Fibre bundle structures of Schubert varieties

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January 17, 2014
1. Schubert varieties and fibre bundles

2. Smooth Schubert varieties and fibre bundles

3. Billey-Postnikov decompositions
Schubert varieties

- Let $G$ be a semi-simple Lie group over $\mathbb{C}$.

  Fix $T \subseteq B \subseteq G$ a maximal torus and Borel subgroup of $G$.

  Let $P \subseteq G$ be a parabolic subgroup containing $B$.

- Let $W := N(T)/T$ denote the Weyl group of $G$.

  Let $W_P \subseteq W$ be the Weyl group of $P$.

- Let $G/P$ be the partial flag variety.

  For any $w \in W^P \simeq W/W_P$ (min length coset rep), we have the Schubert variety

$$X^P_w := \overline{BwP}/P \subseteq G/P.$$
Schubert varieties: type A

- $G = SL(\mathbb{C}^n)$ with $\mathbb{C}^n = \text{Span}_\mathbb{C}\{e_1, \ldots, e_n\}$.

  $T=$diagonal matrices, $B=$upper triangular matrices.

- $W = S_n$ permutation matrices.

- $G/B$ is the full flag variety

  \[ \{ V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim(V_i) = i \}. \]

- For any permutation matrix $w \in W$, we have the Schubert variety

  \[ X^B_w = \{ V_\bullet \in G/B \mid \dim(V_i \cap E_j) \geq \text{rk}(w[i,j]) \} \]

  where $E_j = \text{Span}\{e_1, \ldots, e_j\}$. 
Let $Q$ be parabolic subgroup containing $P$ and consider the projection

$$G/P \to G/Q$$

with fibre isomorphic to $Q/P$.

For $w \in W^P$, there is a unique parabolic decomposition $w = vu$ where $v \in W^Q$, $u \in W^P \cap W_Q$ and induced projection

$$X^P_w \to X^Q_v$$

with generic fibre isomorphic to $X^P_u$.

Remark: In general, not all fibres are isomorphic to $X^P_u$. 

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Example. Consider

\[ G/P = \{ V_\bullet = (V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4) \} \]

\[ G/Q = \{ V_3 \subset \mathbb{C}^4 \} \]

with projection \( \pi(V_\bullet) = V_3 \).

- If \( w = s_1 s_2 s_3 s_1 \), then
  \[ X_w^P = \{ V_\bullet \mid V_2 \subset E_3 \} \]
  where \( E_3 \) is a fixed 3-dim subspace.

- \( w = vu = (s_1 s_2 s_3)(s_1) \), and \( \pi(X_w^P) = X_v^Q = G/Q \) with fibre
  \[ \pi^{-1}(V_3) = \{ (V_1, V_2) \mid V_1 \subset V_2 \subseteq V_3 \cap E_3 \} \]
  \[ \cong \begin{cases} 
  X_{s_1}^P & \text{dim}(V_3 \cap E_3) = 2 \\
  X_{s_1 s_2 s_1}^P & V_3 = E_3
  \end{cases} \]
**Question:** What makes a Schubert variety $X_w^P$ smooth?

**Theorem: Ryan (87), Wolper (89)**

Let $G/P$ be a type A flag variety.

The Schubert variety $X_w^P$ is smooth if and only if there exists a parabolic subgroup $Q$ containing $P$ and $w = vu$ with $v \in W^Q$ and $u \in W^P \cap W_Q$ such that:

- $X_v^Q$ and $X_u^P$ are smooth Schubert varieties.
- The projection $X_w^P \to X_v^Q$ is locally trivial with fibre isomorphic to $X_u^P$.

Moreover, $Q$ can be chosen to be a maximal parabolic containing $P$.
(i.e $G/Q$ is a Grassmannian and $Q/P$ has one less step then $G/P$)

Hence $X_w^P$ can be written as a sequence of fibrations with each base isomorphic to a smooth Schubert variety of a Grassmannian.
Example. Consider

\[ G/P = \{ V_\bullet = (V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4) \} \]

\[ G/Q = \{ V_2 \subset \mathbb{C}^4 \} \]

with projection \( \pi(V_\bullet) = V_2 \).

- If \( w = s_1 s_2 s_3 s_1 \), then
  \[ X^P_w = \{ V_\bullet \mid V_2 \subset E_3 \} \]

  is smooth

- \( w = vu = (s_1 s_2)(s_3 s_1) \), and \( X^Q_{s_1 s_2} = \{ V_2 \subset E_3 \} \) with fibre

  \[ \pi^{-1}(V_2) = \{(V_1, V_3) \mid V_1 \subset V_2, V_2 \subset V_3 \} \cong X^P_{s_1 s_3} \cong \mathbb{CP}^1 \times \mathbb{CP}^1 \]
**Question:** What about other finite types?

**Theorem: R-Slofstra (14)**

Let $G/P$ be a flag variety of any finite type.

The Schubert variety $X^P_w$ is (rationally) smooth if and only if there exists a parabolic subgroup $Q$ containing $P$ and $w = vu$ with $v \in W^Q$ and $u \in W^P \cap W_Q$ such that:

- $X^Q_v$ and $X^P_u$ are (rationally) smooth Schubert varieties.
- The projection $X^P_w \to X^Q_v$ is locally trivial with fibre isomorphic to $X^P_u$.

Moreover, $Q$ can be chosen to be a maximal parabolic containing $P$. In other words, $G/Q$ is a **generalized Grassmannian** of appropriate type.

