Discount rates: the equilibrium approach

Elyès Jouini

PIMS, 2008
The classical model

- A time horizon $T$ and a filtered probability space $(\Omega, F, (F_t)_{t \in [0, T]}, P)$
The classical model

- A time horizon \( T \) and a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\)
- Agents indexed by \( i = 1, \ldots, N \),

The classical model

- Equilibrium approach
- Elyès Jouini

Introduction

The classical model

Beliefs heterogeneity

Aggregation

Discount factors

Discount rates

Long term

Specific settings

How to aggregate experts opinions

Impact of stochastic shifts
The classical model

- A time horizon $T$ and a filtered probability space $(\Omega, F, (F_t)_{t\in[0,T]}, P)$
- Agents indexed by $i = 1, ..., N$
  - current endowment at time $t$ denoted by $e_t^i$
The classical model

- A time horizon $T$ and a filtered probability space $(\Omega, F, (F_t)_{t \in [0, T]}, P)$
- Agents indexed by $i = 1, \ldots, N$,
  - current endowment at time $t$ denoted by $e_t^i$
  - VNM utility function of the form

$$E \left[ \int_0^T u_i(t, c_t(\omega)) \, dt \right]$$
The equilibrium

\[ e^* \equiv \sum_{i=1}^{N} e^{*i} \]

\[ de_t^* = \mu_t e_t^* dt + \sigma_t e_t^* dW_t \quad e_0^* = 1 \]

### Definition (Arrow-Debreu equilibrium)

A positive price process \( q^* \) and optimal consumption plans \( (y^{*i})_{i=1,...,N} \) s.t. markets clear, i.e. \( \sum_{i=1}^{N} y^{*i} = e^* \) with

\[ y^{*i} = \arg \max_u E \left[ \int_0^T u_i (t, c_t) \, dt \right] \]

\[ E \left[ \int_0^T q_t (y^{*i}_t - e_t) \, dt \right] \leq 0 \]

- Characterized by

\[ u'_i \left( t, y^{*i}_t \right) = \lambda_i q^*_t \]
The representative agent

**Theorem (Negishi)**

Let us consider $u$ defined by

$$u(t, x) = \max \sum \lambda_i u_i(t, x_i).$$

The equilibrium price $q^*$ is an equilibrium price in the economy with 1 agent (representative agent) with an initial wealth $e^*$.

- The equilibrium is characterized by

  $$u'(t, e^*_t) = q^*_t.$$
The riskless asset

- We consider an asset with a (riskless) dynamics

\[ dS_t^0 = r_t(t, \omega)S_t^0 \, dt \]
The riskless asset

- We consider an asset with a (riskless) dynamics

\[ dS_t^0 = r_t(t, \omega)S_t^0 \, dt \]

- We have \( S_0^0 = E[q_t S_t] \)
We consider an asset with a (riskless) dynamics

\[ dS_t^0 = r_t(t, \omega) S_t^0 \, dt \]

We have \( S_0^0 = E[q_t S_t] \)

More generally, for \( B \in F_s \)

\[ E[1_B(q_t S_t - q_s S_s)] = 0 \] (no arbitrage)
The riskless asset

- We consider an asset with a (riskless) dynamics
  \[ dS^0_t = r_t(t, \omega)S^0_t dt \]
- We have \( S^0_0 = E[q_t S_t] \)
- More generally, for \( B \in F_s \)
  \[ E[1_B(q_t S_t - q_s S_s)] = 0 \quad \text{(no arbitrage)} \]
- \( qS^0 \) is a martingale and
  \[ r_t = -\mu q^* \]
Short-term rate

- Power utility functions

\[ u(t, c) = \exp \left( - \int_0^t \rho_s \, ds \right) \times c^{1 - \frac{1}{\eta}} \]
Short-term rate

- **Power utility functions**
  \[ u(t, c) = \exp \left( - \int_0^t \rho_s ds \right) \times c^{1 - \frac{1}{\eta}} \]

- **Short rate**
  \[
  r_t = \rho_{\text{time preference rate}} + \frac{1}{\eta} \mu_{\text{wealth effect}} - \frac{1}{2} \frac{1}{\eta} \left( 1 + \frac{1}{\eta} \right) \sigma^2_{\text{precautionary saving}}
  \]
Discount factor and rate

\[ A_t = E[q_t] \] (Discount factor)

\[ R_t = -\frac{1}{t} \ln E[q_t] \] (Discount rate)

- If all the parameters are constant and no risk

\[ R = \rho + \frac{1}{\eta} \mu \] (Ramsey)

- time preference rate
- wealth effect
Discount factor and rate

\[ A_t = E[q_t] \quad \text{(Discount factor)} \]
\[ R_t = -\frac{1}{t} \ln E[q_t] \quad \text{(Discount rate)} \]

- If all the parameters are constant and no risk

\[ R = \rho + \frac{1}{\eta} \mu \quad \text{(Ramsey)} \]

- If \( \sigma \neq 0 \)

\[ R = \rho + \frac{1}{\eta} \mu - \frac{1}{2\eta} \left(1 + \frac{1}{\eta}\right) \sigma^2 \quad \text{precautionary saving} \]
Beliefs heterogeneity

- Agent $i$ maximizes $E^{Q^i} \left[ \int_0^T u_i (t, c_t (\omega)) \, dt \right]$ with

$$\frac{dQ^i}{dP} = M_T^i \quad \text{and} \quad dM_t^i = \delta_t^i M_t^i \, dW_t$$
Beliefs heterogeneity

- Agent $i$ maximizes $E^{Q^i} \left[ \int_0^T u_i(t, c_t(\omega)) \, dt \right]$ with
  \[ \frac{dQ^i}{dP} = M^i_T \quad \text{and} \quad dM^i_t = \delta^i_t M^i_t \, dW_t \]

- From agent $i$ point of view
  \[ de^*_t = \mu^i_t e^*_t \, dt + \sigma^i_t e^*_t \, dW^Q^i_t \quad e^*_0 = 1 \]
  \[ \mu^i_t = \mu_t + \delta^i_t \sigma_t \]
Beliefs heterogeneity

Agent $i$ maximizes $E^{Q^i} \left[ \int_0^T u_i (t, c_t (\omega)) \, dt \right]$ with
\[
\frac{dQ^i}{dP} = M_T^i \quad \text{and} \quad dM_t^i = \delta_t^i M_t^i dW_t
\]

