-b}{b} \wedge 1\right)\right| \boldsymbol{\mu}^{\boldsymbol{w}}(dw) &\leq \int_b^{2b} \left|\ln\left(\frac{w-b}{b}\right)\right| \frac{\lambda}{(\rho-\gamma+\lambda)b}\, e^{-\frac{\lambda(w-b)}{(\rho-\gamma+\lambda)b}}\, dw \\
&< \frac{\lambda}{\rho-\gamma+\lambda} \int_0^1 |\ln(x)|\, dx \qquad \text{(E.7)} \\
&= \frac{\lambda}{\rho-\gamma+\lambda},
\end{aligned}$$

from which (E.3) follows.

Step 3 Since the process $\{w_{T_k}\}_{k\geq 1}$ is Markov ergodic by Step 1, with invariant measure $\boldsymbol{\mu}^{\boldsymbol{w}}$ over $[b, w^p]$, and since the function g_∞ is $\boldsymbol{\mu}^{\boldsymbol{w}}$–integrable by Step 2, it follows from the strong law of large numbers for Markov ergodic processes (Stout (1974, Theorem 3.6.7)) that the sequence $\left(\frac{1}{n}\sum_{k=1}^{n} g_\infty(w_{T_k})\right)_{n\geq 1} = \left(\frac{1}{n}\sum_{k=1}^{n} \ln\left(\frac{w_{T_k}-b}{b} \wedge 1\right)\right)_{n\geq 1}$ converges $\mathbf{P}$–almost surely to

$$\int_{[b,w^p]} g_\infty(w)\, \boldsymbol{\mu}^{\boldsymbol{w}}(dw) = \int_{[b,w^p]} \ln\left(\frac{w-b}{b} \wedge 1\right) \boldsymbol{\mu}^{\boldsymbol{w}}(dw) = \int_{[b,2b)} \ln\left(\frac{w-b}{b}\right) \boldsymbol{\mu}^{\boldsymbol{w}}(dw),$$

where the second equality follows from the fact that $g_\infty = 0$ over $[2b, w^p]$. Applying Fact 2 to the sequence $(Y_n)_{n\geq 1} = \left(\frac{1}{n}\sum_{k=1}^{n} \ln\left(\frac{w_{T_k}-b}{b} \wedge 1\right)\right)_{n\geq 1}$ and to the family $(n_t)_{t\geq 0} = (N_{t-})_{t\geq 0}$, and using the fact that $\frac{N_{t-}}{t}$ converges $\mathbf{P}$–almost surely to λ as t goes to ∞ by the strong law of large numbers for the Poisson process, one then obtains that $\frac{1}{t}\sum_{k=1}^{N_{t-}} \ln\left(\frac{w_{T_k}-b}{b} \wedge 1\right)$ converges $\mathbf{P}$–almost surely to $\lambda \int_{[b,2b)} \ln\left(\frac{w-b}{b}\right) \boldsymbol{\mu}^{\boldsymbol{w}}(dw)$ as t goes to ∞. ■

Claim 2 *Let* $\boldsymbol{\mu}^{\boldsymbol{w}+}$ *be the unique invariant measure associated to the process* $\{w_{T_k^+}\}_{k\geq 1}$, *and* *let* $\boldsymbol{\lambda}$ *be the exponential distribution with parameter* λ. *Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^{T_{N_t-}} 1_{\{w_s > w^i\}}\, ds = 1 - \lambda \int_{[b,w^i)\times\mathbb{R}_+} \left\{ \left[\frac{1}{\rho} \ln\!\left(\frac{\rho w^i + \lambda b}{\rho w + \lambda b} \right) \right] \wedge s \right\} \boldsymbol{\mu}^{\boldsymbol{w}_+} \otimes \boldsymbol{\lambda}(dw, ds),$$

$\mathbf{P}$*–almost surely.*

Proof. The proof goes through a sequence of steps.

Step 1 For each $w \in [b, w^i)$, define t_{w,w^i} to be the minimum amount of time it takes for the process $\{w_t\}_{t\geq 0}$ to transit from w to w^i, that is, from (42):

$$t_{w,w^i} = \frac{1}{\rho} \ln\!\left(\frac{\rho w^i + \lambda b}{\rho w + \lambda b} \right). \tag{E.8}$$

For each $k \geq 1$, consider the integral $I_k = \int_{T_{k-1}}^{T_k} 1_{\{w_s > w^i\}}\, ds$, where $T_0 = 0$ by convention. According to (42), two cases must be distinguished. Suppose first that $w_{T_{k-1}^+} \geq w^i$. Then $w_s > w^i$ for all $s \in (T_{k-1}, T_k]$, and therefore $I_k = T_k - T_{k-1}$. Suppose next that $w_{T_{k-1}^+} < w^i$. If $T_k - T_{k-1} \leq t_{w_{T_{k-1}^+}, w^i}$, then $w_s \leq w^i$ for all $s \in (T_{k-1}, T_k]$, and therefore $I_k = 0$. Finally, if $T_k - T_{k-1} > t_{w_{T_{k-1}^+}, w^i}$, then $w_s > w^i$ for all $s \in (T_{k-1} + t_{w_{T_{k-1}^+}, w^i}, T_k]$, and therefore $I_k = T_k - T_{k-1} - t_{w_{T_{k-1}^+}, w^i}$. Summing over $k = 1, \ldots, n$ and rearranging, one obtains that

$$\frac{1}{n} \int_0^{T_n} 1_{\{w_s > w^i\}}\, ds = \frac{1}{n} \sum_{k=1}^n (T_k - T_{k-1}) - \frac{1}{n} \sum_{k=1}^n \left[t_{w_{T_{k-1}^+}, w^i} \wedge (T_k - T_{k-1}) \right] 1_{\{w_{T_{k-1}^+} < w^i\}} \tag{E.9}$$

for all $n \geq 1$.

Step 2 Observe from (42) that the process $\{Z_k\}_{k\geq 1} = \{(w_{T_{k-1}^+}, T_k - T_{k-1})\}_{k\geq 1}$ is Markov. Let $Q : [b, w^p] \times \mathbb{R}_+ \times \mathcal{B}([b, w^p] \times \mathbb{R}_+) \to [0, 1]$ denote the associated transition function, where $\mathcal{B}([b, w^p] \times \mathbb{R}_+)$ is the Borel σ–field over $[b, w^p] \times \mathbb{R}_+$. From (42) again, one has $Z_{k+1} = (h(Z_k), T_{k+1} - T_k)$ for all $k \geq 1$, where the function $h : [b, w^p] \times \mathbb{R}_+ \to [b, w^p - b]$ is defined by

$$h(w,t) = \begin{cases} \left[\left(w + \frac{\lambda b}{\rho}\right) e^{\rho t} - \frac{\lambda b}{\rho} - b \right] \vee b & \text{if } w \in [b, w^i) \text{ and } t \leq t_{w,w^i}, \\ \left\{ \left[\left(w^i + \frac{\lambda b}{\rho - \gamma}\right) e^{(\rho-\gamma)(t - t_{w,w^i})} - \frac{\lambda b}{\rho-\gamma} \right] \wedge w^p - b \right\} \vee b & \text{if } w \in [b, w^i) \text{ and } t > t_{w,w^i}, \\ \left\{ \left[\left(w + \frac{\lambda b}{\rho - \gamma}\right) e^{(\rho-\gamma)t} - \frac{\lambda b}{\rho-\gamma} \right] \wedge w^p - b \right\} \vee b & \text{if } w \in [w^i, w^p] \text{ and } t \geq 0, \end{cases}$$

with t_{w,w^i} as defined in (E.8). Since Z_k and $T_{k+1} - T_k$ are independent and $T_{k+1} - T_k$ has distribution $\boldsymbol{\lambda}$ for all $k \geq 1$, this in turn implies that $Q((w,t), A) = \boldsymbol{\lambda}(A_{h(w,t)})$ for all

