Bowen factors of Markov shifts and surface diffeomorphisms

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The Mathematical Legacy of Rufus Bowen

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Outline

1 Introduction
   - Symbolic dynamics from Bowen to Sarig
   - Consequences for classification and periodic orbits

2 Background on Markov shifts
   - Spectral decomposition and their entropy
   - Periodic points

3 Factors of Markov shifts
   - Pathologies
   - Bowen factors

4 Results
   - Almost Borel classification
   - Periodic points

5 Conclusion - Some questions
The classical setting: uniform hyperbolicity

**Theorem (Sinai, Bowen)**

Any Axiom-A diffeomorphism \( f : M \to M \) has a Markov partition with small diameter \((f, \Omega(f))\) is a Holder continuous factor of a subshift of finite type with good properties:

(i) finite-to-one; (ii) described by a relation on the alphabet:

We will now discuss the quotient map \( \pi: \Sigma_A \to \Omega_i \) defined by \( \pi(x) = \cap_{j=1}^{\infty} f^{-j} R x_j \).

The Markov partition \( \mathcal{C} = \{R_1, \ldots, R_n\} \) is taken so that \( \max_{1 \leq j \leq n} \text{diam}(R_j) \) is less than an expansivity constant. Define a relation \( \sim \) on \( \{1, \ldots, n\} \) by \( j \sim k \) iff \( R_j \cap R_k \neq \emptyset \). Define \( \sim \) on \( \Sigma_A \) by \( x \sim y \) iff \( x \sim y \) for all \( r \in \mathbb{Z} \). It is easy to prove that \( \pi(x) = \pi(y) \) precisely if \( x \sim y \).

**Consequences**

The factor is an isomorphism wrt any ergodic measure with support \( \Omega(f) \)

Finitely many ergodic measures maximizing the entropy (mme) \( \mu_1, \ldots, \mu_r \)

Each \( \mu_i \equiv \text{Bernoulli} \times \mathbb{Z}/p_i \mathbb{Z} \)

\[ |\{x = f^n x\}| \sim (p_1 + \cdots + p_r) e^{nh_{\text{top}}(f)} \text{ for } n \in \text{lcm}(p_1, \ldots, p_r) \mathbb{Z} \]

In fact, \( \zeta_f(z) = \prod_{\mathcal{O}} (1 - z^{\mathcal{O}})^{-1} \) is rational (Manning)

**Goal:** generalize to surface diffeomorphisms using Sarig’s codings
The classical setting: uniform hyperbolicity

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p. 13 of Bowen, On Axiom A diffeomorphisms (1978); Manning (1971)

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Almost Borel classification

Definition

\( S : X \to X \) and \( T : Y \to Y \) are **almost Borel conjugate mod zero entropy** if \( \exists \) invariant Borel subsets \( X' \subset X \), \( Y' \subset Y \) and a Borel isomorphism \( \psi : X' \to Y' \) s.t.

(i) \( \psi \circ S = T \circ \psi \);

(ii) \( X \setminus X' \) and \( Y' \setminus Y \) carry only measures with zero entropy

Using the Bowen property of Sarig’s coding and Hochman’s almost Borel classification:

**Theorem 1 (Boyle-B)**

Any \( C^{1+} \)-diffeomorphism of a compact surface is almost Borel conjugate mod zero entropy to a Markov shift

Using "magic word" isomorphisms as between almost conjugate SFTs

**Theorem 2 (B)**

Let \( f \) be a \( C^\infty \)-diffeomorphism of a compact surface and \( 0 < \chi < h_{\text{top}}(f) \)
Let \( \mu_1, \ldots, \mu_r \) be mme’s, with \( \mu_i \) isomorphic to Bernoulli \( \times \mathbb{Z}/p_i\mathbb{Z} \)
Let \( p := \text{lcm}(p_1, \ldots, p_r) \)

\[
\lim_{n \to \infty, p|n} \left| \{ x \in M : f^n x = x, \chi\text{-hyperbolic} \} \right| e^{-nh_{\text{top}}(f)} \geq p_1 + \cdots + p_r
\]

Compare Sarig; Kaloshin; Burguet
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Markov shifts

A oriented, countable graph with vertices $\mathcal{V}_G$ and edges $\mathcal{E} \subset \mathcal{V}_G \times \mathcal{V}_G$

