Fibre bundle structures of Schubert varieties

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Schubert varieties

- Let G be a semi-simple Lie group over $\mathbb{C}$.

  Fix $T \subseteq B \subseteq G$ a maximal torus and Borel subgroup of G.

  Let $P \subseteq G$ be a parabolic subgroup containing B.

- Let $W := N(T)/T$ denote the Weyl group of G.

  Let $W_P \subseteq W$ be the Weyl group of P.

- Let G/P be the partial flag variety.

  For any $w \in W^P \simeq W/W_P$ (min length coset rep), we have the Schubert variety

$$X_w^P := \overline{BwP}/P \subseteq G/P.$$

Schubert varieties: type A

- $G = SL(\mathbb{C}^n)$ with $\mathbb{C}^n = \mathrm{Span}_{\mathbb{C}}\{e_1, \ldots, e_n\}$.

  T=diagonal matrices, B=upper triangular matrices.

  $W = S_n$ permutation matrices.

- G/B is the full flag variety

$$\{V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim(V_i) = i\}.$$

- For any permutation matrix $w \in W$, we have the Schubert variety

$$X_w^B = \{V_\bullet \in G/B \mid \dim(V_i \cap E_j) \geq \mathrm{rk}(w[i,j])\}$$

  where $E_j = \mathrm{Span}\{e_1, \ldots, e_j\}$.

Schubert varieties: projection maps

- Let Q be parabolic subgroup containing P and consider the projection

$$G/P \to G/Q$$

with fibre isomorphic to Q/P.

- For $w \in W^P$, there is a unique parabolic decomposition $w = vu$ where $v \in W^Q$, $u \in W^P \cap W_Q$ and induced projection

$$X_w^P \to X_v^Q$$

with generic fibre isomorphic to X_u^P.

Remark: In general, not all fibres are isomorphic to X_u^P

Example. Consider

$$G/P = \{V_\bullet = (V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4)\}$$

$$G/Q = \{V_3 \subset \mathbb{C}^4\}$$

with projection $\pi(V_\bullet) = V_3$.

- If $w = s_1 s_2 s_3 s_1$, then
$$X_w^P = \{V_\bullet \mid V_2 \subset E_3\}$$
where E_3 is a fixed 3-dim subspace.

- $w = vu = (s_1 s_2 s_3)(s_1)$, and $\pi(X_w^P) = X_v^Q = G/Q$ with fibre
$$\pi^{-1}(V_3) = \{(V_1, V_2) \mid V_1 \subset V_2 \subseteq V_3 \cap E_3\}$$
$$\cong \begin{cases} X_{s_1}^P & \dim(V_3 \cap E_3) = 2 \\ X_{s_1 s_2 s_1}^P & V_3 = E_3 \end{cases}$$

Question: What makes a Schubert variety X_w^P smooth?

Theorem: Ryan (87), Wolper (89)

Let G/P be a type A flag variety.

The Schubert variety X_w^P is smooth if and only if there exists a parabolic subgroup Q containing P and $w = vu$ with $v \in W^Q$ and $u \in W^P \cap W_Q$ such that:

- X_v^Q and X_u^P are smooth Schubert varieties.
- The projection $X_w^P \to X_v^Q$ is locally trivial with fibre isomorphic to X_u^P

Moreover, Q can be chosen to be a maximal parabolic containing P.

(i.e G/Q is a Grassmannian and Q/P has one less step then G/P)

Hence X_w^P can be written as a sequence of fibrations with each base isomorphic to a smooth Schubert variety of a Grassmannian.

Example. Consider

$$G/P = \{V_\bullet = (V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4)\}$$

$$G/Q = \{V_2 \subset \mathbb{C}^4\}$$

with projection $\pi(V_\bullet) = V_2$.

- If $w = s_1 s_2 s_3 s_1$, then
$$X_w^P = \{V_\bullet \mid V_2 \subset E_3\}$$
is smooth

- $w = vu = (s_1 s_2)(s_3 s_1)$, and $X^Q_{s_1 s_2} = \{V_2 \subset E_3\}$ with fibre
$$\pi^{-1}(V_2) = \{(V_1, V_3) \mid V_1 \subset V_2,\ V_2 \subset V_3\} \cong X^P_{s_1 s_3} \cong \mathbb{CP}^1 \times \mathbb{CP}^1$$

Question: What about other finite types?

Theorem: R-Slofstra (14)

Let G/P be a flag variety of any finite type.

The Schubert variety X_w^P is (rationally) smooth if and only if there exists a parabolic subgroup Q containing P and $w = vu$ with $v \in W^Q$ and $u \in W^P \cap W_Q$ such that:

- X_v^Q and X_u^P are (rationally) smooth Schubert varieties.
- The projection $X_w^P \to X_v^Q$ is locally trivial with fibre isomorphic to X_u^P

Moreover, Q can be chosen to be a maximal parabolic containing P. In other words, G/Q is a generalized Grassmannian of appropriate type.

Hence X_w^P can be written as a sequence of fibrations with each base isomorphic to a (rationally) smooth Schubert variety of a generalized Grassmannian.

Classification of rationally smooth Schubert varieties

Remark: (Rationally) smooth Schubert varieties of generalized Grassmannians are classified. (G/P where P is maximal parabolic)

- Let W be a Coxeter group with simple generating set S and relations

  $$s^2 = e \quad \text{and} \quad (st)^{m_{st}} = e$$

  for some $m_{st} \in \{2, 3, \ldots, \infty\}$.

