Discount rates: the equilibrium approach

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- Agents indexed by $i = 1, \ldots N$,
  - current endowment at time t denoted by e_t^{*i}
  - VNM utility function of the form

$$E\left[\int_0^T u_i\left(t, c_t\left(\omega\right)\right) dt\right]$$

The equilibrium

$$e^* \equiv \sum_{i=1}^{N} e^{*^i}$$

$$de_t^* = \mu_t e_t^* dt + \sigma_t e_t^* dW_t \qquad e_0^* = 1$$

Definition (Arrow-Debreu equilibrium)

A positive price process q^* and optimal consumption plans $\left(y^{*^i}\right)_{i=1,\ldots,N}$ s.t. markets clear, i.e. $\sum_{i=1}^{N} y^{*^i} = e^*$ with

$$y^{*^i} = \underset{E\left[\int_0^T q_t\left(y_t^i - e_t\right)dt\right] \leq 0}{\arg\max} E\left[\int_0^T u_i\left(t, c_t\right) dt\right]$$

- Characterized by

$$u_i'\left(t, y_t^{*^i}\right) = \lambda_i q_t^*$$

The representative agent

Theorem (Negishi)

Let us consider u defined by

$$u(t,x) = \max_{\sum x_i = x} \sum \lambda_i u_i(t,x_i).$$

The equilibrium price q^* is an equilibrium price in the economy with 1 agent (representative agent) with an initial wealth e^*.

- The equilibrium is characterized by

$$u'(t,e_t^*) = q_t^*.$$

The riskless asset

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- qS^0 is a martingale and

$$r_t = -\mu_{q^*}$$

Short-term rate

- Power utility functions

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- Short rate

$$r_t = \underbrace{\rho}_{\text{time preference rate}} + \underbrace{\frac{1}{\eta}\mu}_{\text{wealth effect}} - \underbrace{\frac{1}{2}\frac{1}{\eta}(1+\frac{1}{\eta})\sigma^2}_{\text{precautionary saving}}$$

Discount factor and rate

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\begin{aligned}
A_t &= E\left[q_t\right] && \text{(Discount factor)} \\
R_t &= -\frac{1}{t}\ln E\left[q_t\right] && \text{(Discount rate)}
\end{aligned}
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- If all the parameters are constant and no risk

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Beliefs heterogeneity

- Agent i maximizes $E^{Q^i}\left[\int_0^T u_i\left(t, c_t\left(\omega\right)\right)dt\right]$ with $\frac{dQ^i}{dP} = M_T^i$ and $dM_t^i = \delta_t^i M_t^i dW_t$

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- From agent i point of view

$$\begin{aligned} de_t^* &= \mu_t^i e_t^* dt + \sigma_t e_t^* dW_t^{Q^i} \qquad e_0^* = 1 \\ \mu_t^i &= \mu_t + \delta_t^i \sigma_t \end{aligned}$$

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- Divergence of opinion about the growth rate
- $u_i\left(t, c_t\left(\omega\right)\right) = D_t^i c^{1-\frac{1}{\eta}}$, with $D_t^i \equiv \exp\left(-\int_0^t \rho^i\left(s, \omega\right) ds\right)$ (heterogeneous time preference rates)

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- Beliefs dispersion $\longrightarrow$ additional risk or uncertainty $\longrightarrow$ lower discount rates ?
- DDR ? Trajectory of the decline ?

Declining discount rate

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- Empirical and experimental evidence: individual hyperbolic discounters.

The equilibrium

Definition

Arrow-Debreu equilibrium : a positive price process q^* and a family of optimal consumption plans $\left(y^{*^i}\right)_{i=1,\dots,N}$ such that markets clear, i.e.

$$\begin{cases} y^{*^i} = y^i\left(q^*, M^i, D^i, e^{*^i}\right) \\ \sum_{i=1}^N y^{*^i} = e^* \end{cases}$$

where

$$y^i(q, M, D, e) = \underset{E\left[\int_0^T q_t\left(y_t^i - e_t\right)dt\right] \le 0}{\arg\max} E\left[\int_0^T M_t D_t u(c_t)\, dt\right].$$

Characterized by

$$D_t^i M_t^i u'\left(y_t^{*^i}\right) = \lambda_i q_t^*$$

Aggregation of individual beliefs and time-preferences

We let N^i denote the individual composite characteristic $M^i D^i$.