**Definition**

The **Markov shift** defined by $G$ is $S_G : X_G \rightarrow X_G$:

$$X_G := \{x \in \mathcal{V}_G^\mathbb{Z} : \forall n \in \mathbb{Z} \ x_n \xrightarrow{G} x_{n+1}\} \text{ with } S_G : (x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}$$

(will drop indices $G$ whenever possible)

Subshifts of finite type (SFT)

A Markov shift is $C^0$-conjugate to some SFT iff $X$ compact iff $G$ can be chosen finite

**Theorem (Spectral decomposition)**

The non-wandering set of a Markov shift $(X, S)$ splits into transitive components

$$\Omega(X_G) = \bigsqcup_{i \in I} X_{G_i} \text{ with } S : X_{G_i} \rightarrow X_{G_i} \text{ (topologically) transitive}$$

Furthermore,

$$X_{G_i} = \bigsqcup_{j=0}^{p_i-1} S^j(Y_i) \text{ with } S^{p_i} : Y_i \rightarrow Y_i \text{ topologically mixing}$$

($G_i)_{i \in I}$ are the strongly connected components of $G$ and $p_i$ are their periods
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Markov shifts – entropy

\( h(S, \nu) \) Kolmogorov-Sinai entropy

**Theorem (Gurevič)**

*For a Markov shift \( S \), the Borel entropy*

\[
    h(S) := \sup_{\mu \in \mathcal{P}(S)} h(S, \mu) \in [0, \infty]
\]

*is the upper growth rate of the periodic orbits through a given vertex*

**Definition**

An **mme** (ergodic invariant probability measure maximizing the entropy) is \( \mu \in \mathcal{P}_{\text{erg}}(S) \) such that \( h(S, \mu) = \sup_{\nu \in \mathcal{P}(S)} h(S, \nu) \)

**Theorem (Gurevič)**

*If \( X \) is transitive then it has at most one mme \( \mu \)*

*In this case, \( X \) is called **positive recurrent (PR)** and \( \mu \) is (fully-supported, Markov) Parry measure, isomorphic to Bernoulli \( \times \mathbb{Z}/p\mathbb{Z} \) (or simply: **periodic-Bernoulli**)*
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Background on Markov shifts

Spectral decomposition and their entropy

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If \( X \) is transitive then it has at most one mme \( \mu \).
In this case, \( X \) is called **positive recurrent (PR)** and \( \mu \) is (fully-supported, Markov) Parry measure, isomorphic to Bernoulli \( \times \mathbb{Z}/p\mathbb{Z} \) (or simply: **periodic-Bernoulli**).
Markov shifts – periodic points

Markov shift $S : X \to X$ defined by graph $G$
Assume **transitive** with $h(S) < \infty$ and period $p$

Classical positive matrix theory yields:

**Theorem**

$[F] := \{x \in X : x_0 \in F\}$ for some finite set $F \neq \emptyset$ of vertices of $G$
$\text{Fix}(S^j, F) := \{x \in X : S^j x = x \text{ and } O(x) \cap [F] \neq \emptyset\}$

If $G$ is not PR, $\lim_{k \to \infty} |\text{Fix}(S^{kp}, F)| e^{-kp h(S)} = 0$
If $G$ is PR, $\lim_{k \to \infty} |\text{Fix}(S^{kp}, F)| e^{-kp h(S)} = p$

**Counter-examples**

Without restricting to a finite set of vertices:
- $\text{Fix}(S^{kp}, G)$ can be infinite
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Factors of Markov shifts – pathology from loss of entropy

\((X, S), (Y, T)\) selfmaps

**Definition**

A **factor map** \(\pi : (X, S) \rightarrow (Y, T)\) is an onto map \(\pi : X \rightarrow Y\) with \(\pi \circ S = T \circ \pi\)

\(S : X \rightarrow X\) is called the extension and \(T : Y \rightarrow Y\) the factor

**Claim:** Without additional assumptions, their factors can be very different

**Pathology 1** Bad MMEs

**MMEs of Markov shifts**

A Markov shift has at most countably many mme’s and each is periodic-Bernoulli (*1)

**Counter-example of Boyle-B**

There are continuous factors of mixing SFTs whose mme’s include *uncountably* many isomorphic copies of an *arbitrary ergodic automorphism* with positive entropy

**Remark (Sarig, applying Ornstein’s theory)**

Any finite-to-one, \(C^0\) factor of a Markov shift satisfies (*1)
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$\mu$ totally ergodic $\iff$ $\text{Per}(S, \mu) = \{1\}$