From agent $i$ point of view

\[
de_t^* = \mu_t^i e_t^* dt + \sigma_t e_t^* dW_t^{Q^i} \quad \text{with} \quad e_0^* = 1
\]
\[
\mu_t^i = \mu_t + \delta_t^i \sigma_t
\]

Divergence of opinion about the growth rate
Beliefs heterogeneity

Agent $i$ maximizes $E^{Q^i} \left[ \int_0^T u_i (t, c_t (\omega)) \, dt \right]$ with
\[
\frac{dQ^i}{dP} = M_T^i \quad \text{and} \quad dM_t^i = \delta_t^i M_t^i dW_t
\]
From agent $i$ point of view
\[
de_t^* = \mu_t^i e_t^* \, dt + \sigma_t e_t^* dW_t^{Q^i} \quad e_0^* = 1
\]
\[
\mu_t^i = \mu_t + \delta_t^i \sigma_t
\]
Divergence of opinion about the growth rate
\[
u_i (t, c_t (\omega)) = D_t^i c^{1 - \frac{1}{\eta}}, \quad \text{with}
\]
\[
D_t^i \equiv \exp \left( - \int_0^t \rho_t^i (s, \omega) \, ds \right) \quad \text{(heterogeneous time preference rates)}
\]
Main questions

- Representative agent? (consensus belief, consensus time preference rate)
Main questions

- Representative agent? (consensus belief, consensus time preference rate)
- Socially efficient discount factor = average of the individually anticipated ones?
Main questions

- Representative agent? (consensus belief, consensus time preference rate)
- Socially efficient discount factor = average of the individually anticipated ones?
- Risk-free rates and discount rates?
Main questions

- Representative agent? (consensus belief, consensus time preference rate)
- Socially efficient discount factor = average of the individually anticipated ones?
- Risk-free rates and discount rates?
- Beliefs dispersion → additional risk or uncertainty → lower discount rates?
Main questions

- Representative agent? (consensus belief, consensus time preference rate)
- Socially efficient discount factor = average of the individually anticipated ones?
- Risk-free rates and discount rates?
- Beliefs dispersion → additional risk or uncertainty → lower discount rates?
- DDR? Trajectory of the decline?
Declining discount rate

- Weitzman (1998) : « To think about the distant future in terms of standard discounting is to have an uneasy intuitive feeling that something is wrong somewhere»
Declining discount rate

- Weitzman (1998): «To think about the distant future in terms of standard discounting is to have an uneasy intuitive feeling that something is wrong somewhere»
- DDR in a deterministic setting: known changes in growth rate and/or in risk aversion
Declining discount rate

- Weitzman (1998) : « To think about the distant future in terms of standard discounting is to have an uneasy intuitive feeling that something is wrong somewhere»
- DDR in a deterministic setting: known changes in growth rate and/or in risk aversion
- DDR with uncertainty
Declining discount rate

- Weitzman (1998) : « To think about the distant future in terms of standard discounting is to have an uneasy intuitive feeling that something is wrong somewhere»
- DDR in a deterministic setting: known changes in growth rate and/or in risk aversion
- DDR with uncertainty
  - Uncertainty on the discount rate itself (certainty equivalent analysis, Weitzman, 1998, 2001)
Declining discount rate

- Weitzman (1998): «To think about the distant future in terms of standard discounting is to have an uneasy intuitive feeling that something is wrong somewhere»

- DDR in a deterministic setting: known changes in growth rate and/or in risk aversion

- DDR with uncertainty
  - Uncertainty on the discount rate itself (certainty equivalent analysis, Weitzman, 1998, 2001)
Declining discount rate

- Weitzman (1998): «To think about the distant future in terms of standard discounting is to have an uneasy intuitive feeling that something is wrong somewhere»
- DDR in a deterministic setting: known changes in growth rate and/or in risk aversion
- DDR with uncertainty
  - Uncertainty on the discount rate itself (certainty equivalent analysis, Weitzman, 1998, 2001)
- Sustainable welfare function à la Chilchinisky (1997) and Li and Löfgren (2000).
Declining discount rate

- **Weitzman (1998)**: «To think about the distant future in terms of standard discounting is to have an uneasy intuitive feeling that something is wrong somewhere»

- **DDR in a deterministic setting**: known changes in growth rate and/or in risk aversion

- **DDR with uncertainty**
  - Uncertainty on the discount rate itself (certainty equivalent analysis, Weitzman, 1998, 2001)

- **Sustainable welfare function à la Chilchinisky (1997) and Li and Löfgren (2000).**

- **Empirical and experimental evidence**: individual hyperbolic discounters.
The equilibrium

**Definition**

Arrow-Debreu equilibrium: a positive price process \( q^* \) and a family of optimal consumption plans \( (y^*)_i \) such that markets clear, i.e.

\[
\begin{cases}
  y^*_i = y^i_q (q^*, M^i, D^i, e^*_i) \\
  \sum_{i=1}^{N} y^*_i = e^*
\end{cases}
\]

where

\[
y^i (q, M, D, e) = \arg \max E \left[ \int_0^T M_t D_t u(c_t) \, dt \right].
\]

Characterized by

\[
D^i_t M^i_t u' (y^*_i) = \lambda_i q^*_t
\]
Aggregation of individual beliefs and time-preferences

We let $N^i$ denote the individual composite characteristic $M^i D^i$.

**Theorem**

We have $q^*_t = N_t u' \left( e^*_t \right)$ with $N = \left[ \sum_{i=1}^{N} \gamma_i \left( N^i_t \right)^{\eta} \right]^{1/\eta}$.

