Multiscale problems for parabolic Bellman - Isaacs equations

Martino Bardi
(joint work with A. Cesaroni and L. Manca)

Department of Pure and Applied Mathematics
University of Padua, Italy

P.I.M.S. - Vancouver, July 20th - 24th, 2009

Plan

1. Recall: Controlled diffusion in a random oscillating medium
2. A different form of randomness: stochastic volatility in finance
3. Controlled diffusion with random parameters: a two-scale model
4. Averaging via Bellman-Isaacs PDEs
5. Examples from finance and marketing

Controlled diffusion in a random oscillating medium

Consider the Ito controlled stochastic differential equation

$$dx_s = f(x_s, \alpha_s)\, ds + \sigma(x_s)\, dW_s, \quad x_0 = x,$$

where W_s is a Brownian motion, and cost functional

$$J(x, \alpha) := E_x \int_0^{\tau_x} l(x_s, \alpha_s)\, ds$$

where τ_x is the exit time of x_s from a given open set Ω.
The value function $v(x) := \inf_\alpha J(x, \alpha)$ is the unique solution of the Dirichlet problem for the (degenerate) elliptic PDE

$$-\frac{1}{2}\mathrm{tr}(\sigma\sigma^T D^2 u) + \max_{a \in A}\{-f \cdot Du - l\} = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Bensoussan - Blankenship 1988 considered the model of highly oscillating random coefficients

$$dx_s = f\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s, \omega\right)\, ds + \sigma\left(\frac{x_s}{\varepsilon}, \omega\right)\, dW_s, \quad x_0 = x,$$

with cost functional

$$J(x,\alpha) := E_x \int_0^{\tau_x} l(x_s, \frac{x_s}{\varepsilon}, \alpha_s, \omega)\, ds$$

where the expectation E_x is taken w.r.t. W_s, NOT w.r.t. ω.
Then the value function $v^\varepsilon(x,\omega)$ is random and solves the stochastic PDE

$$-\frac{1}{2}\mathrm{tr}\left(\sigma\sigma^T\left(\frac{x}{\varepsilon},\omega\right)D^2u\right) + H\left(x, \frac{x}{\varepsilon}, Du, \omega\right) = 0$$

with

$$H(x, y, p, \omega) := \max_{a\in A}\{-f(x, y, a, \omega)\cdot Du - l(x, y, a, \omega)\}.$$

Bensoussan - Blankenship assume $\sigma\sigma^T > 0$ positive definite, so the Bellman equation is uniformly elliptic (and quasilinear).
They assume $\sigma\sigma^T$ and H stationary w.r.t. an ergodic group of translations, and prove that $v^\varepsilon(x,\omega) \to v(x)$ in H^1_0, where v solves

$$-\mathrm{tr}(QD^2 v) + \overline{H}(x, Dv) = 0$$

and there is a formula for the effective matrix Q and Hamiltonian $\overline{H}$.

OPEN QUESTION: is v the value function of an "effective control problem" ?

Caffarelli - Souganidis -Wang 2005 studied fully nonlinear uniformly elliptic PDEs (including general Bellman-Isaacs equations) under the same stationary- ergodic assumption.
Their effective operator has a less explicit representation (and the convergence is different), so the above question is harder in their case.

Bensoussan - Blankenship assume $\sigma\sigma^T > 0$ positive definite, so the Bellman equation is uniformly elliptic (and quasilinear).
They assume $\sigma\sigma^T$ and H stationary w.r.t. an ergodic group of translations, and prove that $v^\varepsilon(x,\omega) \to v(x)$ in H^1_0, where v solves

$$-\mathrm{tr}(QD^2 v) + \overline{H}(x, Dv) = 0$$

and there is a formula for the effective matrix Q and Hamiltonian $\overline{H}$.

OPEN QUESTION: is v the value function of an "effective control problem" ?

Caffarelli - Souganidis -Wang 2005 studied fully nonlinear uniformly elliptic PDEs (including general Bellman-Isaacs equations) under the same stationary- ergodic assumption.
Their effective operator has a less explicit representation (and the convergence is different), so the above question is harder in their case.

Bensoussan - Blankenship assume $\sigma\sigma^T > 0$ positive definite, so the Bellman equation is uniformly elliptic (and quasilinear).
They assume $\sigma\sigma^T$ and H stationary w.r.t. an ergodic group of translations, and prove that $v^\varepsilon(x,\omega) \to v(x)$ in H^1_0, where v solves

$$-\mathrm{tr}(QD^2 v) + \overline{H}(x, Dv) = 0$$

and there is a formula for the effective matrix Q and Hamiltonian $\overline{H}$.

OPEN QUESTION: is v the value function of an "effective control problem" ?

Caffarelli - Souganidis -Wang 2005 studied fully nonlinear uniformly elliptic PDEs (including general Bellman-Isaacs equations) under the same stationary- ergodic assumption.
Their effective operator has a less explicit representation (and the convergence is different), so the above question is harder in their case.

Recall of Bellman-Isaacs equations for controlled diffusions

We consider now the differential game

$$dx_s = f(x_s, \alpha_s, \beta_s)\, ds + \sigma(x_s, \alpha_s, \beta_s)\, dW_s, \quad x_0 = x,$$

where β_s is the control of a second player that wants to MAXIMIZE the cost functional

$$J(t, x, \alpha, \beta) := E_x \left[\int_0^t l(x_s, \alpha_s, \beta_s)\, ds + h(x_t) \right].$$

This includes stochastic control (if B is a singleton).
The lower and upper value functions are defined in terms of nonanticipating strategies.

The (lower) value function is the unique solution of the Cauchy problem for the (possibly degenerate) parabolic PDE

$$\frac{\partial u}{\partial t} + \min_{b \in B} \max_{a \in A} \{L^{a,b} u - l(\cdot, a, b)\} = 0$$

where $L^{a,b}$ is the generator of the diffusion process with constant controls $\alpha_s = a, \beta_s = b$:

$$L^{a,b} u := -\frac{1}{2} \text{trace}(\sigma\sigma^T D^2 u) - f \cdot Du.$$

1 player: P.-L. Lions 1983;
Comparison Principle: R. Jensen 1988, Ishii 1989
2 players: Fleming - Souganidis 1989

A different form of randomness

In practical applications one chooses a form of the system and costs depending on a vector of parameters y:

$$f = f(x, y, a, b), \quad \sigma = \sigma(x, y, a, b), \quad l = l(x, y, a, b)$$

gets some historical values $y_1, ..., y_N$ of the parameters and then estimates $\quad \phi = f, l, \sigma \quad$ by

$$\phi \approx \frac{1}{N} \sum_{i=1}^{N} \phi_i, \quad \phi_i := \phi(x, y_i, a, b),$$

the arithmetic mean of the observed data.

