On the Best Constant in the Moser-Aubin-Onofri Inequality

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Moser-Trudinger inequality

Let S^2 be the 2-dimensional unit sphere and let J_α denote the nonlinear functional on the Sobolev space $H^{1,2}(S^2)$ defined by

$$J_\alpha(u) = \frac{\alpha}{4}\int_{S^2} |\nabla u|^2 \, d\omega + \int_{S^2} u \, d\omega - \ln \int_{S^2} e^u \, d\omega,$$

where $d\omega$ denotes Lebesgue measure on S^2, normalized so that $\int_{S^2} d\omega = 1$.

Moser-Trudinger: For $\alpha \geq 1$,

$$C(\alpha) = \inf\left\{J_\alpha(u);\ u \in H^1(S^2)\right\} > -\infty.$$

Clearly, $C(\alpha) \leq 0$.

Aubin's inequalities

Aubin had shown that by restricting the functional J_α to the "Aubin-submanifold" of $H^1(S^2)$ defined by

$$\mathcal{M} := \left\{ u \in H^1(S^2);\ \int_{S^2} e^u x_j \, dw = 0 \quad \text{for all } 1 \leq j \leq 3 \right\},$$

then it is bounded below (resp., coercive) for $\alpha \geq \frac{1}{2}$ (resp., $\alpha > \frac{1}{2}$) i.e.,

$$\inf_{u \in \mathcal{M}} J_\alpha(u) := A(\alpha) > -\infty.$$

Onofri (later Osgood-Phillips-Sarnak, and Hong): for $\alpha \geq 1$

$$C(1) = A(1) = 0.$$

that is J_α is non-negative on $H^1(S^2)$ provided $\alpha \geq 1$.
Chang-Yang then showed that for some $\epsilon_0 > 0$,

$$\inf_{u \in \mathcal{M}} J_{1-\epsilon_0}(u) := A(1 - \epsilon_0) = 0.$$

They asked

Main conjecture (Chang and Yang)

1. If $\alpha \geq \frac{1}{2}$ then $\inf_{u \in \mathcal{M}} J_\alpha(u) = A(\alpha) = 0$.
2. If $\alpha < \frac{1}{2}$ then $\inf_{u \in \mathcal{M}} J_\alpha(u) = A(\alpha) = -\infty$.

Known results so far:

(1998) Feldman, Froese, Ghoussoub and Gui (2) holds and (1) is true for axially symmetric case and $\alpha \geq \frac{16}{25}$

(2000) Gui-Wei and independently by C.S. Lin
True for axially symmetric case, i.e., $\alpha \geq \frac{1}{2}$.

(2008) Ghoussoub-C.S. Lin
True in general for $\alpha \geq \frac{2}{3}$.

Conformal invariance

For any conformal transformation ψ of S^2, we have:

$$J_1(T_\psi u) = J_1(u) \text{ where } T_\psi(u) = u \circ \psi + \frac{1}{2} \log \det |d\psi|.$$

ψ is defined by $\psi_{P,t}$ where for $P \in S^2$ and $t \geq 1$, choose a frame $(e_1, e_2, e_3 = P)$ then use stereographic coordinates with P at infinity and denote $\psi(z) = \psi_{P,t}(z) = tz$, where

$$\mathbf{x} \equiv z = \cot(\frac{\theta}{2})e^{i\phi} = \frac{x_1 + ix_2}{1 - x_3}.$$

Conformal transformation on S^2 identified with fractional linear transformations $\psi(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ (in $SL(2, \mathbf{C})$) $(\alpha\delta - \beta\gamma = 1)$. Then,

$$\frac{1}{2} \log \det |d\psi(z)| = 2 \log \frac{1 + |z|^2}{|\alpha z + \beta|^2 + |\gamma z + \delta|^2}.$$

Onofri's Proof for $\alpha = 1$

1. For any $u \in H^1(S^2)$, there exists ϕ such that $T_\phi u \in \mathcal{M}$.
2. Since J_1 is conformally invariant, then

$$A(1) := \inf_{u \in \mathcal{M}} J_1(u) = \inf_{u \in H^1(S^2)} J_1(u) := C(1),$$

3. J_1 attains its infimum on $\mathcal{M}$ and Euler-Lagrange equation:

$$\Delta u + 2\left(\frac{e^u}{\int_{S^2} e^u \, dw} - 1 \right) = \sum_{j=1}^{3} \alpha_j x_j e^u \quad \text{on } S^2.$$

4. Kazdan-Warner showed that above equation has solutions only if $\alpha_j = 0$ (no Lagrange multiplier)
5. Conclude by using that the equation

$$\Delta u + e^{2u} = 1 \quad \text{on } S^2$$

has only trivial solution satisfying the Aubin constraint. Metrics with prescribed Gaussian curvature!

