# Abstracts For the PDE workshop on Asymptotic analysis in the calculus of variations and PDEs

July 6-10, 2009

## Mini-course 1

### *Riviere, Tristan (ETH Zurich)*

Integrability by Compensation in the Analysis of Conformally Invariant Problems Conformal invariance plays a significant role in many areas of Physics, such as conformal field theory, renormalization theory, turbulence, general relativity. Naturally, it also plays an important role in geometry: theory of Riemannian surfaces, Weyl tensors, Q-curvature, Yang-Mills fields, etc... We shall be concerned with the study of conformal invariance in analysis. More precisely, we will focus on the study of nonlinear PDEs arising from conformally invariant variational problems (e.g. harmonic maps, prescribed mean curvature surfaces, Yang-Mills equations, amongst others). A transformation is called conformal when it preserves angles, that is, when its differential is a similarity at every point. Unlike in higher dimensions, the group of conformal transformations in two dimensions is very large; it has infinite dimension. In fact, it contains as many elements as there are holomorphic maps. This particularly rich feature motivates us to restrict our attention on the two-dimensional case. Although we shall not be concerned with higher dimension, the reader should know that many of the results presented in these notes can be generalized to any dimension. The first historical instance in which calculus of variations encountered conformal invariance took place early in the twentieth century with the resolution of the Plateau problem. Originally posed by J.-L. Lagrange in 1760, it was solved independently over 150 years later by J. Douglas and T. Rado. In recognition of his work, the former was bestowed the first Fields Medal in 1936 (jointly with L. Alhfors).

## Mini-course 2

### *Mueller, Stefan (Hausdorff Center for Mathematics)*

A fundamental problem in elasticity is to derive theories for lower dimensional objects such as plates, shells or rods from the fully nonlinear three dimensional theory. The usual approach is to make certain assumptions on the three dimensional solutions and then to deduce a lower dimensional theory by formal or rigorous asymptotical analysis. These has lead to large variety of theories, which are sometimes not mutually compatible.

Since the early 90's a new, mathematically rigorous, approach has emerged, which is based on the variational principle and the associated notion of $\Gamma$ convergence. Le Dret and Raoult have used $\Gamma$ convergence to derive a theory for elastic membranes (these have only stretching stiffness, but no bending stiffness and cannot resist compression). In these lectures I will report on work with G. Friesecke (Munich/ Warwick) ,R.D. James (Minnesota) and others to derive a full hierarchy of limiting theories, which are distinguished by the scaling of the elastic energy as a function of thickness.

A key mathematical ingredient is a quantitative rigidity estimate which generalizes results of F. John for deformations with small nonlinear strain. A classical result says that any Lipschitz map from a (bounded) connected set in $\mathbf{R}^n$ to $\mathbf{R}^n$ (we are interested in $n \geq 2$) whose derivative is an element of $SO(n)$ a.e. has in fact constant derivative. The quantative rigidity estimate says that this can be extended to a linear estimate in $L^2$. More precisely for every $u \in W^{1,2}$ and every Lipschitz domain $\Omega$ we have

$$\min_{Q \in SO(n)} \int_\Omega |\nabla u - Q|^2 \leq C(\Omega) \int_\Omega \operatorname{dist}^2(\nabla, SO(n)). \tag{1}$$

Some interesting applications, e.g. in (soft) condensed matter physics or certain growth models in biology correspond to situation which cannot be described by the limiting theories alone and where even the relevant scaling exponents are not known rigorously. At the end of the course I will outline some of these open problems.