QUESTION: is this correct? and why?

Rmk: the data $y_1, ..., y_N$ often look like samples of a stochastic process. How can we model them?

A different form of randomness

In practical applications one chooses a form of the system and costs depending on a vector of parameters y:

$$f = f(x, y, a, b), \quad \sigma = \sigma(x, y, a, b), \quad l = l(x, y, a, b)$$

gets some historical values $y_1, ..., y_N$ of the parameters and then estimates $\quad \phi = f, l, \sigma \quad$ by

$$\phi \approx \frac{1}{N}\sum_{i=1}^{N} \phi_i, \quad \phi_i := \phi(x, y_i, a, b),$$

the arithmetic mean of the observed data.

QUESTION: is this correct? and why?

Rmk: the data $y_1, ..., y_N$ often look like samples of a stochastic process. How can we model them?

A different form of randomness

In practical applications one chooses a form of the system and costs depending on a vector of parameters y:

$$f = f(x, y, a, b), \quad \sigma = \sigma(x, y, a, b), \quad l = l(x, y, a, b)$$

gets some historical values $y_1, ..., y_N$ of the parameters and then estimates $\quad \phi = f, l, \sigma \quad$ by

$$\phi \approx \frac{1}{N} \sum_{i=1}^{N} \phi_i, \quad \phi_i := \phi(x, y_i, a, b),$$

the arithmetic mean of the observed data.

QUESTION: is this correct? and why?

Rmk: the data $y_1, ..., y_N$ often look like samples of a stochastic process. How can we model them?

Example and motivation: Financial models

The evolution of stock S is described by

$$d \log S_s = \gamma \, ds + \sigma \, dW_s$$

but the volatility σ is not really a constant, it rather looks like an ergodic stochastic process, mean-reverting and evolving on a time scale faster than the stock prices, see
Fouque, Papanicolaou, Sircar: Derivatives in financial markets with stochastic volatility, 2000.
Their model for fast stochastic volatility σ is

$$d \log S_s = \gamma \, ds + \sigma(y_s) \, dW_s$$
$$dy_s = \frac{1}{\varepsilon}(m - y_s) \, ds + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

for some $\sigma(\cdot) > 0$ and with correlated $W_.$ and $\tilde{W}_.$.

Example and motivation: Financial models

The evolution of stock S is described by

$$d \log S_s = \gamma\, ds + \sigma\, dW_s$$

but the volatility σ is not really a constant, it rather looks like an ergodic stochastic process, mean-reverting and evolving on a time scale faster than the stock prices, see
Fouque, Papanicolaou, Sircar: Derivatives in financial markets with stochastic volatility, 2000.
Their model for fast stochastic volatility σ is

$$d \log S_s = \gamma\, ds + \sigma(y_s)\, dW_s$$

$$dy_s = \frac{1}{\varepsilon}(m - y_s)\, ds + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

for some $\sigma(\cdot) > 0$ and with correlated $W_.$ and $\tilde{W}_.$.

They show empirical data supporting the theory and discuss several models , mostly for option pricing problems.

Use asymptotic expansions methods for the PDEs associated to the problems.

Most problems have NO control, so it is not hard to justify the formal calculations.

For problems with control the justification can be done by viscosity solutions of the Bellman equation.

They show empirical data supporting the theory and discuss several models , mostly for option pricing problems.

Use asymptotic expansions methods for the PDEs associated to the problems.

Most problems have NO control, so it is not hard to justify the formal calculations.

For problems with control the justification can be done by viscosity solutions of the Bellman equation.

Merton portfolio optimization problem

Invest β_s in the stock S_s, $1 - \beta_s$ in a bond with interest rate r.
Then the wealth x_s evolves as

$$d\,x_s = (r + (\gamma - r)\beta_s)x_s\,ds + x_s\beta_s\,\sigma(y_s)\,dW_s$$
$$dy_s = \frac{1}{\varepsilon}(m - y_s)\,ds + \frac{\nu}{\sqrt{\varepsilon}}d\tilde{W}_s$$

and want to maximize the expected utility at time t, $E[h(x_t)]$ for some h increasing and concave. The HJB equation is

$$\frac{\partial V^\varepsilon}{\partial t} - rxV_x^\varepsilon - \max_b \left\{(\gamma - r)bxV_x^\varepsilon + \frac{b^2x^2\sigma^2}{2}V_{xx}^\varepsilon\right\} = \frac{(m-y)V_y^\varepsilon + \nu^2 V_{yy}^\varepsilon}{\varepsilon}$$

QUESTIONS:
Is the limit as $\varepsilon \to 0$ a Merton problem with constant volatility $\overline{\sigma}$?

If so, is $\overline{\sigma}$ an average of $\sigma(\cdot)$?

Merton portfolio optimization problem

Invest β_s in the stock S_s, $1 - \beta_s$ in a bond with interest rate r.
Then the wealth x_s evolves as

$$d\,x_s = (r + (\gamma - r)\beta_s)x_s\,ds + x_s\beta_s\,\sigma(y_s)\,dW_s$$
$$dy_s = \frac{1}{\varepsilon}(m - y_s)\,ds + \frac{\nu}{\sqrt{\varepsilon}}d\tilde{W}_s$$

and want to maximize the expected utility at time t, $E[h(x_t)]$ for some h increasing and concave. The HJB equation is

$$\frac{\partial V^\varepsilon}{\partial t} - rxV_x^\varepsilon - \max_b \left\{(\gamma - r)bxV_x^\varepsilon + \frac{b^2x^2\sigma^2}{2}V_{xx}^\varepsilon\right\} = \frac{(m-y)V_y^\varepsilon + \nu^2 V_{yy}^\varepsilon}{\varepsilon}$$

QUESTIONS:
Is the limit as $\varepsilon \to 0$ a Merton problem with constant volatility $\overline{\sigma}$?

If so, is $\overline{\sigma}$ an average of $\sigma(\cdot)$?