Metrics with prescribed Gaussian curvature $K(x)$

If metric $g = e^{2u} g_0$ with Gaussian curvature K, then,

$$\Delta u + K(x) e^{2u} = 1 \quad \text{on } S^2 \tag{1}$$

where $\Delta = \Delta_{g_0}$ is Laplacian related to standard metric g_0.

$$F_K(u) = \int_{S^2} |\nabla u|^2 \, d\omega + 2 \int_{S^2} u \, d\omega - \ln \int_{S^2} K(x) e^{2u} \, d\omega, \tag{2}$$

1. When $u = \frac{1}{2} \ln |J_\psi|$ for ψ conformal transformation, then $\Delta u + e^{2u} = 1$ and $K = 1$, and when $K = c$, $u = \frac{1}{2} \ln |J_\psi| + \ln c$ is another non-trivial solution with $F_c(u) = \ln c$.
2. One can show that if u is a local minimum then there is a solution only if K is constant. Hence, the need to find a saddle.
3. (K, u) satisfy (1) if and only if $(K \circ \psi, T_\psi(u)$ satisfies (1) for any conformal transformation.

Determinant of Laplacians

Metric $g \to$ Laplacian $\Delta_g \to$ eigenvalues $(\lambda_i^g)_i \to$ Determinant

$$\det \Delta_g = \Pi_i \lambda_i^g.$$

Not so fast! But one can write the determinant as

$$\det \Delta_g = e^{-\xi'(0)} \quad \text{where } \xi(s) = \Sigma_{k>0} \frac{1}{\lambda_k^s} \quad \text{for } Re(s) > 1.$$

Ray-Singer-Polyakov gives when $g = e^{2u} g_0$ and equal volume,

$$\ln \frac{det \Delta_g}{det \Delta_{g_0}} = -\frac{1}{12\pi} \int_{S^2} (2u + \nabla u|^2) \, d\omega.$$

1. Right hand side is invariant under conformal transformation.
2. Onofri's inequality implies that among all conformal metrics on S^2 with equal volume, $\ln \det \Delta_{g_0}$ is maximum.

Connections to Navier-Stokes equations

$$\Delta u - (u \cdot \nabla)u = \nabla p \quad \mathrm{div} u = 0 \in \mathrm{on}\, \mathbf{R}^3$$

scales under $u \to \lambda u(\lambda x)$.
What are then the solutions that are invariant under such a scaling?
Explicit examples are the Landau solutions. Anything else?

Sverak (2009): Only -1-homogeneous solutions of the stationary Navier-Stoke equation on $\mathbf{R}^3$ are the Landau solutions (i.e, axially symmetric and $u = 0$ if weak and smooth away from the origin).

Proof: $u(x) = \mathbf{v}(\mathbf{x}) + f(x)\mathbf{x}$ (where $\mathbf{v}(\mathbf{x})$ tangent to S^2).
After some work, one can show that $v = \nabla\phi$ and

$$\Delta\phi + 2e^{\phi} = 2 \quad \text{on } S^2.$$

Axially symmetric case

Let θ and φ denote the usual angular coordinates on the sphere, and define $x = cos(\theta)$. Axially symmetric functions depend on x only. For such functions the above conjecture reduces to the following one-dimensional inequality:

$$\frac{1}{2}\int_{-1}^{1}(1-x^2)|g'(x)|^2\,dx + 2\int_{-1}^{1} g(x)\,dx - 2\ln\frac{1}{2}\int_{-1}^{1} e^{2g(x)}dx \geq 0,$$

for any g on $(-1, 1)$ satisfying

$$\int_{-1}^{1}(1-x^2)|g'(x)|^2dx < \infty \text{ and } \int_{-1}^{1} e^{2g(x)}xdx = 0.$$

Conjecture proved by
(1998) Feldman, Froese, Ghoussoub and Gui (True for $\alpha \geq \frac{16}{25}$)
(2000) Gui-Wei and independently by Lin (for $\alpha \geq \frac{1}{2}$).

