Global Newtonian limit for the Relativistic Boltzmann Equation near Vacuum

Robert Mills Strain III (University of Pennsylvania)

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Relativistic Boltzmann Equation

- The relativistic Boltzmann Equation

$$p^{\mu}\partial_{\mu} f = \mathcal{C}(f,f),$$

- The "transport term" is a lorentz inner product with signature $(-+++)$

$$p^{\mu}\partial_{\mu} = p_0\partial_t + p\cdot\nabla_x$$

Here $p_0 = \sqrt{c^2+|p|^2}$ is the relativistic energy.

- The "collision operator" is $\mathcal{C}(f,h) = \mathcal{C}_g(f,h) - \mathcal{C}_l(f,h)$.
- With Gain term

$$\mathcal{C}_g(f,h) = \frac{c}{2}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}W(p,q|p',q')f(p')h(q')$$

And Loss term

$$\mathcal{C}_l(f,h) = \frac{c}{2}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}W(p',q'|p,q)f(p)h(q)$$

Transition Rate

- The kernel $W(p,q|p',q')$ is called the transition rate:

$$W(p,q|p',q') = s\sigma(g,\theta)\delta^{(4)}(p^\mu + q^\mu - p^{\mu\prime} - q^{\mu\prime}),$$

- $\sigma(g,\theta)$ is the differential cross-section or scattering kernel.
- $p^\mu = (-p_0, p)$ and $q^\mu = (-q_0, q)$ are relativistic four-vectors: $p, q \in \mathbb{R}^3$. The Lorentz inner product is then

$$p^\mu q_\mu = -p_0 q_0 + p \cdot q.$$

- The energy in the center-of-momentum system is

$$s = -(p^\mu + q^\mu)(p_\mu + q_\mu) \geq 0$$

- Lastly we define the relative momentum

$$g^2 = (p^\mu - q^\mu)(p_\mu - q_\mu) \geq 0$$

And the scattering angle θ: $\cos\theta = (p^\mu - q^\mu)(p'_\mu - q'_\mu)/g^2$

Physical Cross Section

- The Collisional Cross Sections can be computed via Quantum Field Theory (QFT), (e.g. Peskin & Schroeder 1995)
- They can not be computed from a Scattering Problem because there is no widely accepted theory of relativistic N-Body dynamics (or 2-Body).
- Short Range Interactions:

$$\sigma \equiv \text{constant}.$$

This “hard-ball” cross section is the relativistic analouge of the hard-sphere kernel in the Newtonian case.

Physical Cross Section (Cont...)

- Møller Scattering: electron-electron scatttering:

$$\sigma = r_0^2 \frac{1}{u^2(u^2-1)^2}\left\{\frac{(2u^2-1)^2}{\sin^4\theta} - \frac{2u^4-u^2-\frac{1}{4}}{\sin^2\theta} + \frac{1}{4}(u^2-1)^2\right\}.$$

  where the magnitude of total four-momentum

$$u = \frac{\sqrt{s}}{2mc}$$

  and $r_0 = \frac{e^2}{4\pi mc^2}$ is the classical electron radius.

- Compton Scattering: photon-electron scattering.

$$\sigma = \frac{1}{2} r_0^2 (1-\xi)\left\{1 + \frac{1}{4}\frac{\xi^2(1-\cos\theta)^2}{1-\frac{1}{2}\xi(1-\cos\theta)} + \left(\frac{1-(1-\frac{1}{2}\xi)(1-\cos\theta)}{1-\frac{1}{2}\xi(1-\cos\theta)}\right)^2\right\}$$

  where

$$\xi = 1 - \frac{m^2c^2}{s}.$$

Glassey-Strauss reduction of the collision integrals

- Glassey and Strauss (1993) reduction

$$\mathcal{C}(f,h) = \int_{\mathbb{R}^3\times S^2} \frac{s\sigma(g,\theta)}{p_0 q_0} B(p,q,\omega)[f(p')h(q') - f(p)h(q)]d\omega dq$$

- with kernel

$$B(p,q,\omega) \equiv c\frac{(p_0+q_0)^2 p_0 q_0 \left|\omega\cdot\left(\frac{p}{p_0}-\frac{q}{q_0}\right)\right|}{[(p_0+q_0)^2-(\omega\cdot[p+q])^2]^2}.$$