Furthermore, $N = BDM$ with

$$dM_t = \delta M_t dW_t, \quad \delta = \sum_{i=1}^{N} \tau_i \delta^i$$

$$dB_t = \rho_B B_t dt, \quad \rho_D = \sum_{i=1}^{N} \tau_i \rho^i$$

$$\rho_B = \frac{\eta - 1}{2} \left( \sum_{i=1}^{N} \tau_i \left( \delta^i \right)^2 - \delta_M^2 \right) = \frac{\eta - 1}{2} \text{Var}^{\tau} (\delta)$$
Consensus Arrow-Debreu prices and consensus socially efficient discount factors

$q^{i\ast}$ equilibrium price if agent $i$ only

**Corollary**

We have

$$q_t^{\ast} = \left[ \sum_{i=1}^{N} \gamma_i \left( q_t^{i\ast} \right)^{\eta} \right]^{1/\eta}$$
Consensus Arrow-Debreu prices and consensus socially efficient discount factors

- $q^*_i$ equilibrium price if agent $i$ only

**Corollary**

*We have*

$$q^*_t = \left[ \sum_{i=1}^{N} \gamma_i \left( q^*_i \right)^\eta \right]^{1/\eta}$$

- $A_t \equiv E[q^*_t]$, discount factor
Consensus Arrow-Debreu prices and consensus socially efficient discount factors

- $q^{i*}$ equilibrium price if agent $i$ only

**Corollary**

*We have*

$$q_t^* = \left[ \sum_{i=1}^{N} \gamma_i (q_t^{i*})^\eta \right]^{1/\eta}$$

- $A_t \equiv E[q_t^*]$, discount factor
- $A_t^i \equiv E[q_t^{i*}]$, discount factor if agent $i$ only
Consensus Arrow-Debreu prices and consensus socially efficient discount factors

- $q^*_i$ equilibrium price if agent $i$ only

**Corollary**

*We have*

$$q^*_t = \left[ \sum_{i=1}^{N} \gamma_i \left( q^*_i \right)^{\eta} \right]^{1/\eta}$$

- $A_t \equiv E[q^*_t]$, discount factor
- $A^i_t \equiv E[q^*_t]$, discount factor if agent $i$ only

Can the socially efficient discount factor $A_t$ be represented as an average of the individual $A^i_t$?
Consensus socially efficient discount factors

**Theorem**

*If* \( \delta^i \equiv \delta \) *and* \( \rho^i (s, \omega) \equiv \rho^i (s) \),

\[
A_t = \left[ \sum_{i=1}^{N} \gamma_i (A_t)^{\eta} \right]^{1/\eta}.
\]
Consensus socially efficient discount factors

Theorem

- **If** \( \delta^i \equiv \delta \) **and** \( \rho^i (s, \omega) \equiv \rho^i (s) \),
  \[
  A_t = \left[ \sum_{i=1}^{N} \gamma_i (A^i_t)^{\eta} \right]^{1/\eta}.
  \]

- **If** \( \eta = 1 \), \( A_t = \sum_{i=1}^{N} \gamma_i (A^i_t) \)
Consensus socially efficient discount factors

Theorem

- If $\delta^i \equiv \delta$ and $\rho^i (s, \omega) \equiv \rho^i (s)$,

  $$A_t = \left[ \sum_{i=1}^{N} \gamma_i (A_t^i)^{\eta} \right]^{1/\eta}.$$  

- If $\eta = 1$,  

  $$A_t = \sum_{i=1}^{N} \gamma_i (A_t^i)$$

- If $\eta \neq 1$,
Consensus socially efficient discount factors

Theorem

- If $\delta^i \equiv \delta$ and $\rho^i (s, \omega) \equiv \rho^i (s)$,
  \[ A_t = \left[ \sum_{i=1}^{N} \gamma_i \left( A_t^i \right)^{\eta} \right]^{1/\eta}. \]

- If $\eta = 1$, \[ A_t = \sum_{i=1}^{N} \gamma_i \left( A_t^i \right) \]
- If $\eta \neq 1$,
  \[ A_t \leq \left[ \sum_{i=1}^{N} \gamma_i \left( A_t^i \right)^{\eta} \right]^{1/\eta} \text{ for } \eta < 1, \]
Consensus socially efficient discount factors

Theorem

- If $\delta^i \equiv \delta$ and $\rho^i (s, \omega) \equiv \rho^i (s)$,

$$A_t = \left[ \sum_{i=1}^{N} \gamma_i (A_t^i)^{\eta} \right]^{1/\eta}.$$ 

- If $\eta = 1$, 
  $$A_t = \sum_{i=1}^{N} \gamma_i (A_t^i)$$ 

- If $\eta \neq 1$,
  - $A_t \leq \left[ \sum_{i=1}^{N} \gamma_i (A_t^i)^{\eta} \right]^{1/\eta}$ for $\eta < 1$,
  - $A_t \geq \left[ \sum_{i=1}^{N} \gamma_i (A_t^i)^{\eta} \right]^{1/\eta}$ for $\eta > 1$
Consensus socially efficient discount factors

**Theorem**

- If $\delta^i \equiv \delta$ and $\rho^i (s, \omega) \equiv \rho^i (s)$,

  $$A_t = \left[ \sum_{i=1}^{N} \gamma_i \left( A_t^i \right)^\eta \right]^{1/\eta}.$$

- If $\eta = 1$, $A_t = \sum_{i=1}^{N} \gamma_i \left( A_t^i \right)$

- If $\eta \neq 1$,

  - $A_t \leq \left[ \sum_{i=1}^{N} \gamma_i \left( A_t^i \right)^\eta \right]^{1/\eta}$ for $\eta < 1$,
  - $A_t \geq \left[ \sum_{i=1}^{N} \gamma_i \left( A_t^i \right)^\eta \right]^{1/\eta}$ for $\eta > 1$

  *equality only when divergence is deterministic ($N_i / N_j$ is deterministic for all $i, j$).*
The equilibrium approach is compatible with Weitzman’s assumption (arithmetic average discount factor) if
Consensus socially efficient discount factors
Main results

- The equilibrium approach is compatible with Weitzman’s assumption (arithmetic average discount factor) if
  1. Logarithmic utility functions
The equilibrium approach is compatible with Weitzman’s assumption (arithmetic average discount factor) if

1. Logarithmic utility functions
2. Each scenario/expertise corresponds to a subjective discount factor (different $\mu'_i$s or $\rho'_i$s)
Consensus socially efficient discount factors
Main results

- The equilibrium approach is compatible with Weitzman’s assumption (arithmetic average discount factor) if
  1. Logarithmic utility functions
  2. Each scenario/expertise corresponds to a subjective discount factor (different $\mu_i$ or $\rho_i$)
  3. Well chosen weights

Beliefs heterogeneity can be interpreted as more risk/uncertainty or less information: same impact on the trade-off between today’s consumption and future consumption (Gollier-Kimball 1996, Gollier, 2000)
Consensus socially efficient discount factors
Main results