Modelling random parameters

A process $\tilde{y}_\tau$ is ergodic with invariant measure μ if for all measurable ϕ

$$\lim_{T\to+\infty} E\left[\frac{1}{T}\int_0^T \phi(\tilde{y}_\tau)\, d\tau\right] = \int \phi(y)\, d\mu(y) =: E[\phi].$$

Define $y_t^\varepsilon := \tilde{y}_{t/\varepsilon}$. Suppose you observe y_t^ε at the times $t = i/N$, $i = 1, ..., N$. Want to estimate quantities depending on y_t^ε (e.g., the system and cost $\phi = f, \sigma, l$) by

$$\frac{1}{N}\sum_{i=1}^{N} \phi_i, \quad \phi_i := \phi(x, y_{i/N}^\varepsilon, a, b).$$

For N large and ε small, setting $\tau = t/\varepsilon$ we get

$$\frac{1}{N}\sum_{i=1}^{N} \phi_i \approx \int_0^1 \phi(y_t^\varepsilon)\, dt = \varepsilon \int_0^{1/\varepsilon} \phi(\tilde{y}_\tau)\, d\tau \approx E[\phi].$$

Modelling random parameters

A process $\tilde{y}_\tau$ is ergodic with invariant measure μ if for all measurable ϕ

$$\lim_{T\to+\infty} E\left[\frac{1}{T}\int_0^T \phi(\tilde{y}_\tau)\, d\tau\right] = \int \phi(y)\, d\mu(y) =: E[\phi].$$

Define $y_t^\varepsilon := \tilde{y}_{t/\varepsilon}$. Suppose you observe y_t^ε at the times $t = i/N$, $i = 1, ..., N$. Want to estimate quantities depending on y_t^ε (e.g., the system and cost $\phi = f, \sigma, l$) by

$$\frac{1}{N}\sum_{i=1}^{N}\phi_i, \quad \phi_i := \phi(x, y^\varepsilon_{i/N}, a, b).$$

For N large and ε small, setting $\tau = t/\varepsilon$ we get

$$\frac{1}{N}\sum_{i=1}^{N}\phi_i \approx \int_0^1 \phi(y_t^\varepsilon)\, dt = \varepsilon\int_0^{1/\varepsilon}\phi(\tilde{y}_\tau)\, d\tau \approx E[\phi].$$

Modelling random parameters

A process $\tilde{y}_\tau$ is ergodic with invariant measure μ if for all measurable ϕ

$$\lim_{T\to+\infty} E\left[\frac{1}{T}\int_0^T \phi(\tilde{y}_\tau)\, d\tau\right] = \int \phi(y)\, d\mu(y) =: E[\phi].$$

Define $y_t^\varepsilon := \tilde{y}_{t/\varepsilon}$. Suppose you observe y_t^ε at the times $t = i/N$, $i = 1, ..., N$. Want to estimate quantities depending on y_t^ε (e.g., the system and cost $\phi = f, \sigma, l$) by

$$\frac{1}{N}\sum_{i=1}^N \phi_i, \quad \phi_i := \phi(x, y_{i/N}^\varepsilon, a, b).$$

For N large and ε small, setting $\tau = t/\varepsilon$ we get

$$\frac{1}{N}\sum_{i=1}^N \phi_i \approx \int_0^1 \phi(y_t^\varepsilon)\, dt = \varepsilon \int_0^{1/\varepsilon} \phi(\tilde{y}_\tau)\, d\tau \approx E[\phi].$$

Conclusion:
The arithmetic mean of data is a good approximation of a function of the random parameters if

- there are many data, and
- the parameters are an ergodic process evolving on a time scale much faster than the state variables.

QUESTION:
What are the right quantities to average?

The system data f, σ and cost l themselves or something else?

Conclusion:
The arithmetic mean of data is a good approximation of a function of the random parameters if

- there are many data, and
- the parameters are an ergodic process evolving on a time scale much faster than the state variables.

QUESTION:
What are the right quantities to average?

The system data f, σ and cost l themselves or something else?

Two-scale model of DGs with random parameters

If $\tilde{y}_\tau$ solves

(FS) $$d\tilde{y}_\tau = g(\tilde{y}_\tau)\,d\tau + \nu(\tilde{y}_\tau)\,dW_\tau,$$

and $y_s = \tilde{y}_{s/\varepsilon}$, we get the two-scale system

(2SS) $$\begin{aligned} dx_s &= f(x_s, y_s, \alpha_s, \beta_s)\,ds + \sigma(x_s, y_s, \alpha_s, \beta_s)\,dW_s \qquad x_s \in \mathbf{R}^n, \\ dy_s &= \frac{1}{\varepsilon} g(y_s)\,ds + \frac{1}{\sqrt{\varepsilon}}\nu(y_s)\,dW_s, \qquad y_s \in \mathbf{R}^m, \end{aligned}$$

Want to understand the limit as $\varepsilon \to 0$:

a Singular Perturbation problem.

Main assumption: some form of ergodicity of the fast subsystem (FS).

The PDE formulation

V^ε solves

$$\begin{cases} \frac{\partial V^\varepsilon}{\partial t} + \mathcal{H}\left(x, y, D_x V^\varepsilon, D^2_{xx} V^\varepsilon, \frac{1}{\sqrt{\varepsilon}} D^2_{xy} V^\varepsilon\right) - \frac{1}{\varepsilon}\mathcal{L} V^\varepsilon = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^m \\ V^\varepsilon(0, x, y) = h(x, y) \qquad \text{in } \mathbf{R}^n \times \mathbf{R}^m, \end{cases}$$

$$\mathcal{H}(x, y, p, X, Z) := \min_{b \in B} \max_{a \in A} \left\{ -\text{tr}(\sigma\sigma^T X) - f \cdot p - l - \text{tr}(\sigma\nu Z^T) \right\}$$

$$\mathcal{L} := \text{trace}(\nu\nu^T D^2_{yy}) + g \cdot D_y.$$

It is a Singular Perturbation or Penalization problem for the B-I PDE. Since all the derivatives w.r.t. y are penalized we expect

- $V^\varepsilon(t, x, y) \to V(t, x)$
- the limit V satisfy a PDE in lower dimension n instead of $n + m$.

The PDE formulation

V^ε solves

$$\begin{cases} \frac{\partial V^\varepsilon}{\partial t} + \mathcal{H}\left(x, y, D_x V^\varepsilon, D^2_{xx} V^\varepsilon, \frac{1}{\sqrt{\varepsilon}} D^2_{xy} V^\varepsilon\right) - \frac{1}{\varepsilon}\mathcal{L} V^\varepsilon = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^m \\ V^\varepsilon(0, x, y) = h(x, y) \qquad \text{in } \mathbf{R}^n \times \mathbf{R}^m, \end{cases}$$

$$\mathcal{H}(x, y, p, X, Z) := \min_{b \in B} \max_{a \in A} \left\{ -\mathrm{tr}(\sigma\sigma^T X) - f \cdot p - l - \mathrm{tr}(\sigma\nu Z^T) \right\}$$

$$\mathcal{L} := \mathrm{trace}(\nu\nu^T D^2_{yy}) + g \cdot D_y .$$

It is a Singular Perturbation or Penalization problem for the B-I PDE. Since all the derivatives w.r.t. y are penalized we expect

- $V^\varepsilon(t, x, y) \to V(t, x)$
- the limit V satisfy a PDE in lower dimension n instead of $n + m$.