## *Alama, Stan (McMaster University)*

### Gamma-convergence results for superconducting thin films

We start with Gamma-convergence of the full three-dimensional Ginzburg-Landau functional with applied magnetic field, for a thin domain of varying thickness converging to a planar domain. We identify three interesting regimes, depending on the strength of the applied magnetic field in terms of the film thickness. The subcritical regime includes the result of Chapman-Du-Gunzburger, in which the functional Gamma-converges to a simpler two-dimensional energy with constant, orthogonally applied magnetic field. In the critical regime, we discover two curious phenomena involving the component of the magnetic field parallel to the limiting plane. We then study the limiting energies in each regime in the London limit to describe concentration of vortices. We show that, near the lower critical field, vortices may concentrate on points or on curves in the domain, depending on the choice of applied field and film profile. This represents two separate collaborations, one with Bronsard and Galvao-Sousa, and the other with Bronsard and Millot.

## *Braides, Andrea (University of Rome "Tor Vergata", Italy)*

**Asymptotic analysis of discrete systems**

Continuum limits of discrete (lattice) systems for vanishing lattice size starting from simple atomistic interactions exhibit a variety of multi-scale behaviors, which sometimes depart from their analog continuous counterparts, sometimes simplify and clarify the ansatz to be assumed in the continuum. I present two examples of such a situation.

1. Defects in linear media. A linear medium is simply modeled by quadratic mass-spring interactions; defects are introduced as the possibility of breaking such bonds at a given threshold (truncated quadratic potential). We (partly) characterize all possible limit of such discrete systems (which turns out to be a family of free-discontinuity energies), showing in particular that they are independent of the percentage of defects, contrary to the analogous G- closure problem for mixtures of quadratic interactions and to the random case (which instead exhibits a percolation threshold);

2. Pattern-depending energies in spin systems. A simple system of nearest and next-to-nearest spin interaction may give rise (in dependence of the interaction parameters) to interfacial energies parameterized on the set of ground states, with the necessity to introduce a sew set of "phases", and a "phase-transition" energy.

Works in colaboration with LAura SIgalotti and Marco Cicalese, respectively.

## *Esposito, Pierpaolo* (Universita' degli Studi Roma Tre)

**Some second and fourth order elliptic PDE's arising in MEMS models**

A MEMS (Micro Electro-Mechanical System) device can be idealized as a membrane kept fixed along the boundary which deflects towards an upper conducting plane. In the stationary regime, the MEMS model is described by a fourth-order nonlinear eigenvalue problem with Dirichlet boundary condition and a singular nonlinear term. In the small thickness limit, it reduces to a second-order problem which has been largely studied in recent years and some precise results on its solution set are now available. For a small but finite thickness, much less is known and the analysis of the minimal branch on the unit ball has been carried over just recently. In a joint work with Cowan, Ghoussoub and Moradifam, we have considered the compactness of the minimal branch which holds exactly for dimensions $N \leq 8$. This can be also regarded as a regularity result in low dimension for weak semi-stable solutions of the corresponding fourth-order equation.

## *Fonseca, Irene (Carnegie Mellon University)*

### Variational Methods in Imaging

Deblurring , denoising, inpainting and recolorization of images are fundamental problems in image processing and have given rise in the past few years to a vast variety of techniques and methods touching different fields of mathematics. Among them, variational methods based on the minimization of certain energy functionals have been successfully employed to treat a fairly general class of image restoration problems. The underlying theoretical challenges are common to the variational formulation of problems in other areas (e.g. materials science). Here first order RGB variational problems for recolorization will be analyzed, and the use of second order variational problems to eliminate the staircasing effect will be validated.

## *Garroni, Adriana*

### A variational derivation for a line tension model for dislocations

The main mechanism for crystal plasticity is the formation and motion of a special class of defects, the dislocations. These are topological defects in the crystalline structure that can be identify with lines on which energy concentrates. In recent years there has been a considerable effort for the derivation of models that describe these objects at different scales (from an energetic and a dynamical point of view). The results obtained mainly concern special geometries, as one dimensional models, reduction to straight dislocations, the activation of only one slip system, etc.

We will consider a fully three dimensional discrete model for crystal plasticity that accounts for the presence of dislocations. By means of Gamma we obtain, at suitable scales, a line tension energy. The characterization of the line tension energy density requires a relaxation result for energies defined on curves.