- The equilibrium approach is compatible with Weitzman’s assumption (arithmetic average discount factor) if
  1. Logarithmic utility functions
  2. Each scenario/expertise corresponds to a subjective discount factor (different $\mu_i$'s or $\rho_i$'s)
  3. Well chosen weights

- In general,
The equilibrium approach is compatible with Weitzman’s assumption (arithmetic average discount factor) if

1. Logarithmic utility functions
2. Each scenario/expertise corresponds to a subjective discount factor (different $\mu'_i$s or $\rho'_i$s)
3. Well chosen weights

In general,

1. the right concept of average is the $\eta$—average
Consensus socially efficient discount factors
Main results

- The equilibrium approach is compatible with Weitzman’s assumption (arithmetic average discount factor) if
  1. Logarithmic utility functions
  2. Each scenario/expertise corresponds to a subjective discount factor (different $\mu'_i$s or $\rho'_i$s)
  3. Well chosen weights

- In general,
  1. the right concept of average is the $\eta$—average
  2. The average is a weighted average
Consensus socially efficient discount factors

Main results

- The equilibrium approach is compatible with Weitzman’s assumption (arithmetic average discount factor) if
  1. Logarithmic utility functions
  2. Each scenario/expertise corresponds to a subjective discount factor (different $\mu'_i$s or $\rho'_i$s)
  3. Well chosen weights

- In general,
  1. The right concept of average is the $\eta$–average
  2. The average is a weighted average
  3. $A$ can not be reduced to this average : there is an aggregation bias (upward or downward depending on $\eta$)
Consensus socially efficient discount factors
Main results

- The equilibrium approach is compatible with Weitzman’s assumption (arithmetic average discount factor) if
  1. Logarithmic utility functions
  2. Each scenario/expertise corresponds to a subjective discount factor (different $\mu_i$ or $\rho_i$)
  3. Well chosen weights

- In general,
  1. the right concept of average is the $\eta$–average
  2. The average is a weighted average
  3. $A$ can not be reduced to this average: there is an aggregation bias (upward or downward depending on $\eta$)

- Beliefs heterogeneity can be interpreted as more risk/uncertainty or less information: same impact on the trade-off between today’s consumption and future consumption (Gollier-Kimball 1996, Gollier, 2000)
Consensus risk-free rates and consensus socially efficient discount rates

Consensus risk-free rates

Theorem

\[
rf = \sum_{i=1}^{N} \tau_i \rho^i + \frac{1}{\eta} \left( \mu + \delta M \sigma \right) - \frac{1}{2} \frac{\left( 1 + \frac{1}{\eta} \right)}{2\eta} \sigma^2 - \frac{\eta - 1}{2} \text{Var}^\tau (\delta)
\]

Aggregation time pref

Agg exp. wealth

Stand prec sav

Agg bias

\[
= \sum_{i=1}^{N} \tau_i (r^i)^f - \frac{1}{2} (\eta - 1) \text{Var}^\tau (\delta)
\]

\[
= r^f (\text{standard}) + \sum_{i=1}^{N} \tau_i \rho^i + \frac{1}{\eta} \left( \sum_{i=1}^{N} \tau_i \delta^i \right) \sigma - \frac{\eta - 1}{2} \text{Var}^\tau (\delta)
\]

- Average level (patience, pessimism)
Consensus risk-free rates and consensus socially efficient discount rates
Consenus risk-free rates

Theorem

\[ r^f = \sum_{i=1}^{N} \tau_i \rho^i + \frac{(\mu + \delta_M \sigma)}{\eta} - \frac{\left(1 + \frac{1}{\eta}\right)}{2\eta} \sigma^2 - \frac{\eta - 1}{2} \text{Var}^\tau(\delta) \]

\[ = \sum_{i=1}^{N} \tau_i \left(r^i\right)^f - \frac{1}{2} (\eta - 1) \text{Var}^\tau(\delta) \]

\[ = r^f(\text{standard}) + \sum_{i=1}^{N} \tau_i \rho^i + \frac{1}{\eta} \left(\sum_{i=1}^{N} \tau_i \delta^i\right) \sigma - \frac{\eta - 1}{2} \text{Var}^\tau(\delta) \]

- Average level (patience, pessimism)
- Correlation
Consensus risk-free rates and consensus socially efficient discount rates

Consensus risk-free rates

**Theorem**

\[
rf = \sum_{i=1}^{N} \tau_i \rho^i + \frac{(\mu + \delta M \sigma)}{\eta} - \frac{(1 + \frac{1}{\eta})}{2\eta} \sigma^2 - \frac{\eta - 1}{2} \text{Var}^\tau(\delta)
\]

\[
= \sum_{i=1}^{N} \tau_i (r^i)^f - \frac{1}{2} (\eta - 1) \text{Var}^\tau(\delta)
\]

\[
= rf(\text{stand}) + \sum_{i=1}^{N} \tau_i \rho^i + \frac{1}{\eta} \left( \sum_{i=1}^{N} \tau_i \delta^i \right) \sigma - \frac{\eta - 1}{2} \text{Var}^\tau(\delta)
\]

- Average level (patience, pessimism)
- Correlation
- Beliefs dispersion (depends on $\eta > 1, \eta < 1$). For $\eta > 1$: more risk $\Rightarrow$ more saving $\Rightarrow$ downward pressure on $r_f$. 

**Equilibrium approach**

Elyès Jouini

**Introduction**

The classical model

Beliefs heterogeneity

Aggregation

Discount factors

Discount rates

Long term

Specific settings

How to aggregate experts opinions

Impact of stochastic shifts
Consensus risk-free rates and consensus socially efficient discount rates

Consensus risk-free rates

**Theorem**

\[
rf = \sum_{i=1}^{N} \tau_i \rho^i + \left( \frac{\mu + \delta M \sigma}{\eta} \right) - \frac{\left(1 + \frac{1}{\eta}\right)}{2\eta} \sigma^2 - \frac{\eta - 1}{2} \text{Var}^{\tau} (\delta) \\
= \sum_{i=1}^{N} \tau_i (r^i)^f - \frac{1}{2} (\eta - 1) \text{Var}^{\tau} (\delta) \\
= rf (\text{standard}) + \sum_{i=1}^{N} \tau_i \rho^i + \frac{1}{\eta} \left( \sum_{i=1}^{N} \tau_i \delta^i \right) \sigma - \frac{\eta - 1}{2} \text{Var}^{\tau} (\delta)
\]

- Average level (patience, pessimism)
- Correlation
- Beliefs dispersion (depends on \( \eta > 1, \eta < 1 \)). For \( \eta > 1 \):
  - more risk \( \Rightarrow \) more saving \( \Rightarrow \) downward pressure on \( rf \)
Long term considerations

**Theorem**

Suppose that for all $i$, the individual asymptotic discount rate $R^i_\infty \equiv \lim_{t \leq T; t, T \to \infty} R^T_{t, i}$ exists. Moreover, we suppose $\gamma (T) \geq \epsilon > 0$ for $R^l_\infty = \inf \{ R^i_\infty; i = 1, ... N \}$.