Linear averaging of the data

Assume (FS) is ergodic with invariant measure μ.
Denote with $\langle \phi \rangle := \int \phi(y) d\mu(y)$.

Theorem [Kushner, book 1990]

If there is only one player (B = singleton), the system has $\sigma = \sigma(x, y)$ possibly degenerate but independent of the control, and

$$f(x, y, a) = f_0(x, y) + f_1(x, a), \quad l(x, y, a) = l_0(x, y) + l_1(x, a), \quad h = h(x)$$

then the linear averaging of the data is the correct limit, i.e.,

$$\lim_{\varepsilon \to 0} V^\varepsilon(t, x, y) = V(t, x) := \inf_{\alpha.} E \left[\int_0^t \langle l \rangle (x_s, \alpha_s) \, ds + h(x_t) \right],$$

$$dx_s = \langle f \rangle (x_s, \alpha_s) \, ds + \langle \sigma\sigma^T \rangle^{1/2} (x_s) \, dW_s$$

Remarks:

- Kushner proof is by probabilistic methods, hard to extend to differential games;
- Merton problem is not covered because the system has the term $x_s \beta_s \sigma(y_s)\, dW_s$, where β_s is the control;
- the splitting assumption $f = f_0(x, y) + f_1(x, a)$ is also not satisfied in some models (see later);
- the results says that the limit "effective" PDE is

$$\frac{\partial V}{\partial t} - \text{trace}(\langle \sigma\sigma^T \rangle D^2_{xx} V) + \max_{a \in A} \{-\langle f \rangle \cdot D_x V - \langle l \rangle\} = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n$$

Remarks:

- Kushner proof is by probabilistic methods, hard to extend to differential games;
- Merton problem is not covered because the system has the term $x_s \beta_s \sigma(y_s)\, dW_s$, where β_s is the control;
- the splitting assumption $f = f_0(x, y) + f_1(x, a)$ is also not satisfied in some models (see later);
- the results says that the limit "effective" PDE is

$$\frac{\partial V}{\partial t} - \text{trace}(\langle \sigma \sigma^T \rangle D^2_{xx} V) + \max_{a \in A} \{ -\langle f \rangle \cdot D_x V - \langle l \rangle \} = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n$$

Remarks:

- Kushner proof is by probabilistic methods, hard to extend to differential games;
- Merton problem is not covered because the system has the term $x_s \beta_s \sigma(y_s)\, dW_s$, where β_s is the control;
- the splitting assumption $f = f_0(x, y) + f_1(x, a)$ is also not satisfied in some models (see later);
- the results says that the limit "effective" PDE is

$$\frac{\partial V}{\partial t} - \text{trace}(\langle \sigma \sigma^T \rangle D^2_{xx} V) + \max_{a \in A} \{ -\langle f \rangle \cdot D_x V - \langle l \rangle \} = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n$$

Remarks:

- Kushner proof is by probabilistic methods, hard to extend to differential games;
- Merton problem is not covered because the system has the term $x_s\beta_s\sigma(y_s)\,dW_s$, where β_s is the control;
- the splitting assumption $f = f_0(x, y) + f_1(x, a)$ is also not satisfied in some models (see later);
- the results says that the limit "effective" PDE is

$$\frac{\partial V}{\partial t} - \text{trace}(\langle \sigma\sigma^T \rangle D^2_{xx} V) + \max_{a \in A} \{-\langle f \rangle \cdot D_x V - \langle l \rangle\} = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n$$

The PDE approach

$$\begin{cases} \frac{\partial V^\varepsilon}{\partial t} + \mathcal{H}\left(x, y, D_x V^\varepsilon, D^2_{xx} V^\varepsilon, \frac{1}{\sqrt{\varepsilon}} D^2_{xy} V^\varepsilon\right) - \frac{1}{\varepsilon}\mathcal{L} V^\varepsilon = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^m \\ V^\varepsilon(0, x, y) = h(x, y) \qquad \text{in } \mathbf{R}^n \times \mathbf{R}^m, \end{cases}$$

1. Look for effective $\overline{H}$ and $\overline{h}$ such that the candidate limit problem is

$$\frac{\partial V}{\partial t} + \overline{H}\left(x, D_x V, D^2_{xx} V\right) = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n, \qquad V(0, x) = \overline{h}(x)$$

2. Prove the convergence $V^\varepsilon(t, x, y) \to V(t, x)$ solution of the effective Cauchy problem.

3. Interpret the limit PDE as a Bellman-Isaacs equation and find a limiting effective control-game problem.

The PDE approach

$$\begin{cases} \frac{\partial V^\varepsilon}{\partial t} + \mathcal{H}\left(x, y, D_x V^\varepsilon, D^2_{xx} V^\varepsilon, \frac{1}{\sqrt{\varepsilon}} D^2_{xy} V^\varepsilon\right) - \frac{1}{\varepsilon}\mathcal{L} V^\varepsilon = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^m \\ V^\varepsilon(0, x, y) = h(x, y) \qquad \text{in } \mathbf{R}^n \times \mathbf{R}^m, \end{cases}$$

1. Look for effective $\overline{H}$ and $\overline{h}$ such that the candidate limit problem is

$$\frac{\partial V}{\partial t} + \overline{H}\left(x, D_x V, D^2_{xx} V\right) = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}^n, \qquad V(0, x) = \overline{h}(x)$$

2. Prove the convergence $V^\varepsilon(t, x, y) \to V(t, x)$ solution of the effective Cauchy problem.

3. Interpret the limit PDE as a Bellman-Isaacs equation and find a limiting effective control-game problem.

Theorem [periodic case, O. Alvarez - M.B. 2007, Mem. A.M.S. 2010]

In the general DG model assume all data are $\mathbb{Z}^m$-periodic in y and $\nu\nu^T(y) > 0$.