## *Gui, Chengfeng (University of Connecticut)*

### Quadruple Junction Solutions in the Entire Three Dimensional Space

In this talk, I will discuss the quadruple junction solutions in the entire three dimensional space to a vector-valued Allen-Cahn equation which models multiple phase separation. The solution is the basic profile of the local structure near a quadruple junction in three dimensional crystalline material under the generalized Allen-Cahn model, and is the three dimensional counterpart of triple junction solution which is two dimensional. I will start with one dimensional heteroclinic solutions, and describe how we can construct higher dimensional solutions from the lower dimensional ones, and explain the complications and difficulties in constructing such solution in three dimensions.

## *Jerrard, Bob (University of Toronto)*

**Dynamics of topological defects in semilinear wave equations.**

A vigorous line of research, dating back at least to the mid 70s, establishes various ways in which topological defects (such as domain walls and vortex filaments) in solutions of semilinear elliptic PDEs are related to minimal surfaces in some scaling limit. Analogous questions about semilinear hyperbolic equations arise in the physics literature in descriptions of objects known as "cosmic strings", which have not yet been observed but are believed to be likely to exist in our universe. We describe some recent work on these problems, joint with Alberto Montero. Our results can be very loosely described as giving a rigorous derivation of the equation for the dynamics of cosmic strings, deduced formally by cosmologists about 30 years ago.

## *Kristensen, Jan (University of Oxford)*

**On boundary regularity in variational problems**

It is well-known that minimizers of regular variational problems can be nondifferentiable and even unbounded in high dimensions. They are however partially regular, meaning that they are smooth outside some small relatively closed set (the singular set; points in its complement are regular points). Until recently the estimates for the Hausdorff dimension of the singular set were too weak to allow us to infer existence of even one regular boundary point. In this talk we discuss results on higher differentiability that address this issue, and that in particular imply partial regularity of minimizers up to the boundary for a general class of variational problems. The talk is based on joint work with Rosario Mingione (Parma).

## *Malchiodi, Andrea*

**Variational theory for Lioville equations with singular data**

We study a class of equations which arises in the study of Chern-Simons vortices, and which presents singular sources in the right-hand side. We discuss some related Moser-Trudinger type inequalities and variational schemes.

## *Niethammer, Barbara (U of Oxford)*

**Evolution in dilute diblock-copolymer systems**

We study a free boundary problem describing the micro phase separation of diblock copolymer melts in the regime that one component has small volume fraction such that the phases separate into an ensemble of small spheres. An asymptotic analysis within a gradient flow setting can be used to derive mean-field equations for the evolution of particle centers and radii. It turns out that on a time scale of the order of the average volume of the spheres, the evolution is dominated by coarsening and subsequent stabilization of the radii of the spheres, whereas migration becomes relevant on a later time scale. We also show how this analysis can be made rigorous. (Based on joint work with M. Helmers, X Ren and Y. Oshita)

## *Pacard , Frank*

### Geometric aspects of the Allen-Cahn and other semilinear elliptic equations

I will try to explain the analogies between the theory of Complete constant mean curvature surfaces with Delaunay ends (in Euclidean 3-space) and a certain class of solutions for semilinear elliptic equations such as : the Allen- Cahn equation (which has its origin in the modeling of phase transition) or the equation which arises in the study of stationary waves for the nonlinear Schoedinger equation. These analogies lead to the construction of new solutions to these equations and to the analysis of the corresponding moduli spaces.

## *Peletier, Mark (Technische Universiteit Eindhoven)*

### Many-spike structures in block copolymers

Block copolymers owe their technological importance to their ability to form structures with a wide variety of patterns. The simplest form is a diblock copolymer, which consists of two subpolymers (A and B 'blocks') grafted together. If the two blocks repel each other, phase separation ensues, causing A-rich and B-rich regions to appear. At the same time, the chemical bond between the two parts limits the spatial scale of separation, giving rise to many different patterns, all with a well-defined length scale.