- Post-Collisional Momentum:

$$p' = p + a(p,q,\omega)\omega, \quad q' = q - a(p,q,\omega)\omega,$$

- where: $a(p,q,\omega) = \frac{2(p_0+q_0)p_0 q_0\left\{\omega\cdot\left(\frac{q}{q_0}-\frac{p}{p_0}\right)\right\}}{(p_0+q_0)^2-\{\omega\cdot(p+q)\}^2}$
- The energies: $p_0' = p_0 + N_0$ and $q_0' = q_0 - N_0$:

$$N_0 \equiv \frac{2\omega\cdot(p+q)\{p_0(\omega\cdot q) - q_0(\omega\cdot p)\}}{(p_0+q_0)^2-\{\omega\cdot(p+q)\}^2}$$

Center of Momentum Reduction of the Collision Integrals

- Lorentz Transformations grant another reduction:

$$\mathcal{C}(f,h) = \int_{\mathbb{R}^3\times S^2} v_c\ \sigma(g,\theta)\ [f(p')h(q') - f(p)h(q)]\ d\omega dq.$$

- $v_c = v_c(p,q)$ is the Møller velocity:

$$v_c(p,q) \equiv \frac{c}{2}\sqrt{\left|\frac{p}{p_0} - \frac{q}{q_0}\right|^2 - \frac{1}{c^2}\left|\frac{p}{p_0}\times\frac{q}{q_0}\right|^2} = \frac{c}{4}\frac{g\sqrt{s}}{p_0 q_0}.$$

- The post collisional momentum can be written:

$$p' = \frac{p+q}{2} + g\left(\omega + (\gamma-1)(p+q)\frac{(p+q)\cdot\omega}{|p+q|^2}\right),$$

$$q' = \frac{p+q}{2} - g\left(\omega + (\gamma-1)(p+q)\frac{(p+q)\cdot\omega}{|p+q|^2}\right),$$

where $\gamma = (p_0+q_0)/\sqrt{s}$.

Center of Momentum Reduction (continued...)

- The energies are

$$p_0' = \frac{p_0 + q_0}{2} + \frac{g}{\sqrt{s}}\omega \cdot (p+q),$$

$$q_0' = \frac{p_0 + q_0}{2} - \frac{g}{\sqrt{s}}\omega \cdot (p+q).$$

- These will be the coordinates we use. As far as we know this is the first time the coordinates are used in a mathematically oriented paper.
- More generally can do this reduction with any Lorentz Transformation Λ and obtain

$$P' = \frac{1}{2}\Lambda^{-1}\begin{pmatrix} s \\ g\omega \end{pmatrix}, \quad Q' = \frac{1}{2}\Lambda^{-1}\begin{pmatrix} s \\ -g\omega \end{pmatrix},$$

Need only

$$\Lambda(P+Q) = (\sqrt{s}, 0, 0, 0)^t,$$

Lorentz Transformations Mapping Into $p + q = 0$

$$\Lambda(P+Q) = (\sqrt{s}, 0, 0, 0), \ \Lambda(P-Q) = (0, 0, 0, g)$$

$$\Lambda = \begin{pmatrix} \frac{p_0+q_0}{\sqrt{s}} & -\frac{p_1+q_1}{\sqrt{s}} & -\frac{p_2+q_2}{\sqrt{s}} & -\frac{p_3+q_3}{\sqrt{s}} \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ 0 & \frac{(p\times q)_1}{|p\times q|} & \frac{(p\times q)_2}{|p\times q|} & \frac{(p\times q)_3}{|p\times q|} \\ \frac{p_0-q_0}{g} & -\frac{p_1-q_1}{g} & -\frac{p_2-q_2}{g} & -\frac{p_3-q_3}{g} \end{pmatrix}.$$

$$\Lambda^1_0 = \frac{2|p\times q|}{\sqrt{s}g} = \frac{|p\times q|}{\sqrt{(p^\mu q_\mu)^2 - c^4}}.$$

$$\Lambda^1_i = \frac{2\left(p_i\left\{p_0 - q_0 p^\mu q_\mu\right\} + q_i\left\{q_0 - p_0 p^\mu q_\mu\right\}\right)}{\sqrt{s}g|p\times q|} \quad (i = 1, 2, 3).$$

Formal Newtonian Limit in the Center of Momentum

- The collision operator converges (as $c \to \infty$) to

$$\mathcal{Q}_\infty(f,g) = \frac{1}{2}\int_{\mathbb{R}^3\times S^2} |p-q|[f(p')g(q') - f(p)g(q)]d\omega dq.$$

  This is again when $\sigma = 1$... (other cross sections will give other limits.)