Then the fast subsystem (FS) has a unique invariant measure μ and

$$V^\varepsilon(t,x,y) \to V(t,x) \quad \text{as } \varepsilon \to 0, \quad \text{locally uniformly}$$

and V is the unique solution of

$$\begin{cases} \frac{\partial V}{\partial t} + \int \mathcal{H}\left(x, y, D_x V, D^2_{xx} V, 0\right) \, d\mu(y) = 0 & \text{in } \mathbf{R}_+ \times \mathbf{R}^n \\ V(0,x) = \int h(x,y) \, d\mu(y) \end{cases}$$

Note the very simple formulas

$$\overline{\mathcal{H}}(x,p,X) = \langle \mathcal{H}(x,\cdot,p,X,0) \rangle, \qquad \overline{h}(x) = \langle h(x,\cdot) \rangle.$$

Main theorem: unbounded fast variables

To fit the financial models we assume

$$|f(x,y,a,b)|, |\sigma(x,y,a,b)| \leq C|x|$$

so $\mathbf{R}_+{}^n$ is invariant for x_s. Assume also

$$|g(y)|, |\nu(y)| \leq C(1+|y|).$$

Main assumption on (FS)

$\nu\nu^T(y) > 0$ and there exists $w \in C(\mathbf{R}^d)$, $k > 0$, $R_0 > 0$:

(L) $$-\mathcal{L}w \geq k \quad \forall\, |y| > R_0, \qquad \lim_{|y|\to+\infty} w(y) = +\infty$$

Proposition

(L) $\Longrightarrow$ $\exists\,!$ invariant measure μ for (FS).

Example 1. The Ornstein-Uhlenbeck process

$$d\tilde{y}_\tau = (m - \tilde{y}_\tau)\, d\tau + \sqrt{2}\,\nu\, dW_\tau$$

(m, ν constant) satisfies (L) with $w(y) = |y|^2$ and $\mu \sim \mathcal{N}(m, \nu\nu^T)$ is Gaussian.

It is also mean-reverting, i.e., the drift pulls the process back to its mean value m.

Example 2. $\nu(y)$ bounded and $\lim_{|y|\to\infty} g(y) \cdot y = -\infty \implies$ (L) "if $\tilde{y}_\tau$ gets very large the drift of (FS) pulls it back".

Example 3. $\limsup_{|y|\to\infty} \left[g(y) \cdot y - \operatorname{tr} \nu\nu^T(y)\right] < 0 \implies$ (L).

Remark. The proof relies on results by Hasminskii 1980.
P.L. Lions - Musiela (2002 unpublished) say that (L) is essentially equivalent to the ergodicity of (FS).

Example 1. The Ornstein-Uhlenbeck process

$$d\tilde{y}_\tau = (m - \tilde{y}_\tau)\, d\tau + \sqrt{2}\, \nu\, dW_\tau$$

(m, ν constant) satisfies (L) with $w(y) = |y|^2$ and $\mu \sim \mathcal{N}(m, \nu\nu^T)$ is Gaussian.

It is also mean-reverting, i.e., the drift pulls the process back to its mean value m.

Example 2. $\nu(y)$ bounded and $\lim_{|y|\to\infty} g(y) \cdot y = -\infty \implies$ (L)
"if $\tilde{y}_\tau$ gets very large the drift of (FS) pulls it back".

Example 3. $\limsup_{|y|\to\infty} \left[g(y) \cdot y - \mathrm{tr}\, \nu\nu^T(y)\right] < 0 \implies$ (L).

Remark. The proof relies on results by Hasminskii 1980.
P.L. Lions - Musiela (2002 unpublished) say that (L) is essentially equivalent to the ergodicity of (FS).

Example 1. The Ornstein-Uhlenbeck process

$$d\tilde{y}_\tau = (m - \tilde{y}_\tau)\, d\tau + \sqrt{2}\, \nu\, dW_\tau$$

(m, ν constant) satisfies (L) with $w(y) = |y|^2$ and $\mu \sim \mathcal{N}(m, \nu\nu^T)$ is Gaussian.

It is also mean-reverting, i.e., the drift pulls the process back to its mean value m.

Example 2. $\nu(y)$ bounded and $\lim_{|y|\to\infty} g(y) \cdot y = -\infty \implies$ (L)
"if $\tilde{y}_\tau$ gets very large the drift of (FS) pulls it back".

Example 3. $\limsup_{|y|\to\infty} \left[g(y) \cdot y - \mathrm{tr}\, \nu\nu^T(y)\right] < 0 \implies$ (L).

Remark. The proof relies on results by Hasminskii 1980.
P.L. Lions - Musiela (2002 unpublished) say that (L) is essentially equivalent to the ergodicity of (FS).

Example 1. The Ornstein-Uhlenbeck process

$$d\tilde{y}_\tau = (m - \tilde{y}_\tau)\, d\tau + \sqrt{2}\, \nu\, dW_\tau$$

(m, ν constant) satisfies (L) with $w(y) = |y|^2$ and $\mu \sim \mathcal{N}(m, \nu\nu^T)$ is Gaussian.

It is also mean-reverting, i.e., the drift pulls the process back to its mean value m.

Example 2. $\nu(y)$ bounded and $\lim_{|y|\to\infty} g(y) \cdot y = -\infty \implies$ (L) "if $\tilde{y}_\tau$ gets very large the drift of (FS) pulls it back".

Example 3. $\limsup_{|y|\to\infty} \left[g(y) \cdot y - \operatorname{tr} \nu\nu^T(y)\right] < 0 \implies$ (L).

Remark. The proof relies on results by Hasminskii 1980.
P.L. Lions - Musiela (2002 unpublished) say that (L) is essentially equivalent to the ergodicity of (FS).

Theorem [unbounded case, M.B., Cesaroni, Manca 2009]

Under the assumptions above

$$V^{\varepsilon}(t,x,y) \to V(t,x) \quad \text{as } \varepsilon \to 0, \quad \text{locally uniformly}$$

and V is the unique solution of

$$\begin{cases} \frac{\partial V}{\partial t} + \int \mathcal{H}\left(x, y, D_x V, D_{xx}^2 V, 0\right) \, d\mu(y) = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+{}^n \\ V(0,x) = \int h(x,y) \, d\mu(y) \end{cases}$$

The conclusion is the same as in the periodic case.
Here μ is the unique invariant measure of the fast subsystem (FS) implied by (L).

Ingredients of the proof:

- Liouville property: (L) $\Longrightarrow$ any bounded subsolution of $-\mathcal{L}u = 0$ is constant.