When $\alpha < 1/2$

Consider the functions

$$\begin{aligned} g(x) &= c\ln(1-x) \quad \text{for} \quad 0 < x < 1-\epsilon \\ g(x) &= c\ln(\epsilon) \quad \text{for} \quad 1-\epsilon < x < 1. \end{aligned}$$

extended as even functions to the whole interval $(-1,1)$. It is clear that $g \in \mathcal{M}_r$ and a calculation shows that for small ϵ

$$I_\alpha(g) = 2\alpha c^2 |\ln(\epsilon)| - \ln\left(-\frac{\epsilon^{2c+1}}{2c+1} + \frac{1}{2c+1} + \epsilon^{2c+1}\right) + O(1)$$

If $2c+1 < 0$, this becomes

$$I_\alpha(g) = p(c)|\ln(\epsilon)| + O(1),$$

where $p(c) = 2\alpha c^2 + 2c + 1$. Now suppose $\alpha < 1/2$. Then the discriminant of $p(c)$, namely $4 - 8\alpha$, is positive. Hence $p(c)$ has real roots and must be negative for some value of c. For this value of c, $2c+1 < p(c) < 0$, so $I_\alpha(g)$ tends to $-\infty$ as ϵ becomes small.

Proof of FFGG for axially symmetric case and $\alpha \geq \frac{2}{3}$

Step 1: Let $G(x) = (1-x^2)g'(x)$ where g is any critical point of

$$I_\alpha(g) = \alpha \int_{-1}^{1} (1-x^2)|g'(x)|^2\,dx + 2\int_{-1}^{1} g(x)\,dx - 2\ln\frac{1}{2}\int_{-1}^{1} e^{2g(x)}dx$$

restricted to $\mathcal{M}_r$. Then

$$(\star) \qquad \begin{cases} \alpha G' - 1 + \frac{2}{\lambda}e^{2g} = 0 \\ (1-x^2)G'' + \frac{2}{\alpha}G - 2GG' = 0 \\ G(-1) = G(1) = 0. \end{cases}$$

Step 2: Integrate $(\star)$ on $(-1,1)$, plus a few integration by parts yield:

$$\left(\frac{2}{\alpha} - 2\right)\int_{-1}^{1} G(x)\,dx = 0$$

G is then orthogonal to the first eigenspace of $\frac{d}{dx}((1-x^2)\frac{d}{dx})$ on $H^1(-1,1)$. Since the second eigenvalue is 2, we have

$$2\int_{-1}^{1} |G(x)|^2 \leq \int_{-1}^{1} (1-x^2)|G'(x)|^2\,dx.$$

Proof continued

We have so far

$$2\int_{-1}^{1}|G(x)|^2 \leq \int_{-1}^{1}(1-x^2)|G'(x)|^2\,dx. \qquad (1)$$

Step 3: Multiply equation by G and integrate by parts twice, we get that

$$\int_{-1}^{1}(1-x^2)|G'(x)|^2\,dx = \left(\frac{2}{\alpha}-1\right)\int_{-1}^{1}|G(x)|^2. \qquad (2)$$

So, by comparing (1) and (2), we get that either $\alpha \leq 2/3$ or that G and hence g is identically 0.

Gui-Wei's proof for $\alpha \geq \frac{1}{2}$ is an iteration of this process.

Connection to other conjectures

Conjecture 1: Suppose $\frac{1}{2} \le \alpha < 1$, then any solution of

$$\alpha \Delta u + e^{2u} - 1 = 0 \text{ in } S^2 \qquad (\star)$$

is identically 0 on S^2.

Let Π denote the stereographic projection $S^2 \to \mathbb{R}^2$ with respect to the North pole $N = (0,0,1)$: $\Pi(x) := \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right)$.
Suppose u is a solution of $(\star)$ and set

$$\tilde{u}(y) := u(\Pi^{-1}(y)) \quad \text{for } y \in \mathbb{R}^2.$$

Then $\tilde{u}$ satisfies

$$\Delta \tilde{u} + \frac{8\pi}{\alpha} J(y) \left(e^{\tilde{u}} - \frac{1}{4\pi} \right) = 0 \quad \text{in } \mathbb{R}^2,$$

where $J(y) := \left(\frac{2}{1+|y|^2}\right)^2$ is the Jacobian of Π. By letting