- The variables in this σ-representation are

$$p' = \frac{p+q}{2} + \frac{1}{2}|p-q|\omega, \quad q' = \frac{p+q}{2} - \frac{1}{2}|p-q|\omega,$$

- The newtonian limit is again the Classical Boltzmann equation (now in σ-representation):

$$\partial_t f + p\cdot\nabla_x f = \mathcal{C}_\infty(f,f)$$

Previous Results for the relativistic Boltzmann Equation

- Glassey & Strauss (1991)
  - compute $\frac{\partial(p',q')}{\partial(p,q)} = -\frac{p_0' q_0'}{p_0 q_0}$
- Dudyński and Ekiel-Jeżewska (1992)
  - large data Diperna-Lions renormalized solutions
- Glassey & Strauss (1993, 1995)
  - *Global Stability of* e^{-p_0} *in* $\mathbb{T}^3_x$, $\mathbb{R}^3_x$,
- Andréasson (1996) & Wennberg (1997)
  - *Regularity of the Gain Term*
- Andréasson, Calogero, Illner, (2004)
  - *blowup for gain-term-only*,

Previous Results for the relativistic Boltzmann Equation

- Calogero (2004)
  - *In* $\mathbb{T}^3_x$, *Local in Time uniform existence, Newtonian limit,*
- Glassey (2006)
  - *Global solutions to the Cauchy problem for the relativistic Boltzmann equation with near-vacuum data with* $c = 1$,
- Ha, Kim, Lee, Noh (2007)
  - Conditional L^1 scattering:

$$\|f^{\#}(t) - f_{+}(t)\|_{L^1_{x,p}} \to 0, \quad t \to \infty$$

  Existence not known in strong enough space for scattering.
- S (2009)
  - *Global Existence Near Vacuum in* $\mathbb{R}^3_x$ *uniform in* $c \geq 1$,
  Validity of Global Newtonian Limit,
  Solution space sufficient for Scattering

Solutions near Vaccum

- Illner-Shinbrot (1984) method for constructing solutions to the Newtonian System near Vacuum.
- Important Newtonian Symmetry:

$$|x+tv|^2+|x+tu|^2=|x+tv'|^2+|x+tu'|^2$$

- Follows from Newtonian conservation of energy:

$$|v|^2+|u|^2=|v'|^2+|u'|^2$$

$$v+u=v'+u'$$

- This apparently fails under special relativity:

$$p_0+q_0=p_0'+q_0'$$

$$p+q=p'+q'$$

- These relativistic Symmetries make it hard to find a positive dispersive quantity as above.
  May not exist. Many have looked for it.

Solutions near Vaccum (S-2009)

- Important Relativistic Invariant:

$$c^3\frac{t^2}{p_0}+\frac{p_0}{c}|x|^2+c^3\frac{t^2}{q_0}+\frac{q_0}{c}|x+t(\hat{p}-\hat{q})|^2$$
$$=c^3\frac{t^2}{q_0'}+\frac{q_0'}{c}\left|x+t(\hat{p}-\hat{q}')\right|^2+c^3\frac{t^2}{p_0'}+\frac{p_0'}{c}\left|x+t(\hat{p}-\hat{p}')\right|^2.$$

- Difficulty: Temporal components due to coupling of space and time via Lorentz Invariance.
- Problem: the transport operator doesn't appear to produce the kind of time decay rates that seem to be required to exploit this symmetry.
- Angular Cut-Off a-la Grad used to handle temporal components of invariant,
  Cut-Off disappears in Newtonian Limit.