Then, $R_\infty \equiv \lim_{t \leq T; t, T \to \infty} R^T_t = \inf \{ R^i_\infty, i = 1, ..., N \}$.

- The aggregation bias vanishes in the long run
Long term considerations

Theorem

Suppose that for all $i$, the individual asymptotic discount rate $R^i_\infty \equiv \lim_{t \leq T; t, T \to \infty} R^T_{t;i}$ exists. Moreover, we suppose $\gamma_I(T) \geq \epsilon > 0$ for $R^l_\infty = \inf \{ R^i_\infty; i = 1, \ldots, N \}$. Then,

$$R^\infty \equiv \lim_{t \leq T; t, T \to \infty} R^T_t = \inf \{ R^i_\infty, i = 1, \ldots, N \}.$$

- The aggregation bias vanishes in the long run
- The relevant asymptotic behavior is the one with the lowest discount rate
Long term considerations

Theorem

Suppose that for all $i$, the individual asymptotic discount rate 
\[ R^i_{\infty} \equiv \lim_{t \leq T; t, T \to \infty} R^T_{t; i} \] exists. Moreover, we suppose 
\[ \gamma(T) \geq \varepsilon > 0 \] for \[ R^l_{\infty} = \inf \{ R^i_{\infty}; i = 1, \ldots, N \} \). Then,

\[ R_{\infty} \equiv \lim_{t \leq T; t, T \to \infty} R^T_t = \inf \{ R^i_{\infty}, i = 1, \ldots, N \} \].

- The aggregation bias vanishes in the long run
- The relevant asymptotic behavior is the one with the lowest discount rate
  - Homogeneous beliefs: lowest rate of impatience
    (Gollier-Zeckhauser in a deterministic setting)
Long term considerations

**Theorem**

Suppose that for all $i$, the individual asymptotic discount rate $R_i^\infty \equiv \lim_{t \leq T; t, T \to \infty} R_T^{T;i}$ exists. Moreover, we suppose $\gamma_l(T) > \varepsilon > 0$ for $R_\infty^l = \inf \{R_i^\infty; i = 1, \ldots, N\}$. Then,

$$R_\infty \equiv \lim_{t \leq T; t, T \to \infty} R_T^T = \inf \{R_i^\infty, i = 1, \ldots, N\}.$$

- The aggregation bias vanishes in the long run
- The relevant asymptotic behavior is the one with the lowest discount rate
  - Homogeneous beliefs: lowest rate of impatience (Gollier-Zeckhauser in a deterministic setting)
  - Homogeneous time preference rate: most pessimistic rate
Long term considerations

Theorem

Suppose that for all $i$, the individual asymptotic discount rate $R^i_\infty \equiv \lim_{t \leq T; t, T \to \infty} R^T_t;i$ exists. Moreover, we suppose 

$$\gamma_I(T) \geq \varepsilon > 0 \text{ for } R^I_\infty = \inf \{ R^i_\infty ; i = 1, \ldots, N \}. Then,$$

$$R_\infty \equiv \lim_{t \leq T; t, T \to \infty} R^T_t = \inf \{ R^i_\infty ; i = 1, \ldots, N \}.$$

- The aggregation bias vanishes in the long run
- The relevant asymptotic behavior is the one with the lowest discount rate
  - Homogeneous beliefs: lowest rate of impatience (Gollier-Zeckhauser in a deterministic setting)
  - Homogeneous time preference rate: most pessimistic rate
- $\gamma_I(T) \geq \varepsilon > 0$
Specific settings
Logarithmic case

\[ A_t = \sum_i \gamma_i A_t^i \text{ (arithmetic average)} \]
Specific settings
Logarithmic case

- \( A_t = \sum_i \gamma_i A_t^i \) (arithmetic average)

\[
R_T^t = \mu - \sigma^2 - \frac{1}{t} \log \left[ \sum_i \gamma_i^T \exp \left( (\rho^i + \sigma \delta^i) t \right) \right]
\]

\[
R_0 = \mu - \sigma^2 + \sum_i w_i \left( \rho^i + \sigma \delta^i \right)
\]

\[
R_\infty = \mu - \sigma^2 + \inf \left( \rho^i + \sigma \delta^i \right)
\]

- \( \gamma_i^T = \frac{w_i \rho^i (1 - \exp - \rho^i T)^{-1}}{\sum_j w_j \rho^j (1 - \exp - \rho^j T)^{-1}} \)
Specific settings
Logarithmic case

- $A_t = \sum_i \gamma_i A_t^i$ (arithmetic average)

$$R_t^T = \mu - \sigma^2 - \frac{1}{t} \log \left[ \sum_i \gamma_i^T \exp\left(\rho^i + \sigma \delta^i\right) t \right]$$

$$R_0 = \mu - \sigma^2 + \sum_i w_i \left( \rho^i + \sigma \delta^i \right)$$

$$R_\infty = \mu - \sigma^2 + \inf \left( \rho^i + \sigma \delta^i \right)$$

$$\gamma_i^T = \frac{w_i \rho^i (1 - \exp - \rho^i T)^{-1}}{\sum_j w_j \rho^j (1 - \exp - \rho^j T)^{-1}}$$

- $R_0 \geq R_\infty$ and $R_t^T$ decreases with $t$
Specific settings
Logarithmic case

\( w_1 = w_2, \delta_1 = -\delta_2 \)

different levels of \( \delta \)

\( R_t \downarrow \) with \( \delta \), pessim. limit

same starting point

\( \text{cov}(w, \delta) > 0 \)