$(w,t,A) \in [b,w^p] \times \mathbb{R}_+ \times \mathcal{B}([b,w^p] \times \mathbb{R}_+)$, where $A_{w'} = \{t' \in \mathbb{R}_+ \mid (w',t') \in A\}$ is the w'–section of A for all $w' \in [b,w^p]$. Now, let $A \in \mathcal{B}([b,w^p] \times \mathbb{R}_+)$ be Markov invariant for $\{Z_k\}_{k\geq 1}$, that is $Q((w,t),A) = 1$ for all $(w,t) \in A$. Then $\boldsymbol{\lambda}(A_{h(w,t)}) = 1$. Moreover, since $(h(w,t),t') \in A$ if $t' \in A_{h(w,t)}$, one has $Q((h(w,t),t'),A) = 1$ and thus $\boldsymbol{\lambda}(A_{h(h(w,t),t')}) = 1$ for all $t' \in A_{h(w,t)}$. For each $(w,t) \in A$, consider the set $h(h(w,t),A_{h(w,t)})$. It follows from the definition of h that $h(h(w,t),A_{h(w,t)}) \subset [[h(w,t)-b] \vee b, w^p - b]$. One now shows that $h(h(w,t),A_{h(w,t)})$ has full Lebesgue measure in $[[h(w,t)-b] \vee b, w^p - b]$. Observe first that the mapping $h(h(w,t),\cdot)$ is increasing over $\mathbb{R}_+$, with $h(h(w,t),0) = [h(w,t)-b] \vee b$ and $h(h(w,t),t') = w^p - b$ for $t' \geq t_{b,w^p}$, with t_{b,w^p} as defined in (E.2). Thus one only needs to check that $h(h(w,t),\mathbb{R}_+ \backslash A_{h(w,t)})$ has Lebesgue measure 0. This follows from the fact that $\mathbb{R}_+ \backslash A_{h(w,t)}$ has $\boldsymbol{\lambda}$–measure 0, and thus has Lebesgue measure 0 since these two measures are mutually absolutely continuous, along with the fact that $h(h(w,t),\cdot)$ is increasing and absolutely continuous over any interval of the form $[0,n]$, $n \geq 1$, and thus maps sets of Lebesgue measure 0 onto sets of Lebesgue measure 0 (Rudin (1986, Theorem 7.18)). Since $h(h(w,t),A_{h(w,t)})$ has full Lebesgue measure in $[[h(w,t)-b] \vee b, w^p - b]$ for any Markov invariant set A and all $(w,t) \in A$, one has

$$h(h(w_1,t_1),A_{1,h(w_1,t_1)}) \cap h(h(w_2,t_2),A_{2,h(w_2,t_2)}) \neq \emptyset$$

for any Markov invariant sets A_1 and A_2 and for all $(w_1,t_1) \in A_1$ and $(w_2,t_2) \in A_2$. As $\boldsymbol{\lambda}(A_{1,w''}) = \boldsymbol{\lambda}(A_{2,w''}) = 1$ for all $w'' \in h(h(w_1,t_1),A_{1,h(w_1,t_1)}) \cap h(h(w_2,t_2),A_{2,h(w_2,t_2)})$, one gets that $A_{1,w''} \cap A_{2,w''} \neq \emptyset$ for any such w'', so that $A_1 \cap A_2 \neq \emptyset$. Hence there are no disjoint Markov invariant sets, and $\{Z_k\}_{k\geq 1}$ is Markov ergodic (Stout (1974, Definition 3.6.8)). To complete this step of the proof, one shows that $\{Z_k\}_{k\geq 1}$ has a stationary initial distribution. Proceeding as in Step 1 of the proof of Claim 1, it is easy to check that the process $\{w_{T_k^+}\}_{k\geq 1}$ has a unique stationary initial distribution. That is, letting $P_+ : [b,w^p] \times \mathcal{B}([b,w^p]) \to [0,1]$ denote the associated transition function, there exists a unique probability measure $\boldsymbol{\mu}^{\boldsymbol{w}_+}$ over $[b,w^p]$ such that, for each $A \in \mathcal{B}([b,w^p])$,

$$\boldsymbol{\mu}^{\boldsymbol{w}_+}(A) = \int_{[b,w^p]} P_+(w,A)\, \boldsymbol{\mu}^{\boldsymbol{w}_+}(dw). \tag{E.10}$$

Since $Z_k = (w_{T_{k-1}^+}, T_k - T_{k-1})$ for all $k \geq 1$, and since $w_{T_{k-1}^+}$ and $T_k - T_{k-1}$ are independent for all $k \geq 1$, a natural guess for the invariant measure associated to $\{Z_k\}_{k\geq 1}$ is the product measure $\boldsymbol{\mu}^{\boldsymbol{w}_+} \otimes \boldsymbol{\lambda}$. To verify this, let $E_1 \times E_2$ be a measurable rectangle in $\mathcal{B}([b,w^p] \times \mathbb{R}_+)$. Then, for each $k \geq 1$, one has

$$\int_{[b,w^p]\times\mathbb{R}_+} Q((w,t),E_1\times E_2)\,\boldsymbol{\mu}^{\boldsymbol{w}+}\otimes\boldsymbol{\lambda}(dw,dt)$$

$$= \int_{[b,w^p]\times\mathbb{R}_+} 1_{\{h(w,t)\in E_1\}}\,\boldsymbol{\lambda}(E_2)\,\boldsymbol{\mu}^{\boldsymbol{w}+}\otimes\boldsymbol{\lambda}(dw,dt)$$

$$= \boldsymbol{\lambda}(E_2)\int_{[b,w^p]}\boldsymbol{\mu}^{\boldsymbol{w}+}(dw)\int_{\mathbb{R}_+} 1_{\{h(w,t)\in E_1\}}\,\boldsymbol{\lambda}(dt)$$

$$= \boldsymbol{\lambda}(E_2)\int_{[b,w^p]}\mathbf{P}\,[h(w,T_k-T_{k-1})\in E_1]\,\boldsymbol{\mu}^{\boldsymbol{w}+}(dw)$$

$$= \boldsymbol{\lambda}(E_2)\int_{[b,w^p]}\mathbf{P}\big[w_{T_k^+}\in E_1\,|\,w_{T_{k-1}^+}=w\big]\,\boldsymbol{\mu}^{\boldsymbol{w}+}(dw)$$

$$= \boldsymbol{\lambda}(E_2)\int_{[b,w^p]} P_+(w,E_1)\,\boldsymbol{\mu}^{\boldsymbol{w}+}(dw)$$

$$= \boldsymbol{\lambda}(E_2)\,\boldsymbol{\mu}^{\boldsymbol{w}+}(E_1)$$

$$= \boldsymbol{\mu}^{\boldsymbol{w}+}\otimes\boldsymbol{\lambda}(E_1\times E_2),$$

where the first equality follows from the definition of the transition function Q, the second from Tonelli's theorem, the third from the fact that $T_k - T_{k-1}$ has distribution $\boldsymbol{\lambda}$, the fourth from the independence of $w_{T_{k-1}^+}$ and $T_k - T_{k-1}$, the fifth from the definition of the transition function P_+, the sixth from (E.10), and the last from the definition of the product measure $\boldsymbol{\mu}^{\boldsymbol{w}+}\otimes\boldsymbol{\lambda}$. A standard monotone class argument then implies that

$$\boldsymbol{\mu}^{\boldsymbol{w}+}\otimes\boldsymbol{\lambda}(A) = \int_{[b,w^p]\times\mathbb{R}_+} Q((w,t),A)\,\boldsymbol{\mu}^{\boldsymbol{w}+}\otimes\boldsymbol{\lambda}(dw,dt)$$

for all $A \in \mathcal{B}([b,w^p]\times\mathbb{R}_+)$, so that $\boldsymbol{\mu}^{\boldsymbol{w}+}\otimes\boldsymbol{\lambda}$ is an invariant measure associated to $\{Z_k\}_{k\geq 1}$. Since $\{Z_k\}_{k\geq 1}$ is Markov ergodic, this invariant measure is in fact unique (Stout (1974, Theorem 3.6.7)).