Theorem

We have $q_t^* = N_t u'(e_t^*)$ *with* $N = \left[\sum_{i=1}^N \gamma_i \left(N_t^i\right)^\eta\right]^{1/\eta}$.
Furthermore, $N = BDM$ *with*

$$
\begin{aligned}
dM_t &= \delta_M M_t dW_t, \qquad \delta_M = \sum_{i=1}^N \tau_i \delta^i \\
dB_t &= \rho_B B_t dt, \qquad \rho_D = \sum_{i=1}^N \tau_i \rho^i \\
\rho_B &= \frac{\eta - 1}{2} \left(\sum_{i=1}^N \tau_i \left(\delta^i\right)^2 - \delta_M^2 \right) = \frac{\eta - 1}{2} Var^\tau (\delta)
\end{aligned}
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Consensus Arrow-Debreu prices and consensus socially efficient discount factors

- q^{i^*} equilibrium price if agent i only

Corollary

We have

$$q_t^* = \left[\sum_{i=1}^{N} \gamma_i \left(q_t^{i^*} \right)^{\eta} \right]^{1/\eta}$$

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- $A_t \equiv E\left[q_t^*\right]$, discount factor
- $A_t^i \equiv E\left[q_t^{i^*}\right]$, discount factor if agent i only
- Can the socially efficient discount factor A_t be represented as an average of the individual A_t^i?

Consensus socially efficient discount factors

Theorem

- *If* $\delta^i \equiv \delta$ *and* $\rho^i(s,\omega) \equiv \rho^i(s)$,

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  - *equality only when divergence is deterministic* (N_i / N_j *is deterministic for all* i, j).

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- In general,
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  2. The average is a weighted average
  3. A can not be reduced to this average : there is an aggregation bias (upward or downward depending on η)
- Beliefs heterogeneity can be interpreted as more risk/uncertainty or less information : same impact on the trade-off between today's consumption and future consumption (Gollier-Kimball 1996, Gollier, 2000)

Theorem

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\begin{aligned}
r^f &= \underbrace{\sum_{i=1}^{N} \tau_i \rho^i}_{Agg\ time\ pref} + \underbrace{\frac{(\mu + \delta_M \sigma)}{\eta}}_{Agg\ exp.\ wealth} - \underbrace{\frac{\left(1+\frac{1}{\eta}\right)}{2\eta}\sigma^2}_{stand\ prec\ sav} - \underbrace{\frac{\eta-1}{2} Var^{\tau}(\delta)}_{Agg\ bias} \\
&= \sum_{i=1}^{N} \tau_i \left(r^i\right)^f - \frac{1}{2}(\eta - 1)\, Var^{\tau}(\delta) \\
&= r^f(stdard) + \sum_{i=1}^{N} \tau_i \rho^i + \frac{1}{\eta}\left(\sum_{i=1}^{N} \tau_i \delta^i\right)\sigma - \frac{\eta-1}{2} Var^{\tau}(\delta)
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- Average level (patience, pessimism)

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- Beliefs dispersion (depends on $\eta > 1, \eta < 1$). For $\eta > 1$: more risk $\Rightarrow$ more saving $\Rightarrow$ downward pressure on r^f

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Long term considerations

Theorem

Suppose that for all i, the individual asymptotic discount rate $R_\infty^i \equiv \lim_{t \leq T; t,T \to \infty} R_t^{T,i}$ exists. Moreover, we suppose $\gamma_I(T) \geq \varepsilon > 0$ for $R_\infty^I = \inf\{R_\infty^i; i = 1, \ldots N\}$). Then,

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  - Homogeneous time preference rate : most pessimistic rate

Long term considerations

Theorem

Suppose that for all i, the individual asymptotic discount rate $R_\infty^i \equiv \lim_{t \leq T; t, T \to \infty} R_t^{T,i}$ exists. Moreover, we suppose $\gamma_I(T) \geq \varepsilon > 0$ for $R_\infty^I = \inf \{R_\infty^i ; i = 1, ...N\}$). Then,

$$R_\infty \equiv \lim_{t \leq T; t, T \to \infty} R_t^T = \inf \{R_\infty^i , i = 1, ..., N\} .$$

- The aggregation bias vanishes in the long run
- The relevant asymptotic behavior is the one with the lowest discount rate
  - Homogeneous beliefs : lowest rate of impatience (Gollier-Zeckhauser in a deterministic setting)
  - Homogeneous time preference rate : most pessimistic rate
- $\gamma_I(T) \geq \varepsilon > 0$

Specific settings
Logarithmic case

- $A_t = \sum_i \gamma_i A_t^i$ (arithmetic average)