\( R_t \uparrow \) with \( \delta \) for small \( t \)

\( R_t \downarrow \) with \( \delta \) for large \( t \)

\( \neq \) starting point
Specific settings
Power utility functions, \( \eta < 1 \)

- The equilibrium discount rates dominates the averages
Specific settings
Power utility functions, \( \eta < 1 \)

- The equilibrium discount rates dominates the averages
- The \( \eta \)-average is a better approx., the distance is due to beliefs disp. and this effect may last for centuries
Specific settings
Power utility functions, $\eta<1$

- The equilibrium discount rates dominates the averages
- The $\eta$–average is a better approx., the distance is due to beliefs disp. and this effect may last for centuries
- The three curves converge to the lowest discount rate
Specific settings
Power utility functions, $\eta > 1$

- The equilibrium discount rates is below the averages
Specific settings
Power utility functions, \( \eta > 1 \)

- The equilibrium discount rates is below the averages
- The \( \eta \)–average is still a better approximation
Specific settings
Power utility functions, \( \eta > 1 \)

- The equilibrium discount rates is below the averages
- The \( \eta \)–average is still a better approximation
- The three curves converge to the lowest discount rate
Let us consider $n$ experts: $(R^i)_{i=1,...,n}$ as in Weitzman (2001)

- $N$ groups of agents,
Divergence of experts’ opinions

Let us consider $n$ experts: $(R^i)_{i=1,...,n}$ as in Weitzman (2001)

- $N$ groups of agents,
- $w_i \equiv \text{relative size of group } i,$
Let us consider $n$ experts: $(R^i)_{i=1,...,n}$ as in Weitzman (2001)

- $N$ groups of agents,
- $w_i \equiv \text{relative size of group } i$,
- $\rho_i \equiv \text{pure time preference rate of the agents in group } i$,
Let us consider $n$ experts: $(R^i)_{i=1,...,n}$ as in Weitzman (2001)

- $N$ groups of agents,
- $w_i \equiv$ relative size of group $i$,
- $\rho_i \equiv$ pure time preference rate of the agents in group $i$,
- $t \equiv$ the time at which a cost or benefit is incurred,
Let us consider \( n \) experts : \((R^i)_{i=1,...,n}\) as in Weitzman (2001)

- \( N \) groups of agents,
- \( w_i \) \( \equiv \) relative size of group \( i \),
- \( \rho_i \) \( \equiv \) pure time preference rate of the agents in group \( i \),
- \( t \) \( \equiv \) the time at which a cost or benefit is incurred,
- \( e_t^* \sim \ln \mathcal{N}((\mu_i - \frac{1}{2}\sigma_i^2)t, \sigma_i^2 t) \),
Divergence of experts’ opinions

Let us consider \( n \) experts : \((R^i)_{i=1,...,n}\) as in Weitzman (2001)

- \( N \) groups of agents,
- \( w_i \equiv \text{relative size of group } i \),
- \( \rho_i \equiv \text{pure time preference rate of the agents in group } i \),
- \( t \equiv \text{the time at which a cost or benefit is incurred} \),
- \( e_t^* \sim \ln \mathcal{N}((\mu_i - \frac{1}{2}\sigma_i^2)t, \sigma_i^2t) \),
- log utility functions
Divergence of experts’ opinions

Let us consider $n$ experts: $(R^i)_{i=1,\ldots,n}$ as in Weitzman (2001)

- $N$ groups of agents,
- $w_i \equiv$ relative size of group $i$,
- $\rho_i \equiv$ pure time preference rate of the agents in group $i$,
- $t \equiv$ the time at which a cost or benefit is incurred,
- $e^*_t \sim \ln \mathcal{N}((\mu_i - \frac{1}{2}\sigma_i^2)t, \sigma_i^2t)$,
- log utility functions
- $R^i \equiv \rho_i + \mu_i - \sigma_i^2$, equilibrium discount rate if the economy was made of group $i$ agents only
the consensus discount rates are averages of the individual rates (as in Weitzman 1998)
The consensus discount rates are averages of the individual rates (as in Weitzman 1998)

- weighted averages, weights proportional to the pure time preference rates,
Equilibrium approach

Elyès Jouini

Introduction

The classical model

Beliefs heterogeneity

Aggregation

Discount factors

Discount rates

Long term

Specific settings

How to aggregate experts opinions

Impact of stochastic shifts

\[ R_t \equiv -\frac{1}{t} \ln \sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} \exp -R_i t, \]

\[ r_t \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp (-r_i t)}{\sum_{j=1}^{N} w_j \rho_j \exp (-r_j t)} r_i. \]

- the consensus discount rates are averages of the individual rates (as in Weitzman 1998)
- weighted averages, weights proportional to the pure time preference rates,
- bias towards the more impatient agents
In the case of homogeneous beliefs \((\mu_i = \mu, \sigma_i = \sigma)\)

\[
    r_t \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp(-\rho_i t)}{\sum_{j=1}^{N} w_j \rho_j \exp(-\rho_j t)} \rho_i + \mu - \sigma^2.
\]

- the expression involves the covariance between \(\rho_i\) and \(\exp(-\rho_i t)\) as in Lengwiler (2005)
In the case of homogeneous beliefs \((\mu_i = \mu, \sigma_i = \sigma)\)

\[
 r_t \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp(-\rho_i t)}{\sum_{j=1}^{N} w_j \rho_j \exp(-\rho_j t)} \rho_i + \mu - \sigma^2.
\]

- the expression involves the covariance between \(\rho_i\) and \(\exp(-\rho_i t)\) as in Lengwiler (2005)
- it gives the expression for the consensus utility discount rate
In the case of homogeneous beliefs \((\mu_i = \mu, \sigma_i = \sigma)\)

\[
r_t \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp(-\rho_i t)}{\sum_{j=1}^{N} w_j \rho_j \exp(-\rho_j t)} \rho_i + \mu - \sigma^2.
\]

- the expression involves the covariance between \(\rho_i\) and \(\exp(-\rho_i t)\) as in Lengwiler (2005)
- it gives the expression for the consensus utility discount rate
- different from the one obtained by Gollier (2005) or Nocetti (2008)
In the case of homogeneous beliefs \((\mu_i = \mu, \sigma_i = \sigma)\)

\[
rt \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{w_i \rho_i \exp(-\rho_i t)}{\sum_{j=1}^{N} w_j \rho_j \exp(-\rho_j t)} \rho_i + \mu - \sigma^2.
\]