Step 3 One finally uses (E.9) to evaluate the limit of the sequence $\big(\frac{1}{n}\int_0^{T_n} 1_{\{w_s>w^i\}}\,ds\big)_{n\geq 1}$. Since the random variables $(T_k - T_{k-1})_{k\geq 1}$ are independently and identically distributed according to the exponential distribution $\boldsymbol{\lambda}$ with parameter λ, it follows from the strong law of large numbers that the sequence $\big(\frac{1}{n}\sum_{k=1}^n (T_k - T_{k-1})\big)_{n\geq 1}$ converges $\mathbf{P}$–almost surely to $\frac{1}{\lambda}$. Next, since the process $\{(w_{T_{k-1}^+}, T_k - T_{k-1})\}_{k\geq 1}$ is Markov ergodic by Step 2, with invariant measure $\boldsymbol{\mu}^{\boldsymbol{w}+} \otimes \boldsymbol{\lambda}$ over $[b, w^p] \times \mathbb{R}_+$, and since the mapping $(w, s) \mapsto (t_{w,w^i} \wedge s) 1_{\{w<w^i\}}$ is measurable, positive and bounded above by $(w, s) \mapsto s$, and hence $\boldsymbol{\mu}^{\boldsymbol{w}+} \otimes \boldsymbol{\lambda}$–integrable, it follows from the strong law of large numbers for Markov ergodic processes (Stout (1974, Theorem 3.6.7)) that the sequence $\left(\frac{1}{n}\sum_{k=1}^{n}\left[t_{w_{T_{k-1}^+},w^i} \wedge (T_k - T_{k-1})\right] 1_{\{w_{T_{k-1}^+}<w^i\}}\right)_{n\geq 1}$ converges $\mathbf{P}$–almost surely to

$$\int_{[b,w^p]\times\mathbb{R}_+} (t_{w,w^i} \wedge s) 1_{\{w<w^i\}}\, \boldsymbol{\mu}^{\boldsymbol{w}+} \otimes \boldsymbol{\lambda}(dw, dt)$$

$$= \int_{[b,w^i)\times\mathbb{R}_+} (t_{w,w^i} \wedge s)\, \boldsymbol{\mu}^{\boldsymbol{w}+} \otimes \boldsymbol{\lambda}(dw, dt)$$

$$= \int_{[b,w^i)\times\mathbb{R}_+} \left\{\left[\frac{1}{\rho}\ln\left(\frac{\rho w^i + \lambda b}{\rho w + \lambda b}\right)\right] \wedge s\right\} \boldsymbol{\mu}^{\boldsymbol{w}+} \otimes \boldsymbol{\lambda}(dw, ds),$$

where the second equality follows from the definition (E.8) of t_{w,w^i}. Applying Fact 2 to the sequence $(Y_n)_{n\geq 1} = \left(\frac{1}{n}\int_0^{T_n} 1_{\{w_s>w^i\}}\, ds\right)_{n\geq 1}$ and to the family $(n_t)_{t\geq 0} = (N_{t^-})_{t\geq 0}$, and using the fact that $\frac{N_{t^-}}{t}$ converges $\mathbf{P}$–almost surely to λ as t goes to ∞ by the strong law of large numbers for the Poisson process, one then obtains that $\frac{1}{t}\int_0^{T_{N_{t^-}}} 1_{\{w_s>w^i\}}\, ds$ converges $\mathbf{P}$–almost surely to $1 - \lambda\int_{[b,w^i)\times\mathbb{R}_+}\left\{\left[\frac{1}{\rho}\ln\left(\frac{\rho w^i+\lambda b}{\rho w+\lambda b}\right)\right] \wedge s\right\} \boldsymbol{\mu}^{\boldsymbol{w}+} \otimes \boldsymbol{\lambda}(dw, ds)$ as t goes to ∞. ■

Claim 3 *One has*

$$\lim_{t\to\infty} \frac{1}{t}\int_{T_{N_{t^-}}}^{t} 1_{\{w_s>w^i\}}\, ds = 0,$$

$\mathbf{P}$*–almost surely.*

Proof. For each $t \geq 0$,

$$0 \leq \frac{1}{t}\int_{T_{N_{t^-}}}^{t} 1_{\{w_s>w^i\}}\, ds \leq 1 - \frac{T_{N_{t^-}}}{t} = \frac{N_{t^-}}{t}\left[\frac{t}{N_{t^-}} - \frac{1}{N_{t^-}}\sum_{k=1}^{N_{t^-}}(T_k - T_{k-1})\right]. \tag{E.11}$$

Applying Fact 2 to the sequence $(Y_n)_{n\geq 1} = \left(\frac{1}{n}\sum_{k=1}^{n}(T_k - T_{k-1})\right)_{n\geq 1}$ and to the family $(n_t)_{t\geq 0} = (N_{t^-})_{t\geq 0}$, and using the fact that $\frac{N_{t^-}}{t}$ converges $\mathbf{P}$–almost surely to λ as t goes to ∞ by the strong law of large numbers for the Poisson process, one then obtains from (E.11) that $\frac{1}{t}\int_{T_{N_{t^-}}}^{t} 1_{\{w_s>w^i\}}\, ds$ converges $\mathbf{P}$–almost surely to 0 as t goes to ∞. ■

Given (E.1) and (E.8), combining Claims 1 to 3 completes the proof of Proposition 4. ■