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There is a finite-to-one, continuous factor of an SFT which has a unique totally ergodic measure with nonzero entropy

Compare:

Remark

A Markov shift has infinitely many totally ergodic measures with nonzero entropy, or none
Factors of Markov shifts – pathology of finite-to-one factors

**Pathology 2** Finite-to-one factors can still be bad "at a period"

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The Bowen relation

\( \pi : (X, S) \to (Y, T) \) factor map with

- \((X, S)\) a symbolic system: \( X \subset A^\mathbb{Z} \) and \( S : (x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}} \)
- \((Y, T)\) arbitrary

Let \([a] := \{ x \in X : x_0 = a \}\) for \( a \in A \)

**Bowen relation (Boyle-B)**

The **Bowen relation** of \( \pi \) is the symmetric relation over \( A \) defined by:

\[ a \sim b \iff \pi([a]) \cap \pi([b]) \neq \emptyset \]

It is of **finite type** if \(|\{ b \in A : a \sim b \}| < \infty\) for each \( a \in A \)

The factor \( \pi : X \to Y \) has the **Bowen property** if, for all \( x, x' \in X \)

\[ \pi(x) = \pi(x') \iff \forall n \in \mathbb{Z} \ x_n \sim x'_n \]

Recall Bowen **On Axiom A diffeomorphisms (1978):**

We will now discuss the quotient map \( \pi : \Sigma_A \to \Omega_i \) defined by \( \pi(x) = \bigcap_{j=-\infty}^{\infty} \bigcup_{j=-\infty}^{n} R_{x_j} \).

The Markov partition \( \mathcal{C} = \{ R_1, \ldots, R_n \} \) is taken so that \( 2 \max_{1 \leq j \leq n} \text{diam}(R_j) \) is less than an expansive constant. Define a relation \( \sim \) on \( \{1, \ldots, n\} \) by \( j \sim k \) iff \( R_j \cap R_k \neq \emptyset \). Define \( \sim \) on \( \Sigma_A \) by \( x \sim y \) iff \( x_r \sim y_r \), \( \forall r \in \mathbb{Z} \). It is easy to prove that \( \pi(x) = \pi(y) \) precisely if \( x \sim y \).
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Bowen factors of finite type

Examples

1. The coding of an Axiom A diffeomorphism induced by Markov partitions defines a finite-to-one Bowen factor

2. Expansive continuous factors, in particular of SFTs (ie, Fried’s finitely presented systems)

3. Any one-block code between two symbolic systems is a Bowen factor. Note: need not preserve entropy, even if of finite type.

#-recurrent set (Sarig)

\( X^\# \) is the set of \( x \in X \) s.t. \( |\{ n \leq 0 : x_n = a \}| = |\{ n \geq 0 : x_n = b \}| = \infty \) for some \( a, b \)

Theorem (Sarig)

Given a surface \( C^{1+}\)-diffeomorphism, and \( \chi > 0 \), let \( \pi_\chi : X_\chi \to M_\chi \) be Sarig’s Hölder continuous factor map with \( X_\chi \) a Markov shift and \( M_\chi \) its \( \chi \)-hyperbolic part \( \pi_\chi \) restricted to \( X^\#_\chi \) is finite-to-one and a Bowen factor of finite type (Boyle-B)
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Almost Borel conjugacy to a Markov shift

$(X, S)$ a Markov shift and $(X_i)_{i \in I}$ be its spectral decomposition

**Theorem (Boyle-B)**

Let $\pi: (X, S) \to (Y, T)$ be a Borel factor of a Markov shift with $h(S) < \infty$

Assume for all $i \in I$: the restriction $\pi|_{X_i^\#}$ is finite-to-one with the Bowen property

Then $(Y, T)$ is almost Borel conjugate modulo zero entropy to a Markov shift

**Main ingredients of the proof**

- Hochman’s almost Borel generator theorem
- Countable unions of Markov shifts are almost Borel conjugate to Markov shifts, etc.
- Low entropy part (injectivity from marker lemma)
- Top entropy part (a.e. injectivity a la Manning)

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- Top entropy part (a.e. injectivity a la Manning)

**Theorem (Boyle-B)**

Any $C^{1+}$-diffeomorphism of a compact surface is almost Borel conjugate mod zero entropy to a Markov shift
Almost Borel classification of $C^{1+}$-diffeos

We are reduced to the classification of Markov shifts (Boyle-B)