Most of these patterns are all but inaccessible by analytical means, even via mean-field theories such as the one by Ohta and Kawasaki. In a limit of highly skewed mass distribution, however, we show that this functional admits structures for which the limiting energy takes a simple form. These consist of many concentrated, near-spherical regions of one phase in a sea of the other phase.

The energy for these structures is dominated by a single-spike term, which penalizes each spike independently. This term drives the system towards spikes of a well-defined size. At the next level the interaction between the spikes is given by a Coulomb interaction potential, giving rise to approximately periodic arrangements.

## *Robert , Frederic*

### The Lin-Ni's conjecture: the influence of the curvature

Lin and Ni conjectured that the Neuman system

$$\left\{ \begin{array}{ll} -\Delta u + \epsilon u = u^{\frac{n+2}{n-2}} & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{array} \right\}$$

admits only the constant solution for small positive $\epsilon$. This conjecture has stimulated and generated intensive contributions in the past decades. In particular, it has been known for long that the conjecture is not valid in small dimensions (except 3). Moreover, recent exemples of Wei show that it is not valid when the mean curvature is negative and for infinite energies. We prove here that the conjecture is valid in the remaining cases. This is joint work with O.Druet and J.Wei.

## *Sere, Eric (Universite Paris-Dauphine)*

### Existence and nonexistence of a ground state for a relativistic model of the atom

This is joint work with Mathieu Lewin and Philippe Gravejat. We consider the reduced Bogoliubov-Dirac-Fock model describing relativistic electrons near an atomic nucleus. We show that positively charged ions and neutral atoms have a ground state, and that there is no ground state when the number of electrons is at least twice the number of protons.

## *Sternberg, Peter (Indiana University)*

### Ginzburg-Landau on thin shells and manifolds

I will discuss results on the Ginzburg-Landau energy in the presence of an applied magnetic field when the superconducting sample occupies a thin neighborhood of a bounded, closed manifold in 3-space. This includes (i) Gamma-convergence to a reduced Ginzburg-Landau model posed on the manifold (ii) Analysis of the limiting problem, focusing on the construction of two-vortex critical points when the manifold is a simply connected surface of revolution and the applied field is constant and vertical (iii) Calculation of the asymptotic value of the first critical field $H_{c1}$ for large values of the Ginzburg-Landau parameter when the manifold is a surface of revolution.

This is joint work with Andres Contreras.

## *Tonegawa, Yoshihiro*

### Existence of mean curvature flow with non-smooth transport term

We establish the existence of time-dependent family of integral varifolds whose velocity is given by its mean curvature plus flow field which belongs to a certain

Sobolev class. The method is based on the phase field approximation and its singular perturbation analysis. The result is applied to a coupled system of two-phase flow equation with surface tension.

## *Wei, Jun-cheng (Chinese University of Hong Kong)*

### Allen-Cahn Equation, De Giorgi Conjecture and Minimal Surfaces

We consider the following Allen-Cahn equation

$$\Delta u + u - u3 = 0 \text{ in } R^N$$

A famous conjecture due to De Giorgi states that any bounded solution with montone property

$$\frac{\partial u}{\partial y_N} > 0$$

must be one-dimensional. Great progress has been achieved in the last 15 years: Ghoussoub-Gui ($N = 2$), Ambrosio-Cabre ($N = 3$), Savin ($N = 4, 5, 6, 7, 8$). In 2008, we constructed a counterexample for dimensions $N \geq 9$, based on the counterexample of minimal graph by Bombieri, De Giorgi and Giusti (1969). We discuss our approach and recent work on the connection between embedded minimal surfaces with finite total curvature in $R3$ and the finite Morse index solutions of Allen-Cahn equation in $R3$. (Joint work with M. del Pino, M.Kowalczyk)