Remark The proofs of Claims 1 and 2 given above directly proceed by establishing that the processes $\{w_{T_k}\}_{k\geq 1}$ and $\{(w_{T_{k-1}^+}, T_k - T_{k-1})\}_{k\geq 1}$ are Markov ergodic, that is, that they have no disjoint invariant sets. Since the existence of an invariant measure can be proven in each case by other means, this allows one to use the strong law of large numbers for Markov ergodic processes (Stout (1974, Theorem 3.6.7)). A slightly different approach consists in first showing that the transition functions associated to these processes satisfy Doeblin's condition (Doob (1953, Chapter V, Section 6)), which ensures the existence of invariant measures. One then establishes that there exists a unique ergodic set and correspondingly a unique invariant measure. Finally, one uses the strong law of large numbers for Markov processes whose transition functions are known to satisfy Doeblin's condition (Doob (1953, Chapter V, Theorem 6.2)). That this is the case of the transition function P of $\{w_{T_k}\}_{k\geq 1}$ is implicit in Step 1 of the proof of Claim 1, where it is shown that it satisfies Condition M in Stokey and Lucas (1989, Chapter 11, Section 4). This condition is stronger than Doeblin's and implies at once that there exists a unique invariant measure. Consider now the process $\{(w_{T^+_{k-1}}, T_k - T_{k-1})\}_{k\geq 1}$ with transition Q over $[b, w^p] \times \mathbb{R}_+$. By definition, the transition Q satisfies Doeblin's condition if there exists a finite measure $\boldsymbol{\varphi}$ over $\mathcal{B}([b, w^p] \times \mathbb{R}_+)$, an integer $\nu \geq 1$ and a number $\varepsilon > 0$ such that, for each $(w, t, A) \in [b, w^p] \times \mathbb{R}_+ \times \mathcal{B}([b, w^p] \times \mathbb{R}_+)$, $Q^\nu((w, t), A) \leq 1 - \varepsilon$ whenever $\boldsymbol{\varphi}(A) \leq \varepsilon$. We now exhibit a triple $(\boldsymbol{\varphi}, \nu, \varepsilon)$ such that this condition holds. To do so, fix some $\varepsilon \in (0, e^{-\lambda t_{b,w^p}})$, with t_{b,w^p} defined as in (E.2), and consider the measure $\boldsymbol{\varphi} = \frac{e^{-\lambda t_{b,w^p}}\varepsilon}{e^{-\lambda t_{b,w^p}} - \varepsilon}\, \boldsymbol{\delta}_{w^p - b} \otimes \boldsymbol{\lambda}$, where $\boldsymbol{\delta}_{w^p-b}$ is the Dirac mass at $w^p - b$. For each $A \in \mathcal{B}([b, w^p] \times \mathbb{R}_+)$, note that $\boldsymbol{\varphi}(A) = \frac{e^{-\lambda t_{b,w^p}}\varepsilon}{e^{-\lambda t_{b,w^p}} - \varepsilon} \boldsymbol{\lambda}(A_{w^p-b})$, where A_{w^p-b} is the $(w^p - b)$–section of A. This implies in particular that $\boldsymbol{\lambda}(A_{w^p-b}) \leq 1 - \frac{\varepsilon}{e^{-\lambda t_{b,w^p}}}$ whenever $\boldsymbol{\varphi}(A) \leq \varepsilon$, so that in this case

$$\begin{aligned}
Q^2((w, t), A) &= 1 - Q^2((w, t), A^c) \\
&\leq 1 - Q^2((w, t), \{w^p - b\} \times \mathbb{R}_+ \cap A^c) \\
&\leq 1 - e^{-\lambda t_{b,w^p}}[1 - \boldsymbol{\lambda}(A_{w^p-b})] \\
&\leq 1 - \varepsilon
\end{aligned}$$

for all $(w, t) \in [b, w^p] \times \mathbb{R}_+$, where the second inequality follows from the definitions of t_{b,w^p} and Q. Thus Q satisfies Doeblin's condition, as claimed. Moreover, observe that for each $(w, t, A) \in [b, w^p] \times \mathbb{R}_+ \times \mathcal{B}([b, w^p] \times \mathbb{R}_+)$,

$$Q^2((w, t), A) \geq e^{-\lambda t_{b,w^p}} \boldsymbol{\lambda}(A_{w^p - b}) = \left(\frac{e^{-\lambda t_{b,w^p}}}{\varepsilon} - 1\right) \boldsymbol{\varphi}(A),$$

which, since $e^{-\lambda t_{b,w^p}} > \varepsilon$, implies that $Q^2((w, t), A) > 0$ whenever $\boldsymbol{\varphi}(A) > 0$. This in turn is a sufficient condition for Q to have a unique ergodic set (Stokey and Lucas (1989, Theorem 11.10)). One can then show as in Step 2 of the proof of Claim 2 that the corresponding unique invariant measure is $\boldsymbol{\mu}^{\boldsymbol{w}+} \otimes \boldsymbol{\lambda}$. Finally, Step 3 of the proof of Claim 2 follows from applying the strong law of large numbers for Markov processes whose transition functions satisfy Doeblin's condition (Doob (1953, Chapter V, Theorem 6.2)).

Proof of Proposition 5. One first checks that (49) holds uniformly in γ whenever c is close enough to 0. Specifically, using the notation of Appendix C, and keeping in mind that $\frac{v_{\beta_0}(b)}{b} = \beta_0 > v'_{\beta_0+}(b)$ by (C.6), suppose that $v_{\beta_0}(b) - bv'_{\beta_0+}(b) \geq c$. Then, for each $\gamma \in (0, r)$, it must be that $f(b) - bf'_+(b) = v_{\beta_\gamma,\gamma}(b) - bv'_{\beta_\gamma,\gamma+}(b) \geq c$ as well. Suppose indeed that the contrary holds for such a γ. Then, since $\beta_\gamma > \beta_0$ by Proposition C.2.2 and $\frac{u_2(b)}{b} = 1 > u'_{2+}(b)$ by (C.10), one would have

$$
\begin{aligned}
c &> v_{\beta_\gamma,\gamma}(b) - bv'_{\beta_\gamma,\gamma+}(b) \\
&= u_1(b) - bu'_{1+}(b) + \beta_\gamma[u_2(b) - bu'_{2+}(b)] \\
&> u_1(b) - bu'_{1+}(b) + \beta_0[u_2(b) - bu'_{2+}(b)] \\
&= v_{\beta_0}(b) - bv'_{\beta_0+}(b),
\end{aligned}
$$

a contradiction. The claim follows. Now, under (49), one has by (48)

$$
\lim_{t\to\infty} \frac{\ln(X_t)}{t} = \lambda \int_{[b,2b)} \ln\left(\frac{w-b}{b}\right) \boldsymbol{\mu}^{\boldsymbol{w}}(dw) + \gamma. \tag{E.12}
$$

The remainder of the proof then consists in constructing appropriate upper and lower bounds for $\int_{[b,2b)} \ln\left(\frac{w-b}{b}\right) \boldsymbol{\mu}^{\boldsymbol{w}}(dw)$. Consider first the upper bound. Writing (C.18) at w^p and using (C.17) along with the fact that v is positive and increasing yields

$$
w^p = \frac{\mu - \lambda C - \gamma c - (r-\gamma)v(w^p) - \lambda[v(w^p) - v(w^p - b)]}{\rho - r} < \frac{\mu - \lambda C}{\rho - r},
$$

uniformly in γ. Let $\overline{w}^p = \frac{\mu - \lambda C}{\rho - r}$ and define auxiliary processes $\{\overline{w}_t\}_{t\geq 0}$ and $\{\bar{l}_t\}_{t\geq 0}$ by

$$
\overline{w}_t = w_0 + \int_0^{t-} \left\{ (\rho \overline{w}_s + \lambda b)\, ds - b\left(\frac{\overline{w}_s - b}{b} \wedge 1\right) dN_s - d\bar{l}_s \right\}, \tag{E.13}
$$

$$
\bar{l}_t = \max\{\overline{w}_0 - \overline{w}^p, 0\} + \int_0^t (\rho \overline{w}^p + \lambda b) 1_{\{\overline{w}_{s+} = \overline{w}^p\}}\, ds \tag{E.14}
$$

for all $t \geq 0$. Observe that the process $\{\overline{w}_t\}_{t\geq 0}$ is independent of γ. It is easy to check from