Hence $X^P_w$ can be written as a sequence of fibrations with each base isomorphic to a (rationally) smooth Schubert variety of a generalized Grassmannian.
Classification of rationally smooth Schubert varieties

Remark: (Rationally) smooth Schubert varieties of generalized Grassmannians are classified. ($G/P$ where $P$ is maximal parabolic)

- Let $W$ be a Coxeter group with simple generating set $S$ and relations
  \[ s^2 = e \quad \text{and} \quad (st)^{m_{st}} = e \]
  for some $m_{st} \in \{2, 3, \ldots, \infty\}$.

- For any $w \in W$, define the support $S(w) := \{ s \in S \mid s \leq w \}$.

- For any subset $J \subseteq S$, let $W_J \subseteq W$ denote the group generated by $J$ and let $W^J$ denote the minimal length coset representatives of $W/W_J$.

- We say $w \in W^J$ is a **maximal element** if it is the unique maximal length element in the set $W^J \cap W_{S(w)}$. 
Let $G/P$ be a generalized Grassmannian with $W_P = W_J$ and $J = S \setminus \{s\}$.

Then $X_P^w$ is rationally smooth if and only if $w$ is a maximal element of $W^P$, or $w$ is one of the following elements:

<table>
<thead>
<tr>
<th>$W$</th>
<th>$s$</th>
<th>$w$</th>
<th>index set</th>
<th>smooth</th>
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<td>$s_1 \ldots s_n$</td>
<td>$n \geq 2$</td>
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<tr>
<td>$B_n$</td>
<td>$s_k$</td>
<td>$u_{n,k+1} s_1 \cdots s_k$</td>
<td>$1 &lt; k &lt; n$</td>
<td>no</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$s_n$</td>
<td>$s_1 \ldots s_n$</td>
<td>$n \geq 2$</td>
<td>no</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$s_1$</td>
<td>$s_k s_{k+1} \cdots s_n s_{n-1} \cdots s_1$</td>
<td>$1 &lt; k \leq n$</td>
<td>yes</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$s_k$</td>
<td>$u_{n,k+1} s_1 \cdots s_k$</td>
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<tr>
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<td>$s_1$</td>
<td>$s_4 s_3 s_2 s_1$</td>
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<td>no</td>
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<tr>
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<tr>
<td>$G_2$</td>
<td>$s_1$</td>
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<td>n/a</td>
<td>no</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$s_2$</td>
<td>$s_1 s_2$</td>
<td>n/a</td>
<td>yes</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$s_2$</td>
<td>$s_2 s_1 s_2, s_1 s_2 s_1 s_2$</td>
<td>n/a</td>
<td>no</td>
</tr>
</tbody>
</table>

Here $u_{n,k}$ denotes the maximal length element in $W^S \setminus \{s_1, s_k\} \cap W^S \setminus \{s_1\}$ when $W$ has type $B_n$ or $C_n$.

If $w$ is a maximal element of $W^P$, then $X_P^w$ is smooth.
We can use previous theorems to enumerate smooth and rationally smooth Schubert varieties in the complete flag variety $G/B$ in classical types.

The generating series for type $A$ is due to Haiman.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A$</th>
<th>$B$ (smooth)</th>
<th>$C$ (smooth)</th>
<th>$B/C$ (r.s.)</th>
<th>$D$</th>
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<td>3832004</td>
<td>3445462</td>
</tr>
</tbody>
</table>
**Definition:** A parabolic decomposition $w = vu$, $v \in W^P$, $u \in W_P$ is a BP (Billey-Postnikov) decomposition if any of the following are true:

1. $u$ is the maximal length element in $[e, w] \cap W_P$.
2. The Poincaré polynomials $P_w(q) = P_{v}^{P}(q) \cdot P_{u}(q)$.
3. $S(v) \cap W_P$ is contained in the left descent set of $u$.

**Theorem: R-Slofstra (14)**

Let $W$ be the Weyl group of $G$. The parabolic decomposition $w = vu$ is a BP decomposition with respect to $P$ if and only if the projection

$$X^B_w \to X^P_v$$

is an algebraic fibre bundle with fibre isomorphic to $X^B_u$.

**Remarks:**

- $X^B_w$ does not have to be rationally smooth.
- The theorem is true for Schubert varieties in $G/P$.
- The theorem is true for Kac-Moody Schubert varieties.
Existence of Billey-Postnikov decompositions

**Definition:** A parabolic decomposition $w = vu$ with respect to $P$ is Grassmannian if $S(w) = S(u) + 1$. (Hence $G/P$ is a generalized Grassmannian)

**Theorem:** Gasharov (98), Billey (98), Billey-Postnikov (05), Oh-Yoo (10)

Let $W$ be a Weyl group of finite type and $w \in W$. If $X^B_w$ is rationally smooth, then either $w$ or $w^{-1}$ has Grassmannian BP decomposition with respect to $J = S(w) \setminus \{s\}$, where $s$ is some leaf of the Dynkin diagram of $S(w)$.

**Theorem:** R-Slofstra (14)

Let $W$ be a Weyl group of finite type and $w \in W$. If $X^B_w$ is rationally smooth, then $w$ has a Grassmannian BP decomposition $w = vu$ with respect to $J = S(w) \setminus \{s\}$ for some $s \in S(w)$ ($s$ is not necessarily a leaf).
Question: If $W$ is an arbitrary Coxeter, do all rationally smooth elements have nontrivial BP-decompositions?

Partial results:

- Affine type A (Billey-Crites (11)).
- $W$ is a Coxeter group with no commuting relations ($m_{st} \geq 3$) (R-Slofstra (12)).
- $W$ is a right angle Coxeter group (R-Slofstra (12)).

Thank you!