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R_t^T &= \mu - \sigma^2 - \frac{1}{t} \log \left[\sum_i \gamma_i^T \exp^{-\left(\rho^i + \sigma\delta^i\right)t} \right] \\
R_0 &= \mu - \sigma^2 + \sum_i w_i \left(\rho^i + \sigma\delta^i\right) \\
R_\infty &= \mu - \sigma^2 + \inf \left(\rho^i + \sigma\delta^i\right) \\
\gamma_i^T &= \frac{w_i \rho^i \left(1 - \exp -\rho^i T\right)^{-1}}{\sum_j w_j \rho^j \left(1 - \exp -\rho^j T\right)^{-1}}
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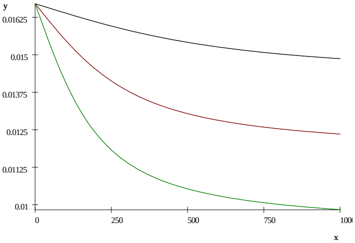
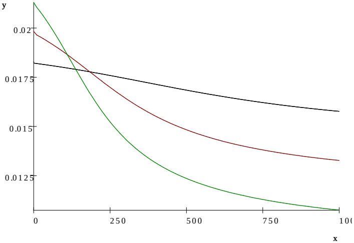
- $R_0 \geq R_\infty$ and R_t^T decreases with t





$w_1 = w_2, \delta_1 = -\delta_2$

different levels of δ

$R_t \searrow$ with δ, pessim. limit

same starting point

$cov(w, \delta) > 0$

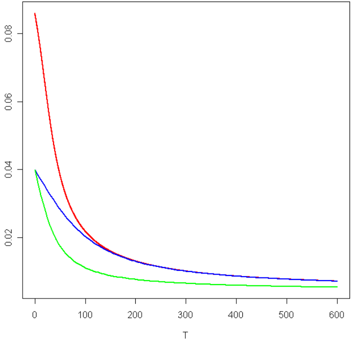
$R_t \nearrow$ with δ for small t

$R_t \searrow$ with δ for large t

$\neq$ starting point

Specific settings

Power utility functions, eta<1



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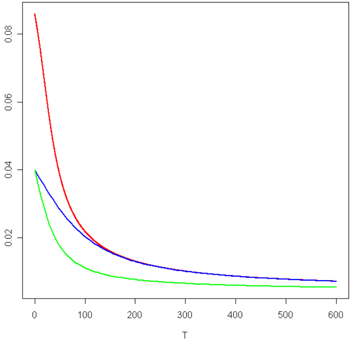

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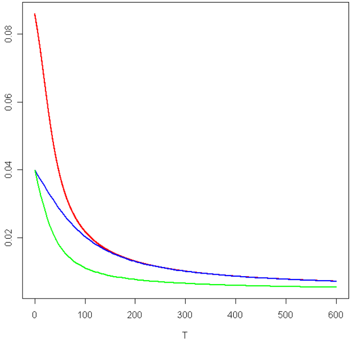

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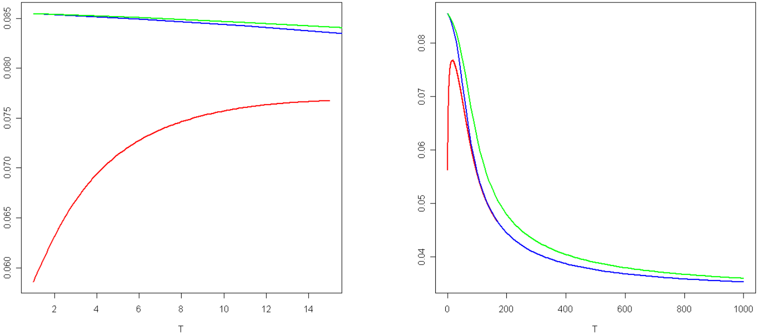

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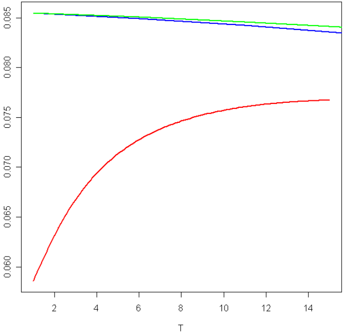

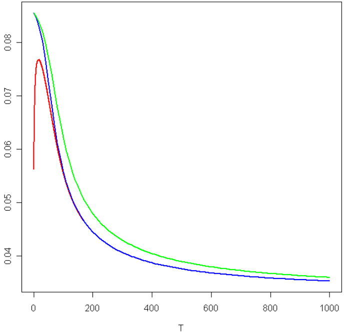

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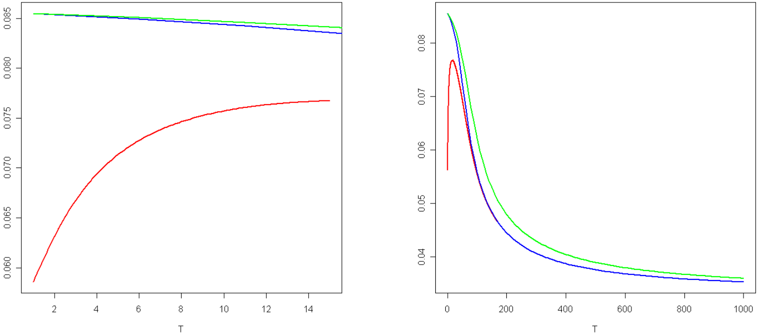