- the expression involves the covariance between \(\rho_i\) and \(\exp(-\rho_i t)\) as in Lengwiler (2005)
- it gives the expression for the consensus utility discount rate
- different from the one obtained by Gollier (2005) or Nocetti (2008)
- our weights are given by \(w_i \rho_i \exp(-\rho_i t)\) instead of \(\lambda_i \exp(-\rho_i t)\)
In the case of homogeneous beliefs \((\mu_i = \mu, \sigma_i = \sigma)\)

\[
r_t \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp(-\rho_i t)}{\sum_{j=1}^{N} w_j \rho_j \exp(-\rho_j t)} \rho_i + \mu - \sigma^2.
\]

- the expression involves the covariance between \(\rho_i\) and \(\exp(-\rho_i t)\) as in Lengwiler (2005)
- it gives the expression for the consensus utility discount rate
- different from the one obtained by Gollier (2005) or Nocetti (2008)
- our weights are given by \(w_i \rho_i \exp(-\rho_i t)\) instead of \(\lambda_i \exp(-\rho_i t)\)
- both approaches coincide if the Pareto weights are proportional to \(w_i \rho_i\).
If $\rho_i$ and $b_i$ are independent,

$$r_t \geq \sum_{i=1}^{N} \frac{w_i \exp - r^i t}{\sum_{j=1}^{N} w_j \exp - r^j t} r^i = r_t^W$$
Corollary

1. $R_t$ and $r_t$ decrease with $t$, 

$\rho_i = \rho$, and normal distribution $N(m, \sigma^2)$ on $b_i = \mu_i \sigma^2$. 

$R_t = \rho + m v^2 t$ (Reinschmidt, 2002) 

If $\rho_i = \gamma(\alpha_1, \beta_1)$ and $b_i = \mu_i \sigma^2$ independent, then 

$r_t = \alpha_1 + 1 \beta_1 + t + \alpha_2 \beta_2 + t$ 

As in 4. and $\beta_1 = \beta_2 = \beta$ then $R_i = \gamma(\alpha, \beta)$. 

With $\alpha = \alpha_1 + \alpha_2$ and $r_t = r_{wt}$.
Corollary

1. $R_t$ and $r_t$ decrease with $t$,

2. The asymptotic equilibrium discount rates are given by the lowest individual discount rate, i.e. $R_\infty = r_\infty = \inf_i r^i = \inf_i R^i$. 
Corollary

1. $R_t$ and $r_t$ decrease with $t$,

2. The asymptotic equilibrium discount rates are given by the lowest individual discount rate, i.e.

$$R_\infty = r_\infty = \inf_i r^i = \inf_i R^i.$$ 

3. $\rho_i = \rho$, and normal distribution $N(m, \nu^2)$ on $b_i = \mu_i - \sigma_i^2$, $R_t = \rho + m - \frac{\nu^2}{2} t$ (Reinschmidt, 2002)
General properties and specific distributions

Corollary

1. \( R_t \) and \( r_t \) decrease with \( t \),

2. The asymptotic equilibrium discount rates are given by the lowest individual discount rate, i.e.
   \[ R_\infty = r_\infty = \inf_i r^i = \inf_i R^i. \]

3. \( \rho_i = \rho \), and normal distribution \( \mathcal{N}(m, \nu^2) \) on
   \( b_i = \mu_i - \sigma_i^2 \), \( R_t = \rho + m - \frac{\nu^2}{2} t \) (Reinschmidt, 2002)

4. If \( \rho_i \sim \gamma(\alpha_1, \beta_1) \) and \( b_i = \mu_i - \sigma_i^2 \sim \gamma(\alpha_2, \beta_2) \) independent, then
   \[ r_t = \frac{\alpha_1 + 1}{\beta_1 + t} + \frac{\alpha_2}{\beta_2 + t} \]
Corollary

1. $R_t$ and $r_t$ decrease with $t$,
2. The asymptotic equilibrium discount rates are given by the lowest individual discount rate, i.e.
   $$R_{\infty} = r_{\infty} = \inf_i r_i = \inf_i R^i.$$
3. $\rho_i = \rho$, and normal distribution $\mathcal{N}(m, \nu^2)$ on
   $$b_i = \mu_i - \sigma_i^2, \quad R_t = \rho + m - \frac{\nu^2}{2} t \quad \text{(Reinschmidt, 2002)}$$
4. If $\rho_i \sim \gamma(\alpha_1, \beta_1)$ and $b_i = \mu_i - \sigma_i^2 \sim \gamma(\alpha_2, \beta_2)$ independent, then
   $$r_t = \frac{\alpha_1 + 1}{\beta_1 + t} + \frac{\alpha_2}{\beta_2 + t}$$
5. As in 4. and $\beta_1 = \beta_2 = \beta$ then $R^i \sim \gamma(\alpha, \beta)$ with
   $$\alpha = \alpha_1 + \alpha_2$$ and $r_t = r_t^W + \frac{1}{\beta + t}$.
Figure: Calibration with two independent gamma distr. on Weitzman (2001)'s data. We assume that the two distributions are homothetic and calibrate in order to fit the mean and the variance of the empirical distribution. Weitzman (2001)'s statistical model corresponds to $\lambda = 1$. We maximize the log-likelihood and obtain $\lambda = 0.4116$. 
Figure: Marginal discount rate curve through our calibration (upper curve) and discount rate curve of Weitzman (2001) (lower curve). The intermediate curve represents, with our calibration, the unweighted average.
<table>
<thead>
<tr>
<th>Time period</th>
<th>Name</th>
<th>Numerical value</th>
<th>Approx. rate</th>
<th>Weitzman’s num. value</th>
<th>Weitzman’s appr. rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Within years 1 to 5 hence</td>
<td>Immediate Future</td>
<td>4.99%</td>
<td>5%</td>
<td>3.89%</td>
<td>4%</td>
</tr>
<tr>
<td>Within years 6 to 25 hence</td>
<td>Near Future</td>
<td>4.23%</td>
<td>4%</td>
<td>3.22%</td>
<td>3%</td>
</tr>
<tr>
<td>Within years 26 to 75 hence</td>
<td>Medium Future</td>
<td>2.82%</td>
<td>3%</td>
<td>2.00%</td>
<td>2%</td>
</tr>
<tr>
<td>Within years 76 to 300 hence</td>
<td>Distant Future</td>
<td>1.50%</td>
<td>1.5%</td>
<td>0.97%</td>
<td>1%</td>
</tr>
<tr>
<td>Within years more than 300 hence</td>
<td>Far-Distant Future</td>
<td>0.16%</td>
<td>0%</td>
<td>0.08%</td>
<td>0%</td>
</tr>
</tbody>
</table>
The gamma distribution case