- For any $w \in W$, define the support $S(w) := \{s \in S \mid s \leq w\}$.

- For any subset $J \subseteq S$, let $W_J \subseteq W$ denote the group generated by J and let W^J denote the minimal length coset representatives of W/W_J.

- We say $w \in W^J$ is a maximal element if it is the unique maximal length element in the set $W^J \cap W_{S(w)}$

Theorem:Lakshmibai-Weyman (90), Brion-Polo (99), Robles (12), Hong-Mok (13)

Let G/P be a generalized Grassmannian with $W_P = W_J$ and $J = S \setminus \{s\}$.

Then X_w^P is rationally smooth if and only if w is a maximal element of W^P, or w is one of the following elements:

W	s	w	index set	smooth
B_n	s_n	$s_1 \dots s_n$	$n \geq 2$	yes
B_n	s_1	$s_k s_{k+1} \cdots s_n s_{n-1} \cdots s_1$	$1 < k \leq n$	no
B_n	s_k	$u_{n,k+1} s_1 \cdots s_k$	$1 < k < n$	no
C_n	s_n	$s_1 \dots s_n$	$n \geq 2$	no
C_n	s_1	$s_k s_{k+1} \cdots s_n s_{n-1} \cdots s_1$	$1 < k \leq n$	yes
C_n	s_k	$u_{n,k+1} s_1 \cdots s_k$	$1 < k < n$	yes
F_4	s_1	$s_4 s_3 s_2 s_1$	n/a	no
F_4	s_2	$s_3 s_2 s_4 s_3 s_4 s_2 s_3 s_1 s_2$	n/a	no
F_4	s_4	$s_1 s_2 s_3 s_4$	n/a	yes
F_4	s_3	$s_2 s_3 s_1 s_2 s_1 s_3 s_2 s_4 s_3$	n/a	yes
G_2	s_1	$s_2 s_1,\ s_1 s_2 s_1,\ s_2 s_1 s_2 s_1$	n/a	no
G_2	s_2	$s_1 s_2$	n/a	yes
G_2	s_2	$s_2 s_1 s_2,\ s_1 s_2 s_1 s_2$	n/a	no

Here $u_{n,k}$ denotes the maximal length element in $W^{S\setminus\{s_1,s_k\}} \cap W_{S\setminus\{s_1\}}$ when W has type B_n or C_n.

If w is a maximal element of W^P, then X_w^P is smooth.

Enumeration of rationally smooth Schubert varieties

We can use previous theorems to enumerate smooth and rationally smooth Schubert varieties in the complete flag variety G/B in classical types.

n	A	B (smooth)	C (smooth)	B/C (r.s.)	D
4	88	116	114	142	108
5	366	490	472	596	490
6	1552	2094	1988	2530	2164
7	6652	9014	8480	10842	9474
8	28696	38988	36474	46766	41374
9	124310	169184	157720	202594	180614
10	540040	735846	684404	880210	788676
11	2350820	3205830	2976994	3832004	3445462

The generating series for type A is due to Haiman.

Definition: A parabolic decomposition $w = vu,\ v \in W^P, u \in W_P$ is a BP (Billey-Postnikov) decomposition if any of the following are true:

1. u is the maximal length element in $[e, w] \cap W_P$.
2. The Poincaré polynomials $\mathsf{P}_w(q) = \mathsf{P}_v^P(q) \cdot \mathsf{P}_u(q)$.
3. $S(v) \cap W_P$ is contained in the left descent set of u.

Theorem: R-Slofstra (14)

Let W be the Weyl group of G. The parabolic decomposition $w = vu$ is a BP decomposition with respect to P if and only if the projection

$$X_w^B \to X_v^P$$

is an algebraic fibre bundle with fibre isomorphic to X_u^B.

Remarks:

- X_w^B does not have to be rationally smooth.
- The theorem is true for Schubert varieties in G/P.
- The theorem is true for Kac-Moody Schubert varieties.

Existence of Billey-Postnikov decompositions

Definition: A parabolic decomposition $w = vu$ with respect to P is Grassmannian if $S(w) = S(u) + 1$. (Hence G/P is a generalized Grassmannian)

Theorem: Gasharov (98), Billey (98), Billey-Postnikov (05), Oh-Yoo (10)

Let W be a Weyl group of finite type and $w \in W$. If X_w^B is rationally smooth, then either w or w^{-1} has Grassmannian BP decomposition with respect to $J = S(w) \setminus \{s\}$, where s is some leaf of the Dynkin diagram of $S(w)$.

Theorem: R-Slofstra (14)

Let W be a Weyl group of finite type and $w \in W$. If X_w^B is rationally smooth, then w has a Grassmannian BP decomposition $w = vu$ with respect to $J = S(w) \setminus \{s\}$ for some $s \in S(w)$ (s is not necessarily a leaf).

Existence of Billey-Postnikov decompositions

Question: If W is an arbitrary Coxeter, do all rationally smooth elements have nontrivial BP-decompositions?

Partial results:

- Affine type A (Billey-Crites (11)).
- W is a Coxeter group with no commuting relations ($m_{st} \geq 3$) (R-Slofstra (12)).
- W is a right angle Coxeter group (R-Slofstra (12)).

Thank you!