$$v(y) := \tilde{u}(y) + \frac{1}{\alpha} \log\left((1+|y|^2)^{-2}\right) + \log\left(\frac{32\pi}{\alpha}\right) \quad \text{for } y \in \mathbb{R}^2,$$

we have that v satisfies

$$\Delta v + (1+|y|^2)^l e^v = 0 \quad \text{in } \mathbb{R}^2, \tag{3}$$

where $l = 2(\frac{1}{\alpha} - 1)$.
Note that equation (3) always has a special axially symmetric solution, namely

$$v^*(y) = -\frac{2}{\alpha}\log(1+|y|^2) + \log(\frac{32\pi}{\alpha}) \quad \text{for } y \in \mathbb{R}^2, \tag{4}$$

Moreover, The Pohozaev idendity yields that for any solution v of (3) we have

$$4 < \beta_l(v) := \frac{1}{2\pi}\int_{\mathbb{R}^2}(1+|y|^2)^l e^v dy < 4(1+l), \tag{5}$$

Open question that would imply Conjecture (1) is:

Conjecture 2: Is v^* the only solution of (3) whenever $l > 0$?
Answer should be: YES if $l \leq 2$ and NO if $l > 2$.
We now know that it is YES for $l \leq 1$ (i.e, $\alpha \geq \frac{2}{3}$)

State of conjecture

- It is indeed the case if $\ell < 0$ (i.e. $\alpha > 1$), since then one can use method of moving planes to show that $v(y)$ is radially symmetric with respect to the origin, and then conclude that $u(x)$ is axially symmetric with any line passing through the origin. Thus $u(x)$ must be a constant function on S^2, hence $u = 0$, and $J_\alpha \geq 0$ on $\mathcal{M}$. One recovers Onofri's inequality.

- When $l > 0$ ($\alpha < 1$), the method of moving planes fails and it is still an open problem whether any solution of (3) is equal to v^* or not. The following uniqueness theorem reduces however the problem to whether any solution of (3) is radially symmetric.

Theorem (B): Suppose $l > 0$ and $v_i(y) = v_i(|y|), i = 1, 2,$ are two solutions of (3) satisfying $\beta_l(v_1) = \beta_l(v_2)$, then $v_1 = v_2$ under one of the following conditions:

$$l \leq 1 \text{ or } l > 1 \text{ and } 2l < \beta_l(v_i) < 2(2 + l) \text{ for } i = 1, 2.$$

In order to show how Theorem B implies the axially symmetric case, assume u is a solution of ($\star$) that is axially symmetric with respect to some direction. By rotating, the direction can be assumed to be $(0,0,1)$. Use again the stereographic projection and set v as in (14) to get

$$\begin{cases} v(y) = -\frac{4}{\alpha}\log|y| + O(1), \\ \frac{1}{2\pi}\int_{\mathbb{R}^2}(1+|y|^2)^l e^v dy = 4+2l. \end{cases} \tag{6}$$

If $l \le 1$, i.e., $\alpha \ge \frac{2}{3}$, then $v = v^*$ by (i) of Theorem B, and then $u \equiv 0$. If $l > 1$, then by noting that

$$2l < 4 + 2l = \beta_l(v) < 4 + 4l,$$

we deduce that $v = v^*$ by (ii) of Theorem B, which again means that $u \equiv 0$.

Key lemma

Lemma (Bandle): *Let Ω be a simply connected domain in $\mathbb{R}^2$, and suppose $g \in C^2(\Omega)$ satisfies*

$$\begin{cases} \Delta g + e^g > 0 \quad in\ \Omega \quad and \\ \int_\Omega e^g dy \leq 8\pi. \end{cases}$$

Consider an open set $\omega \subset \Omega$ such that $\lambda_{1,g}(\omega) \leq 0$, where $\lambda_{1,g}(\omega)$ is the first eigenvalue of the operator $\Delta + e^g$ on $H_0^1(\omega)$. Then, we necessarily have

$$\int_\omega e^g dy > 4\pi.$$

Lemma was first proved in Bandle by using the classical Bol inequality. The strict inequality is due to the fact that $\Delta g + e^g > 0$ in Ω.