(42), (43), (E.13) and (E.14) that $w_t \leq \overline{w}_t$ for all $t \geq 0$. Proceeding as in Claim 1 of the proof of Proposition 4, one can further show that $\{\overline{w}_{T_k}\}_{k\geq 1}$ has a unique stationary initial distribution $\boldsymbol{\mu}^{\overline{\boldsymbol{w}}}$ and that

$$\int_{[b,2b)} \ln\left(\frac{w-b}{b}\right) \boldsymbol{\mu}^{\boldsymbol{w}}(dw) \leq \int_{[b,2b)} \ln\left(\frac{w-b}{b}\right) \boldsymbol{\mu}^{\overline{\boldsymbol{w}}}(dw) < 0,$$

uniformly in γ. Here the strict inequality follows from the fact that for each $k \geq 1$ and $w \in (b, \overline{w}^p]$, there is a strictly positive probability that $\overline{w}_{T_{k+1}} < w$ given that $\overline{w}_{T_{k+1}} \geq w$, which implies in turn that the lower bound of the support of the stationary initial distribution $\boldsymbol{\mu}^{\overline{\boldsymbol{w}}}$ of $\{\overline{w}_{T_k}\}_{k\geq 1}$ is b. Therefore, for γ close enough to 0,

$$\lambda \int_{[b,2b)} \ln\left(\frac{w-b}{b}\right) \boldsymbol{\mu}^{\boldsymbol{w}}(dw) + \gamma < 0,$$

from which (50) follows by (E.12). Consider next the lower bound. By (E.7), one has

$$\int_{[b,2b)} \ln\left(\frac{w-b}{b}\right) \boldsymbol{\mu}^{\boldsymbol{w}}(dw) \geq -\frac{\lambda}{\rho - \gamma + \lambda},$$

uniformly in γ. Therefore, if $\gamma > \frac{\lambda^2}{\rho-\gamma+\lambda}$,

$$\lambda \int_{[b,2b)} \ln\left(\frac{w-b}{b}\right) \boldsymbol{\mu}^{\boldsymbol{w}}(dw) + \gamma > 0,$$

from which (51) follows by (E.12). Hence the result. ■

Proof of Proposition 6. Consider for each $k \geq 1$ the σ–fields

$$\begin{aligned} \mathcal{F}_1^k &= \sigma((w_0, T_1 - T_0), (w_{T_1}, T_2 - T_1), \ldots, (w_{T_{k-1}}, T_k - T_{k-1})), \\ \mathcal{F}_k^\infty &= \sigma((w_{T_{k-1}}, T_k - T_{k-1}), (w_{T_k}, T_{k+1} - T_k), \ldots), \end{aligned} \tag{E.15}$$

and denote by

$$\mathcal{T} = \bigcap_{k=1}^{\infty} \mathcal{F}_k^\infty \tag{E.16}$$

the corresponding tail σ–field. Then the following zero-one law holds.

Claim 4 *For each $E \in \mathcal{T}$, either $\mathbf{P}[E] = 0$ or $\mathbf{P}[E] = 1$.*

Proof. One first shows that for each $\varepsilon > 0$, there exists $n_0 \geq 1$ such that

$$\Delta(k,n,w,t,A) = \mathbf{P}[(w_{T_{k+n-1}}, T_{k+n} - T_{k+n-1}) \in A \,|\, (w_{T_{k-1}}, T_k - T_{k-1}) = (w,t)]$$
$$- \mathbf{P}[(w_{T_{k+n-1}}, T_{k+n} - T_{k+n-1} \in A)] \tag{E.17}$$
$$\leq \varepsilon$$

for all $k \geq 1$, $n \geq n_0$, $(w,t) \in [b, w^p] \times \mathbb{R}_+$ and $A \in \mathcal{B}([b,w^p] \times \mathbb{R}_+)$. A standard monotone class argument implies that it is enough to verify (E.17) for sets $A = \bigcup_{i=1}^m E_1^i \times E_2^i$ that are finite unions of disjoint measurable rectangles in $\mathcal{B}([b,w^p] \times \mathbb{R}_+)$. Now, fix some such set A, and let $\tilde{E}_1^1, \ldots, \tilde{E}_1^{\tilde{m}}$ be the atoms of the field of subsets of $\bigcup_{i=1}^m E_1^i$ generated by $E_1^1, \ldots, E_1^m$. The sets $\tilde{E}_1^1, \ldots, \tilde{E}_1^{\tilde{m}}$ form a partition of $\bigcup_{i=1}^m E_1^i$. Define

$$\tilde{I}^+ = \{i \in \{1, \ldots, \tilde{m}\} \,|\, \mathbf{P}[w_{T_{k+n-1}} \in \tilde{E}_1^i \,|\, (w_{T_{k-1}}, T_k - T_{k-1}) = (w,t)] - \mathbf{P}[w_{T_{k+n-1}} \in \tilde{E}_1^i] \geq 0\}.$$

As in Claim 1 of the proof of Proposition 4, let T^* be the adjoint operator associated to the transition function P of $\{w_{T_k}\}_{k\geq 1}$, and let $\|\cdot\|_{TV}$ be the total variation norm over the space $\Delta([b,w^p])$ of Borel probability measures over $[b,w^p]$. Finally, define h as in Claim 2 of the proof of Proposition 4, and let $\boldsymbol{\mu}_{\boldsymbol{w_{T_k}}}$ be the distribution of w_{T_k}. One then has

$$\Delta(k,n,w,t,A)$$
$$= \sum_{i=1}^m \{\mathbf{P}[w_{T_{k+n-1}} \in E_1^i \,|\, (w_{T_{k-1}}, T_k - T_{k-1}) = (w,t)] - \mathbf{P}[w_{T_{k+n-1}} \in E_1^i]\} \,\boldsymbol{\lambda}[E_2^i]$$
$$\leq \mathbf{P}\left[w_{T_{k+n-1}} \in \bigcup_{i\in\tilde{I}^+} \tilde{E}_1^i \,|\, (w_{T_{k-1}}, T_k - T_{k-1}) = (w,t)\right] - \mathbf{P}\left[w_{T_{k+n-1}} \in \bigcup_{i\in\tilde{I}^+} \tilde{E}_1^i\right]$$
$$\leq \frac{1}{2}\,\|T^{*n-1}(\boldsymbol{\delta}_{[h((w-b)\vee b,t)+b]\wedge w^p}) - T^{*n-1}(\boldsymbol{\mu}_{\boldsymbol{w_{T_k}}})\|_{TV}$$
$$\leq \frac{1}{2}\,(1 - e^{-\lambda t_{b,w^p}})^{n-1}\|\boldsymbol{\delta}_w - \boldsymbol{\mu}_{\boldsymbol{w_{T_k}}}\|_{TV}$$
$$\leq (1 - e^{-\lambda t_{b,w^p}})^{n-1},$$

where the first equality follows from the fact that $T_k - T_{k-1}$ is independent of any random variable measurable with respect to $\mathcal{F}_0^{k-1}$, and thus in particular of $w_{T_{k-1}}$, the first inequality from the definition of $\tilde{I}^+$ and from the assumption that the rectangles that make up A are disjoint, the second inequality from the definitions of T^*, h and $\boldsymbol{\mu}_{\boldsymbol{w_{T_k}}}$, and the third inequality from the fact that, as shown in Claim 1 of the proof of Proposition 4, T^* is a contraction of modulus $1 - e^{-\lambda t_{b,w^p}}$. Thus (E.17) holds as soon as $n_0 \geq 1 + \frac{\ln(\varepsilon)}{\ln(1-e^{-\lambda t_{b,w^p}})}$, uniformly in (k, n, w, t, A). The remainder of the proof closely follows Bártfai and Révész (1967). As in their Example 2, a consequence of condition (E.17) is that for each $\varepsilon > 0$, there exists $n_0 \geq 1$ such that the following mixing property holds:

$$\mathbf{P}[E \,|\, \mathcal{F}_1^k] - \mathbf{P}[E] \leq \varepsilon \tag{E.18}$$

for all $k \geq 1$, $n \geq n_0$, and $E \in \mathcal{F}_{k+n}^\infty$, $\mathbf{P}$–almost surely. Fix some $E \in \mathcal{T}$, so that in particular $E \in \mathcal{F}_{k+n}^\infty$ for all $n \geq n_0$. Since ε is arbitrary, the mixing property (E.18) then implies that $\mathbf{P}[E \,|\, \mathcal{F}_1^k] \leq \mathbf{P}[E]$ for all $k \geq 1$, $\mathbf{P}$–almost surely. From Doob (1953, Chapter VII, Theorem 4.3), it follows that $\mathbf{P}[E \,|\, \bigvee_{k=1}^\infty \mathcal{F}_1^k] \leq \mathbf{P}[E]$, $\mathbf{P}$–almost surely. Since $E \in \mathcal{T} \subset \bigvee_{k=1}^\infty \mathcal{F}_1^k$, one finally has $\mathbf{P}[E] = \int_E \mathbf{P}[E \,|\, \bigvee_{k=1}^\infty \mathcal{F}_1^k] \, d\mathbf{P} \leq \int_E \mathbf{P}[E] \, d\mathbf{P} = \mathbf{P}[E]^2$. Hence the result. ■

From now on, we implicitly suppose that $\lim_{t\to\infty} N_t = \infty$, which is without loss of generality since this event occurs with probability 1.

Claim 5 *Each of the events* $\{\lim_{n\to\infty} X_{T_n} = 0\}$ *and* $\{\lim_{n\to\infty} X_{T_n^+} = \infty\}$ *belongs to* $\mathcal{T}$.

Proof. Consider first $\{\lim_{n\to\infty} X_{T_n} = 0\}$. Fix some $k_0 \geq 1$. For each $n \geq k_0 + 1$, one has

$$\begin{aligned}
X_{T_n} &= X_0 \prod_{k=1}^{N_{T_n^-}} \left(\frac{w_{T_k} - b}{b} \wedge 1 \right) \exp\left(\int_0^{T_n} \gamma 1_{\{w_s > w^i\}} \, ds \right) \\
&= X_0 \prod_{k=1}^{n-1} \left(\frac{w_{T_k} - b}{b} \wedge 1 \right) \\
&\qquad \exp\left(\gamma \left\{ \sum_{k=1}^n (T_k - T_{k-1}) - \sum_{k=1}^n \left[t_{w_{T_{k-1}^+}, w^i} \wedge (T_k - T_{k-1}) \right] 1_{\{w_{T_{k-1}^+} < w^i\}} \right\} \right) \qquad \text{(E.19)} \\
&= X_{T_{k_0}} \prod_{k=k_0}^{n-1} \left(\frac{w_{T_k} - b}{b} \wedge 1 \right) \\
&\qquad \exp\left(\gamma \left\{ \sum_{k=k_0+1}^n (T_k - T_{k-1}) - \sum_{k=k_0+1}^n \left[t_{w_{T_{k-1}^+}, w^i} \wedge (T_k - T_{k-1}) \right] 1_{\{w_{T_{k-1}^+} < w^i\}} \right\} \right)
\end{aligned}$$

with $\prod_\emptyset = 1$ by convention, where the second equality follows from (E.9) and from the fact that $N_{T_n^-} = n - 1$. Since $X_{T_{k_0}}$ is a strictly positive random variable, (E.15) and (E.19) jointly imply that $\{\lim_{n\to\infty} X_{T_n} = 0\} \in \mathcal{F}_{k_0+1}^\infty$. Since k_0 is arbitrary, $\{\lim_{n\to\infty} X_{T_n} = 0\} \in \mathcal{T}$ by

(E.16). The proof for $\{\lim_{n\to\infty} X_{T_n^+} = \infty\}$ is similar, observing that

$$X_{T_n^+} = X_{T_{k_0}^+} \prod_{k=k_0+1}^{n} \left(\frac{w_{T_k} - b}{b} \wedge 1\right)$$
$$\exp\left(\gamma\left\{\sum_{k=k_0+1}^{n} (T_k - T_{k-1}) - \sum_{k=k_0+1}^{n} \left[t_{w_{T_{k-1}^+}, w^i} \wedge (T_k - T_{k-1})\right] 1_{\{w_{T_{k-1}^+} < w^i\}}\right\}\right)$$

and that $X_{T_{k_0}^+}$ is a finite random variable. Hence the result. ■

Claim 6 *One has*

$$\{\lim_{t\to\infty} X_t = 0\} = \{\lim_{n\to\infty} X_{T_n} = 0\},$$
$$\{\lim_{t\to\infty} X_t = \infty\} = \{\lim_{n\to\infty} X_{T_n^+} = \infty\}.$$

Proof. Consider first $\{\lim_{t\to\infty} X_t = 0\}$. For each $\omega \in \{\lim_{t\to\infty} X_t = 0\}$ and $\varepsilon > 0$, there exists $t_0(\omega, \varepsilon) \geq 0$ such that $|X_t(\omega)| \leq \varepsilon$ for all $t \geq t_0(\omega, \varepsilon)$. Since the sequence $(T_n(\omega))_{n\geq 1}$ is strictly increasing and diverges to ∞, there exists $n_0(\omega, \varepsilon) \geq 1$ such that $T_n(\omega) \geq t_0(\omega, \varepsilon)$ and hence $|X_{T_n(\omega)}(\omega)| \leq \varepsilon$ for all $n \geq n_0(\omega, \varepsilon)$. As a result of this, $\omega \in \{\lim_{n\to\infty} X_{T_n} = 0\}$ and thus $\{\lim_{t\to\infty} X_t = 0\} \subset \{\lim_{n\to\infty} X_{T_n} = 0\}$. Conversely, for each $\omega \in \{\lim_{n\to\infty} X_{T_n} = 0\}$ and $\varepsilon > 0$, there exists $n_0(\omega, \varepsilon) \geq 1$ such that $|X_{T_n(\omega)}(\omega)| \leq \varepsilon$ for all $n \geq n_0(\omega, \varepsilon)$. Since the process $\{X_t\}_{t\geq 0}$ is increasing on any random interval $(T_{k-1}, T_k]$, $k \geq 1$, it follows that $|X_t(\omega)| \leq \varepsilon$ for all $t > T_{n_0(\omega,\varepsilon)}$. As a result of this, $\omega \in \{\lim_{t\to\infty} X_t = 0\}$ and thus $\{\lim_{n\to\infty} X_{T_n} = 0\} \subset \{\lim_{t\to\infty} X_t = 0\}$. Hence $\{\lim_{t\to\infty} X_t = 0\} = \{\lim_{n\to\infty} X_{T_n} = 0\}$, as claimed. The proof that $\{\lim_{t\to\infty} X_t = \infty\} = \{\lim_{n\to\infty} X_{T_n^+} = \infty\}$ is similar and is therefore omitted. ■

Combining Claims 4 to 6 completes the proof of Proposition 6. ■

F A Heuristic Analysis of the Non Constant Returns to Scale Case

In this appendix, we relax the constant returns to scale assumption, and provide a heuristic assessment of the robustness of our results to small non-linear perturbations in the private benefits function. Specifically, suppose that the private benefits from shirking are represented by a function

$$B^{\varepsilon}(X) = BX + \varepsilon X \phi(X) \tag{F.1}$$

of firm size X, where ε is a positive number and ϕ a bounded, positive, increasing and differentiable function.[3] In the paper, we consider the constant returns to scale case where $\varepsilon = 0$. To assess the robustness of our analysis to this assumption, we heuristically discuss below what happens when ε is small, but strictly positive. We argue that the key qualitative properties of the optimal contract are upheld for such a small perturbation.