- There exist (smooth) approximate correctors w_δ

$$\delta w_\delta - \mathcal{L} w_\delta + \mathcal{H}(x, y, p, X, 0) = 0, \quad |\delta w_\delta(y)| \le C(1 + |y|^2) \quad \text{in } \mathbf{R}^m,$$

$$\lim_{\delta \to 0} -\delta w_\delta(y) = \langle \mathcal{H}(x, \cdot, p, X, 0) \rangle =: \overline{\mathcal{H}}(x, p, X)$$

- Relaxed semilimits $\overline{V}$ and $\underline{V}$ are idependent of y and sub- and supersolution of the effective PDE by adapting L.C. Evans' Perturbed Test Function Method (as in homogenization);

- $\underline{V} \le \langle h(x, \cdot) \rangle \le \overline{V}$ at $t = 0$ by adapting the method of M.B. - O. Alvarez ARMA 2003 for the periodic case;

- Comparison Principle for the effective Cauchy problem $\Longrightarrow$ $\overline{V} \le \underline{V}$ and therefore the convergence is locally uniform.

Ingredients of the proof:

- Liouville property: (L) $\implies$ any bounded subsolution of $-\mathcal{L}u = 0$ is constant.

- There exist (smooth) approximate correctors w_δ

$$\delta w_\delta - \mathcal{L} w_\delta + \mathcal{H}(x, y, p, X, 0) = 0, \quad |\delta w_\delta(y)| \le C(1 + |y|^2) \quad \text{in } \mathbf{R}^m,$$

$$\lim_{\delta \to 0} -\delta w_\delta(y) = \langle \mathcal{H}(x, \cdot, p, X, 0) \rangle =: \overline{\mathcal{H}}(x, p, X)$$

- Relaxed semilimits $\overline{V}$ and $\underline{V}$ are idependent of y and sub- and supersolution of the effective PDE by adapting L.C. Evans' Perturbed Test Function Method (as in homogenization);

- $\underline{V} \le \langle h(x, \cdot) \rangle \le \overline{V}$ at $t = 0$ by adapting the method of M.B. - O. Alvarez ARMA 2003 for the periodic case;

- Comparison Principle for the effective Cauchy problem $\implies$ $\overline{V} \le \underline{V}$ and therefore the convergence is locally uniform.

Ingredients of the proof:

- Liouville property: (L) $\Longrightarrow$ any bounded subsolution of $-\mathcal{L}u = 0$ is constant.
- There exist (smooth) approximate correctors w_δ

$$\delta w_\delta - \mathcal{L} w_\delta + \mathcal{H}(x, y, p, X, 0) = 0, \quad |\delta w_\delta(y)| \leq C(1 + |y|^2) \quad \text{in } \mathbf{R}^m,$$

$$\lim_{\delta \to 0} -\delta w_\delta(y) = \langle \mathcal{H}(x, \cdot, p, X, 0) \rangle =: \overline{\mathcal{H}}(x, p, X)$$

- Relaxed semilimits $\overline{V}$ and $\underline{V}$ are idependent of y and sub- and supersolution of the effective PDE by adapting L.C. Evans' Perturbed Test Function Method (as in homogenization);
- $\underline{V} \leq \langle h(x, \cdot) \rangle \leq \overline{V}$ at $t = 0$ by adapting the method of M.B. - O. Alvarez ARMA 2003 for the periodic case;
- Comparison Principle for the effective Cauchy problem $\Longrightarrow$ $\overline{V} \leq \underline{V}$ and therefore the convergence is locally uniform.

Ingredients of the proof:

- Liouville property: (L) $\Longrightarrow$ any bounded subsolution of $-\mathcal{L}u = 0$ is constant.

- There exist (smooth) approximate correctors w_δ

$$\delta w_\delta - \mathcal{L} w_\delta + \mathcal{H}(x, y, p, X, 0) = 0, \quad |\delta w_\delta(y)| \le C(1 + |y|^2) \quad \text{in } \mathbf{R}^m,$$

$$\lim_{\delta \to 0} -\delta w_\delta(y) = \langle \mathcal{H}(x, \cdot, p, X, 0) \rangle =: \overline{\mathcal{H}}(x, p, X)$$

- Relaxed semilimits $\overline{V}$ and $\underline{V}$ are idependent of y and sub- and supersolution of the effective PDE by adapting L.C. Evans' Perturbed Test Function Method (as in homogenization);

- $\underline{V} \le \langle h(x, \cdot) \rangle \le \overline{V}$ at $t = 0$ by adapting the method of M.B. - O. Alvarez ARMA 2003 for the periodic case;

- Comparison Principle for the effective Cauchy problem $\Longrightarrow$ $\overline{V} \le \underline{V}$ and therefore the convergence is locally uniform.

Ingredients of the proof:

- Liouville property: (L) $\implies$ any bounded subsolution of $-\mathcal{L}u = 0$ is constant.

- There exist (smooth) approximate correctors w_δ

$$\delta w_\delta - \mathcal{L} w_\delta + \mathcal{H}(x, y, p, X, 0) = 0, \quad |\delta w_\delta(y)| \le C(1 + |y|^2) \quad \text{in } \mathbf{R}^m,$$

$$\lim_{\delta \to 0} -\delta w_\delta(y) = \langle \mathcal{H}(x, \cdot, p, X, 0) \rangle =: \overline{\mathcal{H}}(x, p, X)$$

- Relaxed semilimits $\overline{V}$ and $\underline{V}$ are idependent of y and sub- and supersolution of the effective PDE by adapting L.C. Evans' Perturbed Test Function Method (as in homogenization);

- $\underline{V} \le \langle h(x, \cdot) \rangle \le \overline{V}$ at $t = 0$ by adapting the method of M.B. - O. Alvarez ARMA 2003 for the periodic case;

- Comparison Principle for the effective Cauchy problem $\implies$ $\overline{V} \le \underline{V}$ and therefore the convergence is locally uniform.

The Theorem settles steps 1 and 2.
Step 3: find the effective control-game problem.

Corollary [extends Kushner to games and $h = h(x, y)$]

For split systems, i.e.,

$$\sigma = \sigma(x, y), \quad f = f_0(x, y) + f_1(x, a, b), \quad l = l_0(x, y) + l_1(x, a, b),$$

the linear averaging of the data is the correct limit, i.e.,

$$\lim_{\varepsilon \to 0} V^\varepsilon(t, x, y) = V(t, x) := \inf_{\alpha[\cdot]} \sup_{\beta.} E\left[\int_0^t \langle l \rangle (x_s, \alpha[\beta]_s, \beta_s)\, ds + \langle h \rangle (x_t)\right],$$

$$dx_s = \langle f \rangle (x_s, \alpha[\beta]_s, \beta_s)\, ds + \langle \sigma \sigma^T \rangle^{1/2} (x_s)\, dW_s$$

Proof: under these assumptions $\int d\mu$ and $\min_{b \in B} \max_{a \in A}$ commute

$$\overline{\mathcal{H}} = \int \min_{b \in B} \max_{a \in A} \{\ldots\}\, d\mu(y) = \min_{b \in B} \max_{a \in A} \int \{\ldots\}\, d\mu(y).$$

When can we find an effective control problem?