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Divergence of experts' opinions

Let us consider n experts : $\left(R^i\right)_{i=1,\dots,n}$ as in Weitzman (2001)

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- $R^i \equiv \rho_i + \mu_i - \sigma_i^2$, equilibrium discount rate if the economy was made of group i agents only

$$
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R_t &\equiv -\frac{1}{t}\ln\sum_{i=1}^{N}\frac{w_i\rho_i}{\sum_{j=1}^{N}w_j\rho_j}\exp -R^i t,\\
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- bias towards the more impatient agents

In the case of homogeneous beliefs $(\mu_i = \mu, \sigma_i = \sigma)$

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- both approaches coincide if the Pareto weights are proportional to $w_i \rho_i$.

$$R_t \leq \sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} R^i,$$
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If ρ_i and b_i are independent,

$$r_t \geq \sum_{i=1}^{N} \frac{w_i \exp -r^i t}{\sum_{j=1}^{N} w_j \exp -r^j t} r^i = r_t^W$$

General properties and specific distributions

Corollary

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3. $\rho_i = \rho$, *and normal distribution* $\mathcal{N}\left(m, v^2\right)$ *on* $b_i = \mu_i - \sigma_i^2$, $R_t = \rho + m - \frac{v^2}{2} t$ *(Reinschmidt, 2002)*

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4. *If* $\rho_i \sim \gamma(\alpha_1, \beta_1)$ *and* $b_i = \mu_i - \sigma_i^2 \sim \gamma(\alpha_2, \beta_2)$ *independent, then* $r_t = \frac{\alpha_1 + 1}{\beta_1 + t} + \frac{\alpha_2}{\beta_2 + t}$

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5. *As in 4. and* $\beta_1 = \beta_2 = \beta$ *then* $R^i \sim \gamma(\alpha, \beta)$ *with* $\alpha = \alpha_1 + \alpha_2$ *and* $r_t = r_t^W + \frac{1}{\beta + t}$

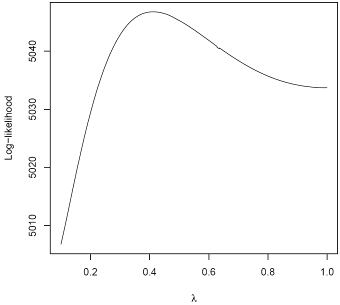

Figure: Calibration with two independent gamma distr. on Weitzman (2001)'s data. We assume that the two distributions are homothetic and calibrate in order to fit the mean and the variance of the empirical distribution. Weitzman (2001)'s statistical model corresponds to $\lambda = 1$. We maximize the log-likelihood and obtain $\lambda = 0.4116$.

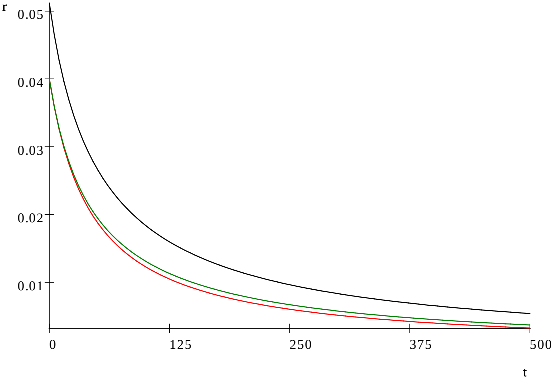

Figure: Marginal discount rate curve through our calibration (upper curve) and discount rate curve of Weitzman (2001) (lower curve). The intermediate curve represents, with our calibration, the unweighted average.

Time period	Name	Numerical value	Approx. rate	Weitzman's num. value	Weitzman's appr. rate
Within years 1 to 5 hence	Immediate Future	4.99%	5%	3.89%	4%
Within years 6 to 25 hence	Near Future	4.23%	4%	3.22%	3%
Within years 26 to 75 hence	Medium Future	2.82%	3%	2.00%	2%
Within years 76 to 300 hence	Distant Future	1.50%	1.5%	0.97%	1%
Within years more than 300 hence	Far-Distant Future	0.16%	0%	0.08%	0%

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- an increase in the variance v_1^2 of the individual pure time preference rates (ρ_i) decreases the marginal discount rate r_t for t large enough.

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3. If all the agents have the same beliefs, then a MLR shift in the distribution of the $\left(R^i\right)$ increases the discount rate R_t for all horizons.

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- The arithmetic average corresponds to a utility maximizing agent that considers each individual belief as a possible scenario while our approach corresponds to a central planner that maximizes the social welfare

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