We need the set of periods of a measure

\[ \text{Per}(f, \mu) := \{ p \geq 1 : e^{2i\pi/p} \in \sigma_{\text{rat}}(f, \mu) \} \text{ for } \mu \in \mathbb{P}_{\text{erg}}(f, \mu) \]

and to maximize entropy at a period:

Corollary (Boyle-B)

For each $p \geq 1$ let:

- $H(p) := \sup^+ \{ h(f, \mu) : \mu \in \mathbb{P}_{\text{erg}}(f), \max \text{Per}(f, \mu) | p \}$
- $M(p) := |\{ \mu \in \mathbb{P}_{\text{erg}}(f) : \max \text{Per}(f, \mu) = p, h(f, \mu) = H(p) \}|$

Then $(H, M)$ is a complete invariant of almost Borel conjugacy mod zero entropy among $C^{1+}$-diffeomorphisms $f$ of compact surfaces
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Almost Borel classification for $C^\infty$ diffeos

A $C^\infty$-diffeo of a compact surface with $h_{\text{top}}(f) > 0$: purely topological invariant

Definition

The homoclinic class of a hyperbolic periodic orbit $O$ is

$$HC(O) := W^s(O) \cap W^u(O)$$

It has period $p \geq 1$ if $HC(O) = \bigcup_{k=0}^{p-1} f^k(A)$ and

$$\text{int}_{HC(O)}(A \cap f^k(A)) = \emptyset$$

for $0 < k < p$ and $f^p : A \to A$ topologically mixing.

Corollary of B-Crovisier-Sarig

Let $(HC(O_j))_{j \in J}$ be the distinct homoclinic classes and $p_j$ their periods.

The previous complete invariant $(H, M)$ of almost Borel conjugacy mod zero entropy satisfies

- $H(p) := \sup_{j:p_j|p} h_{\text{top}}(f, HC(O_j))$
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Example

Among top. mixing surface $C^\infty$ diffeos, the topological entropy is a complete invariant for almost Borel conjugacy mod zero entropy (in fact Borel conjugacy mod periodic points).
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Among top. mixing surface $C^\infty$ diffeos, the topological entropy is a complete invariant for almost Borel conjugacy mod zero entropy (in fact Borel conjugacy mod periodic points)
Lower bounds on periodic points

As for almost conjugacy between SFTs, we have *magic word isomorphisms*

**Theorem (B, in preparation)**

Let $\pi : (X, S) \to (Y, T)$ be a Borel factor of a transitive Markov shift with $h(T) < \infty$

Assume the restriction $\pi|_{X^\#}$ is finite-to-one and a Bowen factor of finite type

Then $\exists$ $X$-word $w$ s.t. $X_w := \{ x \in X : w \text{ occurs i.o. in the past and future} \}$ satisfies:

$$\exists 1 \leq d < \infty \ \forall x \in X_w \ |\pi^{-1}(\pi(x)) \cap X_w| = d$$

Proof: $\deg_\pi(v_1 \ldots v_n, i) := |\{ u_i : u \in A^n, \ u_1 \sim v_1, \ldots, u_n \sim v_n \}|$

**Theorem (B)**

Let $f$ be a $C^\infty$-diffeomorphism of a compact surface and $0 < \chi < h_{\text{top}}(f)$

Let $\mu_1, \ldots, \mu_r$ be its mme’s, with $\mu_i$ isomorphic to Bernoulli $\times \mathbb{Z}/p_i\mathbb{Z}$

Let $p := \text{lcm}(p_1, \ldots, p_r)$

$$\lim_{n \to \infty, p|n} \left| \{ x \in M : f^n x = x, \ \chi\text{-hyperbolic} \} \right| e^{-nh_{\text{top}}(f)} \geq p_1 + \cdots + p_r$$
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Conclusion

The Bowen property gives good control of the (necessary?) failure of injectivity

Number of periodic points

Are there $f$ compact surface $C^\infty$-diffeo and $\chi > 0$ s.t.

$$\limsup_{n \to \infty} |\{x = f^n x, \chi\text{-hyperbolic}\}| e^{-n h_{\text{top}}(f)} = \infty?$$

True Borel conjugacy

For a $C^\infty$-diffeo, each $HC(O)$ is almost Borel conjugate mod zero entropy to a transitive Markov shift

Can this be strengthened to almost Borel mod periodic points?

The dream coding

Is the ($\chi$?)hyperbolic part of a surface diffeomorphism a factor of a Markov shift s.t.

1. the factor is Hölder-continuous
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