There always exist sets A', B', control system $\bar{f}, \bar{\sigma}$, and cost $\bar{l}$ such that

$$\overline{\mathcal{H}} := \int \min_{b \in B} \max_{a \in A} \left\{ -\text{trace}(\sigma\sigma^T D^2_{xx}) - f \cdot D_x - l \right\} d\mu(y)$$

$$= \min_{b' \in B'} \max_{a' \in A'} \left\{ -\text{trace}(\overline{\sigma\sigma}^T D^2_{xx}) - \bar{f} \cdot D_x - \bar{l} \right\}.$$

$$\Longrightarrow V(t,x) := \inf_\alpha \sup_\beta E \left[\int_0^t \bar{l}(x_s, \alpha[\beta]_s, \beta_s) ds + \langle h \rangle (x_t) \right], \ x_s \text{ solving}$$

$$dx_s = \bar{f}(x_s, \alpha[\beta]_s, \beta_s) ds + \bar{\sigma}(x_s, \alpha[\beta]_s, \beta_s) dW_s$$

This can be called an effective control problem - differential game, but it is neither unique nor explicitly related to the original data.
In some cases we can write an explicit formula for it.

When can we find an effective control problem?

There always exist sets A', B', control system $\bar{f}, \overline{\sigma}$, and cost $\bar{l}$ such that

$$\overline{\mathcal{H}} := \int \min_{b\in B} \max_{a\in A} \left\{ -\text{trace}(\sigma\sigma^T D^2_{xx}) - f \cdot D_x - l \right\} d\mu(y)$$

$$= \min_{b'\in B'} \max_{a'\in A'} \left\{ -\text{trace}(\overline{\sigma\sigma}^T D^2_{xx}) - \bar{f} \cdot D_x - \bar{l} \right\}.$$

$\Longrightarrow V(t,x) := \inf_\alpha \sup_\beta E\left[\int_0^t \bar{l}(x_s, \alpha[\beta]_s, \beta_s)ds + \langle h\rangle(x_t)\right]$, x_s solving

$$dx_s = \bar{f}(x_s, \alpha[\beta]_s, \beta_s)ds + \overline{\sigma}(x_s, \alpha[\beta]_s, \beta_s)dW_s$$

This can be called an effective control problem - differential game, but it is neither unique nor explicitly related to the original data.
In some cases we can write an explicit formula for it.

When can we find an effective control problem?

There always exist sets A', B', control system $\bar{f}, \bar{\sigma}$, and cost $\bar{l}$ such that

$$\overline{\mathcal{H}} := \int \min_{b\in B} \max_{a\in A} \left\{ -\text{trace}(\sigma\sigma^T D^2_{xx}) - f \cdot D_x - l \right\} d\mu(y)$$

$$= \min_{b'\in B'} \max_{a'\in A'} \left\{ -\text{trace}(\overline{\sigma\sigma}^T D^2_{xx}) - \bar{f} \cdot D_x - \bar{l} \right\}.$$

$$\Longrightarrow V(t,x) := \inf_\alpha \sup_\beta E\left[\int_0^t \bar{l}(x_s, \alpha[\beta]_s, \beta_s) ds + \langle h \rangle (x_t) \right], \quad x_s \text{ solving}$$

$$dx_s = \bar{f}(x_s, \alpha[\beta]_s, \beta_s) ds + \bar{\sigma}(x_s, \alpha[\beta]_s, \beta_s) dW_s$$

This can be called an effective control problem - differential game, but it is neither unique nor explicitly related to the original data.
In some cases we can write an explicit formula for it.

Merton problem with stochastic volatility

Maximize $E[h(x_t)]$ for the system in $\mathbf{R}^2$

$$dx_s = (r + (\gamma - r)\beta_s)x_s\, ds + x_s\beta_s\, \sigma(y_s)\, dW_s$$

$$dy_s = \frac{1}{\varepsilon}(m - y_s)\, ds + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

with $\gamma > r$, $\sigma > 0$, $\beta_s \in [0, \infty)$,
and W_s, $\tilde{W}_s$ possibly correlated scalar Wiener processes.

Assume the utility h has $h' > 0$ and $h'' < 0$.
Then expect a value function strictly increasing and concave in x, i.e., $V^\varepsilon_x > 0$, $V^\varepsilon_{xx} < 0$. The HJB equation becomes

$$\frac{\partial V^\varepsilon}{\partial t} - rxV^\varepsilon_x + \frac{[(\gamma - r)V^\varepsilon_x]^2}{\sigma^2(y)2V^\varepsilon_{xx}} = \frac{1}{\varepsilon}\left[(m - y)V^\varepsilon_y + \nu^2 V^\varepsilon_{yy}\right] \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+$$

Merton problem with stochastic volatility

Maximize $E[h(x_t)]$ for the system in $\mathbf{R}^2$

$$dx_s = (r + (\gamma - r)\beta_s)x_s\,ds + x_s\beta_s\,\sigma(y_s)\,dW_s$$

$$dy_s = \frac{1}{\varepsilon}(m - y_s)\,ds + \frac{\nu}{\sqrt{\varepsilon}}d\tilde{W}_s$$

with $\gamma > r$, $\sigma > 0$, $\beta_s \in [0, \infty)$,
and W_s, $\tilde{W}_s$ possibly correlated scalar Wiener processes.

Assume the utility h has $h' > 0$ and $h'' < 0$.
Then expect a value function strictly increasing and concave in x, i.e., $V^\varepsilon_x > 0$, $V^\varepsilon_{xx} < 0$. The HJB equation becomes

$$\frac{\partial V^\varepsilon}{\partial t} - rxV^\varepsilon_x + \frac{[(\gamma - r)V^\varepsilon_x]^2}{\sigma^2(y)2V^\varepsilon_{xx}} = \frac{1}{\varepsilon}\left[(m - y)V^\varepsilon_y + \nu^2 V^\varepsilon_{yy}\right] \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+$$

Merton problem with stochastic volatility

Maximize $E[h(x_t)]$ for the system in $\mathbf{R}^2$

$$dx_s = (r + (\gamma - r)\beta_s)x_s\, ds + x_s\beta_s\, \sigma(y_s)\, dW_s$$
$$dy_s = \frac{1}{\varepsilon}(m - y_s)\, ds + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

with $\gamma > r$, $\sigma > 0$, $\beta_s \in [0, \infty)$,
and W_s, $\tilde{W}_s$ possibly correlated scalar Wiener processes.