Denote the principal's value function by F^ε. The Hamilton–Jacobi–Bellman equation now writes as:

$$\begin{aligned} rF^\varepsilon(X_t, W_{t^-}) = X_t(\mu - \lambda C) + \max\{&- X_t\ell_t + (\rho W_{t^-} + \lambda H_t - X_t\ell_t)F^\varepsilon_W(X_t, W_{t^-}) \\ &+ X_t g_t[F^\varepsilon_X(X_t, W_{t^-}) - c] \\ &- \lambda[F^\varepsilon(X_t, W_{t^-}) - F^\varepsilon(X_t x_t, W_{t^-} - H_t)]\}, \end{aligned} \tag{F.2}$$

where the maximization in (F.2) is over the set of controls (g_t, H_t, ℓ_t, x_t) that satisfy

$$\begin{aligned} 0 \leq g_t &\leq \gamma, \\ H_t &\geq \frac{B^\varepsilon(X_t)}{\Delta\lambda}, \\ \ell_t &\geq 0, \\ W_{t^-} - H_t &\geq \frac{B^\varepsilon(X_t x_t)}{\Delta\lambda}. \end{aligned} \tag{F.3}$$

The second of these constraints is the agent's date t incentive compatibility constraint, while the fourth of these constraints, which parallels condition (19) in the paper, expresses the fact that if a loss occurs at date t, reducing by H_t the continuation utility of the agent, it must still be possible to provide incentives after this loss, which requires further reducing the agent's utility by at least $\frac{B^\varepsilon(X_t x_t)}{\Delta\lambda}$, where $X_t x_t$ is the size of the firm after the date t loss.

Optimizing with respect to ℓ_t The first-order condition with respect to ℓ_t is

$$F^\varepsilon_W(X_t, W_{t^-}) \geq -1, \tag{F.4}$$

with equality only if $\ell_t > 0$. Call $W^{p,\varepsilon}(X_t)$ the first value of W_{t^-} at which (F.4) holds as an equality; this corresponds to the payment threshold for a given size X_t. In the constant

[3] These assumptions ensure in particular that B^ε is invertible, and that, in the positive orthant, the graph of B^ε lies in a cone pointed at the origin and whose upper and lower edges cross the axes at the origin only. Since ϕ is bounded, there is no loss of generality in assuming that it is positive: the situation with a negative ϕ could be mimicked by starting from a smaller value of B.

returns to scale case, one has $W^{p,0}(X_t) = X_t w^p$. As in Property 1 of the paper, payments are made only when $W_{t^-} \geq W^{p,\varepsilon}(X_t)$. For the purpose of this heuristic presentation, we shall assume without proof that the mapping $X \mapsto \frac{W^{p,\varepsilon}(X)}{X}$ converges uniformly to w^p as ε goes to 0.

Optimizing with respect to $\boldsymbol{x_t}$ Consider now the case where $W_{t^-} < W^{p,\varepsilon}(X_t)$. Property 2 in the paper states that, in the optimal contract, downsizing is imposed only as the last resort. Let us now examine what happens when $\varepsilon > 0$. Differentiating the objective function on the right-hand side of (F.2) with respect to x_t yields $X_t F_X^\varepsilon(X_t x_t, W_{t^-} - H_t)$. In the limit case where $\varepsilon = 0$, this is equal to

$$f\left(\frac{W_{t^-} - H_t}{X_t x_t}\right) - \frac{W_{t^-} - H_t}{X_t x_t}\, f'\left(\frac{W_{t^-} - H_t}{X_t x_t}\right) > f(b) - b f'_+(b) > 0, \tag{F.5}$$

where, recalling that $b = \frac{B}{\Delta\lambda}$, the first inequality follows from the fact that

$$\frac{W_{t^-} - H_t}{X_t x_t} \geq b + \frac{\varepsilon}{\Delta\lambda}\, \phi(X_t x_t) > b$$

by (F.1) and (F.3) along with the strict concavity of f over $[b, w^p]$, while the second inequality reflects that f vanishes at 0 and is globally concave over $\mathbb{R}_+$ but not differentiable at b. It follows from (F.5) that $F_X^0(X_t x_t, W_{t^-} - H_t)$ is positive and bounded away from 0 over the set of 4-tuples (X_t, x_t, W_{t^-}, H_t) that satisfy (F.3) and $W_{t^-} < W^{p,\varepsilon}(X_t)$. Hence, by continuity, one can reasonably expect that, for ε small enough, $F_X^\varepsilon(X_t x_t, W_{t^-} - H_t) > 0$ for any such 4-tuple; this is for instance the case if the derivative $\frac{\partial F_X^\varepsilon}{\partial \varepsilon}$ is bounded. In that case, it is optimal to let x_t be as large as possible in (F.2). This yields

$$x_t = \frac{(B^\varepsilon)^{-1}(\Delta\lambda(W_{t^-} - H_t))}{X_t} \wedge 1, \tag{F.6}$$

which generalizes Property 2 in the paper, reflecting that, for a given degree of incentives as measured by H_t, downsizing is imposed only when necessary.

Optimizing with respect to $\boldsymbol{H_t}$ Consider again the case where $W_{t^-} < W^{p,\varepsilon}(X_t)$. Property 3 in the paper states that, in the optimal contract, the exposure to risk of the agent is minimized by letting h_t equal the minimal amount b consistent with her exerting effort, or, equivalently, by letting H_t equal $X_t b$. Let us now examine what happens whenever $\varepsilon > 0$. Substituting (F.6) into (F.2), and right-differentiating the objective function on the right-hand side of (F.2) with respect to H_t yields

$$\lambda\big[F^{\varepsilon}_{W}(X_t, W_{t^-}) - F^{\varepsilon}_{W+}(X_t, W_{t^-} - H_t)\big] \tag{F.7}$$

if $B^{\varepsilon}(X_t) < \Delta\lambda(W_{t^-} - H_t)$, and

$$\begin{aligned}\lambda\Bigg[&F^{\varepsilon}_{W}(X_t, W_{t^-}) - F^{\varepsilon}_{W+}((B^{\varepsilon})^{-1}(\Delta\lambda(W_{t^-} - H_t)), W_{t^-} - H_t)\\ &- F^{\varepsilon}_{X}((B^{\varepsilon})^{-1}(\Delta\lambda(W_{t^-} - H_t)), W_{t^-} - H_t)\,\frac{\Delta\lambda}{(B^{\varepsilon})'((B^{\varepsilon})^{-1}(\Delta\lambda(W_{t^-} - H_t)))}\Bigg]\end{aligned} \tag{F.8}$$

if $B^{\varepsilon}(X_t) > \Delta\lambda(W_{t^-} - H_t)$. Examining each case in turn, we argue below that the expressions in (F.7) and (F.8) are negative for ε small enough. In that case, it is optimal to let H_t be as small as possible in (F.2). This yields

$$H_t = \frac{B^{\varepsilon}(X_t)}{\Delta\lambda}, \tag{F.9}$$

which generalizes Property 3 in the paper, reflecting that it is unnecessary to expose the agent to more risk than what is required to provide her incentives to exert effort.