Bol-Alexandroff inequality

Lemma 1: (Nehari) Let σ be subharmonic on $\Omega \subset \mathbb{R}^2$, then

$$\left(\int_{\partial\Omega} e^{\frac{\sigma}{2}}\right)^2 \geq 4\pi \int_{\Omega} e^{\sigma}.$$

Lemma 2: (Bol-Alexandroff) Let Ω be a simply connected domain in $\mathbb{R}^2$, and suppose $u \in C^2(\Omega)$ satisfies

$$\Delta u + e^u > 0 \quad \text{in } \Omega \quad \text{and} \int_{\Omega} e^u dy \leq 8\pi.$$

then for any open set $\omega \subset \Omega$ of class C^1, we have

$$\int_{\partial\omega} e^{\frac{u}{2}} \geq \frac{1}{2} \int_{\omega} e^u dy \Big(8\pi - \int_{\omega} e^u dy\Big).$$

Proof of Ghoussoub-Lin

Theorem: *Suppose* $\frac{2}{3} \leq \alpha < 1$, *then any solution of*

$$\alpha \Delta u + e^{2u} - 1 = 0 \ \ in \ \ S^2, \qquad (\star)$$

is identically 0 *on* S^2.

Sketch: Suppose u is a solution of $(\star)$. Let ξ_0 be a critical point of u, that we can suppose $(0, 0, -1)$. Use stereographic projection Π and let

$v(y) := u(\Pi^{-1}(x)) - \frac{2}{\alpha} \log(1 + |y|^2) + \log(\frac{32\pi}{\alpha})$.

$$\Delta v + (1 + |y|^2)^l e^v = 0 \text{ in } \mathbb{R}^2 \text{ and } \nabla v(0) = 0.$$

The function $\varphi(y) := y_2 \frac{\partial v}{\partial y_1} - y_1 \frac{\partial v}{\partial y_2}$, satisfies

$$\Delta \varphi + (1 + |y|^2)^l e^v \varphi = 0 \quad \text{in } \mathbb{R}^2.$$

If $\varphi \not\equiv 0$, then $\varphi(y) = Q(y)$+higher order terms for $|y| \ll 1$, where $Q(y)$ is a quadratic polynomial of degree m with $m \geq 2$, that is also a harmonic function, i.e., $\Delta Q = 0$. Thus, the nodal line $\{y \,|\, \varphi(y) = 0\}$ divides a small neighborhood of the origin into at least four regions.

Globally, $\mathbb{R}^2$ is therefore divided by the nodal line $\{y \,|\, \varphi(y) = 0\}$ into at least 3 regions, i.e.,

$$\mathbb{R}^2 \setminus \{y \,|\, \varphi(y) = 0\} = \bigcup_{j=1}^{3} \Omega_j.$$

In each component Ω_j, the first eigenvalue of $\Delta + (1 + |y|^2)^l e^v$ being equal to 0. Let $g := \log\big((1 + |y|^2)^l e^v\big)$ so that

$$\Delta g + e^g > 0 \quad \text{in } \mathbb{R}^2,$$

The lemma then implies that for each $j = 1, 2, 3$,

$$\int_{\Omega_j} e^g dy = \int_{\Omega_j} (1 + |y|^2)^l e^v dy > 4\pi.$$

It follows that

$$\frac{8\pi}{\alpha} = \int_{\mathbb{R}^2} (1 + |y|^2)^l e^v dy = \sum_{j=1}^{3} \int_{\Omega_j} (1 + |y|^2)^l e^v dy > 12\pi,$$

which is a contradiction if we had assumed that $\alpha \geq \frac{2}{3}$. Thus we have $\varphi(y) = 0$, i.e., $v(y)$ is axially symmetric, and $u \equiv 0$.

Final remarks

If we further assume that the antipodal of ξ_0 is also a critical point of u, then $\mathbb{R}^2 \setminus \{y \,|\, \varphi(y) = 0\} = \bigcup_{j=1}^{m} \Omega_j$, where $m \geq 4$. The lemma then yields

$$\frac{8\pi}{\alpha} = \int_{\mathbb{R}^2} (1 + |y|^2)^l e^v dy \geq \sum_{j=1}^{m} \int_{\Omega_j} (1 + |y|^2)^l e^v dy > 4m\pi \geq 16\pi,$$

which is a contradiction whenever $\alpha \geq \frac{1}{2}$. By Theorem A, we have again that $u \equiv 0$.
For example, if u is even on S^2 (i.e., $u(z) = u(-z)$ for all $z \in S^2$), then the main theorem holds for $\alpha \geq \frac{1}{2}$.