Assume the utility h has $h' > 0$ and $h'' < 0$.
Then expect a value function strictly increasing and concave in x, i.e., $V_x^\varepsilon > 0$, $V_{xx}^\varepsilon < 0$. The HJB equation becomes

$$\frac{\partial V^\varepsilon}{\partial t} - rxV_x^\varepsilon + \frac{[(\gamma - r)V_x^\varepsilon]^2}{\sigma^2(y)2V_{xx}^\varepsilon} = \frac{1}{\varepsilon}\left[(m - y)V_y^\varepsilon + \nu^2 V_{yy}^\varepsilon\right] \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+$$

By the Theorem, $V^\varepsilon(t,x,y) \to V(t,x)$ as $\varepsilon \to 0$ and V solves

$$\frac{\partial V}{\partial t} - rxV_x + \frac{(\gamma - r)^2 V_x^2}{2V_{xx}} \int \frac{1}{\sigma^2(y)} d\mu(y) = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+$$

So the limit problem is a Merton problem with constant volatility

$$\overline{\sigma} := \left(\int \frac{1}{\sigma^2(y)} \, d\mu(y) \right)^{-1/2}$$

a harmonic average of σ, NOT the linear average!

So if I have N empirical data $\sigma_1, ..., \sigma_N$ of the volatility, in the Black-Scholes formula for option pricing (linear PDE!) I use the arithmetic mean

$$\sigma_a^2 = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2$$

whereas in the Merton problem I use the harmonic mean

$$\sigma_h^2 = \left(\frac{1}{N} \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \right)^{-1} \leq \sigma_a^2.$$

By the Theorem, $V^\varepsilon(t,x,y) \to V(t,x)$ as $\varepsilon \to 0$ and V solves

$$\frac{\partial V}{\partial t} - rxV_x + \frac{(\gamma - r)^2 V_x^2}{2V_{xx}} \int \frac{1}{\sigma^2(y)} d\mu(y) = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}_+$$

So the limit problem is a Merton problem with constant volatility

$$\overline{\sigma} := \left(\int \frac{1}{\sigma^2(y)} \, d\mu(y) \right)^{-1/2}$$

a harmonic average of σ, NOT the linear average!

So if I have N empirical data $\sigma_1, ..., \sigma_N$ of the volatility, in the Black-Scholes formula for option pricing (linear PDE!) I use the arithmetic mean

$$\sigma_a^2 = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2$$

whereas in the Merton problem I use the harmonic mean

$$\sigma_h^2 = \left(\frac{1}{N} \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \right)^{-1} \leq \sigma_a^2.$$

A model in marketing with random parameters

Consider a duopoly: in a market with total sales M the sales of firm 1 are S_s, those of firm 2 are $M - S_s$, and $\alpha_s, \beta_s \geq 0$ are the advertising efforts. Take Lanchester dynamics

$$\dot{S}_s = (M - S_s)\alpha_s - \beta_s S_s$$

and objective functional

$$J = \int_0^t \left(rS_s + \theta {\alpha_s}^2 - {\beta_s}^2 \right) ds,$$

with $\theta > 0$, see Jorgensen and Zaccour, book 2004. If the parameters M, r, θ depend on a O-U process the system becomes

$$\dot{S}_s = (M(y_s) - S_s)\alpha_s - \beta_s S_s, \quad S_0 = x,$$
$$dy_s = \tfrac{1}{\varepsilon}(m - y_s)\, ds + \tfrac{\nu}{\sqrt{\varepsilon}} dW_s, \quad y_0 = y,$$

not split because of the term $M(y_s)\alpha_s$.

The objective functional becomes

$$J^\varepsilon = E\left[\int_0^t \left(r(y_s)S_s + \theta(y_s)\alpha_s^2 - \beta_s^2\right) ds\right]$$

By the Theorem, $V^\varepsilon(t,x,y) \to V(t,x)$ as $\varepsilon \to 0$ and V solves

$$\frac{\partial V}{\partial t} - \int \left(r(y)x + (M(y) - x)^2 \frac{V_x^2}{4} - \frac{x^2 V_x^2}{4\theta(y)}\right) d\mu(y) = 0 \quad \text{in } \mathbf{R}_+ \times \mathbf{R}$$

This is the Isaacs PDE for the game with system

$$\dot{S}_s = \sqrt{\langle M^2\rangle - 2\langle M\rangle S_s + S_s^2}\,\alpha_s - \beta_s S_s$$

that is NOT a Lanchester dynamics (i.e., affine in the state), and objective functional

$$J = \int_0^t \left(\langle r\rangle S_s + \langle \frac{1}{\theta}\rangle^{-1}\alpha_s^2 - \beta_s^2\right) ds$$

that is still linear in state and quadratic in the controls but with different averages of the parameters.

Conclusions

In control and game problems with random parameters driven by a fast ergodic process the limit effective problem can be

1. of the same form and with parameters the historical mean of the random ones (as in uncontrolled problems!)
2. of the same form, but the parameters are obtained by a different averaging of the random ones (as in Merton)
3. of a form different from the original problem (as in the advertising game).

The formula for the effective Hamiltonian is very simple, but there is no general recipe for deducing from it an explicit effective control problem.

Conclusions

In control and game problems with random parameters driven by a fast ergodic process the limit effective problem can be

1. of the same form and with parameters the historical mean of the random ones (as in uncontrolled problems!)
2. of the same form, but the parameters are obtained by a different averaging of the random ones (as in Merton)
3. of a form different from the original problem (as in the advertising game).

The formula for the effective Hamiltonian is very simple, but there is no general recipe for deducing from it an explicit effective control problem.

Conclusions

In control and game problems with random parameters driven by a fast ergodic process the limit effective problem can be

1. of the same form and with parameters the historical mean of the random ones (as in uncontrolled problems!)
2. of the same form, but the parameters are obtained by a different averaging of the random ones (as in Merton)
3. of a form different from the original problem (as in the advertising game).

The formula for the effective Hamiltonian is very simple, but there is no general recipe for deducing from it an explicit effective control problem.

Conclusions

In control and game problems with random parameters driven by a fast ergodic process the limit effective problem can be

1. of the same form and with parameters the historical mean of the random ones (as in uncontrolled problems!)
2. of the same form, but the parameters are obtained by a different averaging of the random ones (as in Merton)
3. of a form different from the original problem (as in the advertising game).

The formula for the effective Hamiltonian is very simple, but there is no general recipe for deducing from it an explicit effective control problem.