Case 1: $\boldsymbol{B^{\varepsilon}(X_t) < \Delta\lambda(W_{t^-} - H_t)}$ Denote by $D^{\varepsilon}_1(X_t, W_{t^-}, H_t)$ the expression in (F.7), divided by λ. In the limit case where $\varepsilon = 0$, this is equal to

$$D^{0}_1(X_t, W_{t^-}, H_t) = f'\bigg(\frac{W_{t^-}}{X_t}\bigg) - f'_+\bigg(\frac{W_{t^-} - H_t}{X_t}\bigg). \tag{F.10}$$

Using the concavity of f along with the fact that

$$H_t \geq bX_t + \frac{\varepsilon}{\Delta\lambda}\,\phi(X_t)$$

by (F.1) and (F.3), and recalling that $w_t = \frac{W_{t^-}}{X_t}$, it follows from (F.10) that

$$D^{0}_1(X_t, W_{t^-}, H_t) \leq f'(w_t) - f'_+(w_t - b). \tag{F.11}$$

Since we have assumed that the mapping $X \mapsto \frac{W^{p,\varepsilon}(X)}{X}$ converges uniformly to w^p as ε goes to 0, $w_t < w^p + O(\varepsilon)$ for ε small enough, uniformly in the pairs (X_t, W_{t^-}) that satisfy $W_{t^-} < W^{p,\varepsilon}(X_t)$. Therefore, since the mapping $w \mapsto f'(w) - f'_+(w - b)$ is negative and bounded away from 0 over $(b, w^p]$ as f is strictly concave over this interval and globally concave over $\mathbb{R}_+$ but not differentiable at b, it follows from (F.11) that, for ε small enough, $D^{0}_1(X_t, W_{t^-}, H_t)$ is also negative and bounded away from 0 over the set of triples (X_t, W_{t^-}, H_t) that satisfy (F.3), $W_{t^-} < W^{p,\varepsilon}(X_t)$ and $B^{\varepsilon}(X_t) < \Delta\lambda(W_{t^-} - H_t)$. Hence, by continuity, one can reasonably expect that, for ε small enough, $D_1^\varepsilon(X_t, W_{t^-}, H_t) < 0$ for any such triple; this is for instance the case if the derivative $\frac{\partial F_W^\varepsilon}{\partial \varepsilon}$ is bounded.

Case 2: $\boldsymbol{B^\varepsilon(X_t) > \Delta\lambda(W_{t^-} - H_t)}$ Denote by $D_2^\varepsilon(X_t, W_{t^-}, H_t)$ the expression in (F.8), divided by λ. In the limit case where $\varepsilon = 0$, this is equal to

$$D_2^0(X_t, W_{t^-}, H_t) = f'\left(\frac{W_{t^-}}{X_t}\right) - \frac{f(b)}{b}. \tag{F.12}$$

An alternative way to see this is that, when $\varepsilon = 0$, the terms in H_t in the objective function on the right-hand side of (F.2), $H_t F_W^0(X_t, W_{t^-}) + F^0((B^0)^{-1}(\Delta\lambda(W_{t^-} - H_t)), W_{t^-} - H_t)$, can be rewritten as $H_t f'\left(\frac{W_{t^-}}{X_t}\right) + (W_{t^-} - H_t)\,\frac{f(b)}{b}$, from which (F.12) follows upon differentiating with respect to H_t. Now, by (F.1) and (F.3),

$$\frac{W_{t^-}}{X_t} \geq b + \frac{\varepsilon}{\Delta\lambda}\,\phi(X_t) > b.$$

Therefore, since f vanishes at 0 and is globally concave over $\mathbb{R}_+$ but not differentiable at b, one has from (F.12)

$$D_2^0(X_t, W_{t^-}, H_t) \leq f'_+(b) - \frac{f(b)}{b} < 0. \tag{F.13}$$

It follows from (F.13) that, for ε small enough, $D_2^0(X_t, W_{t^-}, H_t)$ is negative and bounded away from 0 over the set of triples (X_t, W_{t^-}, H_t) that satisfy (F.3), $W_{t^-} < W^{p,\varepsilon}(X_t)$ and $B^\varepsilon(X_t) > \Delta\lambda(W_{t^-} - H_t)$. Hence, by continuity, one can reasonably expect that, for ε small enough, $D_2^\varepsilon(X_t, W_{t^-}, H_t) < 0$ for any such triple; this is for instance the case if the derivatives $\frac{\partial F_X^\varepsilon}{\partial \varepsilon}$ and $\frac{\partial F_W^\varepsilon}{\partial \varepsilon}$ are bounded.

An important consequence of (F.9) is that downsizing takes place following a loss at date t if and only if $W_{t^-} < \frac{2B^\varepsilon(X_t)}{\Delta\lambda}$, that is, if and only if it is absolutely necessary, in order to maintain limited liability while ensuring incentive compatibility.

Optimizing with respect to $\boldsymbol{g_t}$ Consider again the case where $W_{t^-} < W^{p,\varepsilon}(X_t)$. It follows from (F.2) that it is optimal to let $g_t = \gamma$ if

$$F_X^\varepsilon(X_t, W_{t^-}) > c, \tag{F.14}$$

and $g_t = 0$ otherwise. Let $W^{i,\varepsilon}(X_t) = \inf\{W_{t^-} > B^\varepsilon(X_t) \mid F_W^\varepsilon(X_t, W_{t^-}) > c\}$. Note that, like in the constant returns to scale case, such a value need not exist if c is too high. In the constant returns to scale case, one has $W^{i,0}(X_t) = X_t w^i$ with $w^i < w^p$ whenever it is then strictly optimal to invest, that is if $f(w^p) + w^p > c$. In particular, it is optimal to invest at rate γ as soon as $\frac{W_{t-}}{X_t}$ exceeds w^i. Now consider an arbitrary pair (X_t, W_{t-}) such that $\frac{W_{t-}}{X_t} < \frac{W^{p,\varepsilon}(X_t)}{X_t} = w^p + O(\varepsilon)$, and, as usual, let $w_t = \frac{W_{t-}}{X_t}$. Then, if $w_t > w^i$, one has $F^0_X(X_t, W_{t-}) = f(w_t) - w_t f'(w_t) > c$. Observe that this remains true even if $w_t > w^p$, for then $F^0_X(X_t, W_{t-}) = f(w^p) + w^p > c$ as $f'(w_t) = -1$. Hence, by continuity, one can reasonably expect that, for ε small enough, (F.14) holds for any such pair; this is for instance the case if the derivative $\frac{\partial F^\varepsilon_X}{\partial \varepsilon}$ is bounded. It is then optimal to invest at rate γ at any such pair whenever ε is small enough, which generalizes Property 4 in the paper. In terms of Figure 1 in the paper, this indicates in particular that any straight line $W = Xw$ whose slope w lies strictly between w^i and w^p, and which therefore belongs to the investment region in the constant returns to scale case, also belongs to the investment region in the non constant returns to scale case for ε small enough.

Overall, the above analysis suggests that, if the mapping $(\varepsilon, X, W) \mapsto F^\varepsilon(X, W)$ is not too irregular, then the main qualitative features of the optimal contract under constant returns to scale are robust to small perturbations in the private benefit function. Thus the optimal contract under a small perturbation from constant returns to scale could be depicted on a figure similar to Figure 1 in the paper. The differences would be that the boundary of the downsizing region would be the non-linear function $\frac{B^\varepsilon(X)}{\Delta\lambda}$ of firm size X instead of the linear function Xb, and that the upper and lower boundaries of the investment and no transfers region would be the (presumably non-linear) functions $W^{p,\varepsilon}(X)$ and $W^{i,\varepsilon}(X)$ of firm size X instead of the linear functions Xw^p and Xw^i.

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