- a decrease in the mean $m_2$ or an increase in the variance $\nu_2^2$ of the individual beliefs ($b_i$) decreases the marginal discount rate $r_t$
The gamma distribution case

- a decrease in the mean \( m_2 \) or an increase in the variance \( \nu_2^2 \) of the individual beliefs \((b_i)\) decreases the marginal discount rate \( r_t \).
- same result with a decrease in the mean \( m_1 \) of the individual pure time preference rates \((\rho_i)\).
The gamma distribution case

- a decrease in the mean $m_2$ or an increase in the variance $\nu_2^2$ of the individual beliefs $(b_i)$ decreases the marginal discount rate $r_t$
- same result with a decrease in the mean $m_1$ of the individual pure time preference rates $(\rho_i)$.
- an increase in the variance $\nu_1^2$ of the individual pure time preference rates $(\rho_i)$ decreases the marginal discount rate $r_t$ for $t$ large enough.
The general case

1. If all the agents have the same $\rho_i$, then a FSD (resp. SSD) shift in the distribution of $(R^i)$ increases the discount rate $R_t$ for all horizons.
The general case

1. If all the agents have the same $\rho_i$, then a FSD (resp. SSD) shift in the distribution of $(R_i^t)$ increases the discount rate $R_t$ for all horizons.

2. If all the agents have the same $\rho_i$, then a MLR shift in the distribution of the $(r_i^t)$ increases the marginal discount rate $r_t$ for all horizons.
The general case

1. If all the agents have the same $\rho_i$, then a FSD (resp. SSD) shift in the distribution of $(R^i)$ increases the discount rate $R_t$ for all horizons.

2. If all the agents have the same $\rho_i$, then a MLR shift in the distribution of the $(r^i)$ increases the marginal discount rate $r_t$ for all horizons.

3. If all the agents have the same beliefs, then a MLR shift in the distribution of the $(R^i)$ increases the discount rate $R_t$ for all horizons.
Measured set of agents

- Agent $i$ has a probability measure $Q^i_t$ that represents the distribution of date $t$ aggregate consumption.
Agent $i$ has a probability measure $Q^i_t$ that represents the distribution of date--$t$ aggregate consumption.

Agent $i$ has a pure time preference rate $\rho_i$, a share of total wealth $w_i$ and a log-utility

$$R_t \equiv -\frac{1}{t} \ln \int \frac{w_i \rho_i}{\int w_j \rho_j \, d\nu(j)} \exp(-R^i_t \cdot t) \, d\nu(i)$$
Measured set of agents

- Agent $i$ has a probability measure $Q^i_t$ that represents the distribution of date–$t$ aggregate consumption.
- Agent $i$ has a pure time preference rate $\rho_i$, a share of total wealth $w_i$ and a log-utility:

$$R_t \equiv -\frac{1}{t} \ln \int \frac{w_i \rho_i}{\int w_j \rho_j \, d\nu(j)} \exp \left(-R^i_t \right) \, d\nu(i)$$

- where $R^i_t$ is the equilibrium discount rate that would prevail if the economy was made of agent $i$ only.
Take-home messages

- The socially discount factor is not, in general, an arithmetic average of the individually anticipated ones.
Take-home messages

- The socially discount factor is not, in general, an arithmetic average of the individually anticipated ones
  - $\eta$—average
The socially discount factor is not, in general, an arithmetic average of the individually anticipated ones

- \( \eta \) — average
- weights
The socially discount factor is not, in general, an arithmetic average of the individually anticipated ones.

- $\eta$—average
- weights
- bias related to beliefs and time preference rates dispersion
Take-home messages

- The socially discount factor is not, in general, an arithmetic average of the individually anticipated ones
  - $\eta$—average
  - weights
  - bias related to beliefs and time preference rates dispersion

- The arithmetic average corresponds to a utility maximizing agent that considers each individual belief as a possible scenario while our approach corresponds to a central planner that maximizes the social welfare
Specific cases
Specific cases

- Logarithmic case: weighted arithmetic average (à la Weitzman)
Specific cases

- Logarithmic case: weighted arithmetic average (à la Weitzman)
- Deterministic divergence ($N_i/N_j$ deterministic): weighted $\eta$—average (Gollier-Zeckhauser)
Specific cases

- Logarithmic case: weighted arithmetic average (à la Weitzman)

- Deterministic divergence ($N_i / N_j$ deterministic): weighted $\eta$–average (Gollier-Zeckhauser)

Aggregate pessimism and patience reduces $R$
Specific cases

- Logarithmic case: weighted arithmetic average (à la Weitzman)
- Deterministic divergence ($N_i/N_j$ deterministic): weighted $\eta$–average (Gollier-Zeckhauser)

Aggregate pessimism and patience reduces $R$
Beliefs dispersion reduces $R$ for $\eta > 1$
Specific cases

- Logarithmic case: weighted arithmetic average (à la Weitzman)
- Deterministic divergence ($N_i/N_j$ deterministic): weighted $\eta$—average (Gollier-Zeckhauser)

Aggregate pessimism and patience reduces $R$
Beliefs dispersion reduces $R$ for $\eta > 1$
Long term rate: lowest discount rate
Equilibrium approach

Elyès Jouini

Introduction
The classical model
Beliefs heterogeneity
Aggregation
Discount factors
Discount rates
Long term
Specific settings
How to aggregate experts opinions
Impact of stochastic shifts

Specific cases

- Logarithmic case: weighted arithmetic average (à la Weitzman)
- Deterministic divergence ($N_i / N_j$ deterministic): weighted $\eta$—average (Gollier-Zeckhauser)

- Aggregate pessimism and patience reduces $R$
- Beliefs dispersion reduces $R$ for $\eta > 1$
- Long term rate: lowest discount rate
- Medium term: increasing as well as decreasing yield curves