Glassey's (2006) Theorem

- Glassey gave the first construction of solutions near Vaccum to the relativistic Boltzmann Equation
- Space:

$$e^{p_0}(1+|x\times p|^2)^{1+\delta} f(t,x,p) \leq b_0, \quad 0<\delta<1$$

- Cross Sectional Assumption:

$$\sigma(p,q,\omega) \leq \frac{|\omega\cdot(q\times\hat{p})|\tilde{\sigma}(\omega)}{g(1+g^2)^{\delta+1/2}}$$

And

$$\int_{S^2} d\omega \frac{\tilde{\sigma}(\omega)}{1+|\omega\cdot z|} \leq c|z|^{-1}.$$

- Glassey's (2006) Result: Global existence for near Vacuum Data.

New functional space

- Weight:

$$\rho_c(x,p) = \exp\left(-\alpha p_0 |x|^2/c\right) J^\beta(q), \quad \alpha, \beta > 0.$$

Above $J^\beta(q)$ is the relativistic Maxwellian:

$$J^\beta(q) = \left(4\pi c K_2(c^2)\right)^{-\beta} e^{-\beta c p_0}$$

- Notice that as $c \to \infty$

$$\rho_c(x,p) \to \rho_\infty(x,p) = \exp\left(\alpha |x|^2\right) e^{\beta |p|^2},$$

which is the Newtonian space for Near Vacuum Solutions.

- "Almost invariance" in this space

$$\frac{q_0'}{c}\left|x + t\left(\hat{p} - \hat{q}'\right)\right|^2 + \frac{p_0'}{c}\left|x + t\left(\hat{p} - \hat{p}'\right)\right|^2$$
$$= \frac{p_0}{c}|x|^2 + \frac{q_0}{c}\left|x + t\left(\hat{p} - \hat{q}\right)\right|^2 + \gamma t^2$$

- Deadly Term: $\gamma = c^3\left(\frac{1}{p_0} + \frac{1}{q_0} - \frac{1}{p_0'} - \frac{1}{q_0'}\right)$

- Recall the Bad Term:

$$\gamma = c^3 \left(\frac{1}{p_0} + \frac{1}{q_0} - \frac{1}{p_0'} - \frac{1}{q_0'} \right)$$

If $\gamma \geq 0$, then we would be in buisness.

- In fact we can do better. For $B \geq 0$ and $0 \leq a < 1$ and $t > 0$,

$$h = h(x, p, q, t, c) = \frac{B}{t^2} + a \frac{\beta q_0 \left| x + t \left(\hat{p} - \hat{q} \right) \right|^2 / c}{t^2} \geq 0.$$

- Define the cut-off set

$$\mathcal{B}_c = \left\{ \omega : \gamma \geq -h \right\}.$$

We remark that h can be quite large, and often $\mathcal{B}_c = S^2$.

- To handle this term we introduce a cut-off in the cross section

$$\sigma(\omega) = \sigma(\omega) \mathbf{1}_{\mathcal{B}}(\omega),$$

Here $\mathbf{1}_{\mathcal{B}_c}(\omega)$ is the indicator function of the set $\mathcal{B}_c(\omega)$.

- This is an angular cut-off:

$$\mathcal{B}(\omega) = \{\omega \in S^2 : \gamma \geq -h\}$$

Compare to Grad's angular cutoff...

- Not at all a limitation in the Newtonian Limit ($c \to \infty$):

$$\mathbf{1}_{\mathcal{B}}(\omega) = 1, \quad \forall c \geq c_*(p, q, T)$$

- Otherwise we use a generic collision kernel

$$\sigma(\omega, p, q) \leq \left(A_1 + A_2 g^{-\gamma}\right) \sin^\beta \theta.$$

Above A_1 and A_2 are positive constants. We allow

$$0 \leq \gamma < -3, \quad \beta \geq 0.$$

Calogero (2004) - Local in time Newtonian Limit

- *Local in Time* Existence result. Uniform time of existence $T > 0$ for $c \geq c_0$. Uses Illner - Shinbrot (1984).
- Spatially periodic solutions $x \in \mathbb{T}^3$.
- Uses existence of solultions to the limiting Hard-Sphere Newtonian Boltzmann equation. (Illner - Shinbrot.)
- Establishes (Local) Newtonian Limit as $c \to \infty$.
- To our knowledge, all previous results on classical Newtonian limits for Kinetic Equations are Local in Time.
  e.g. Degond - Palaiseau (1986),
  Asano-Ukai (1986),
  Schaeffer (1986),
  Rendall (1994) Vlasov - Einstein, etc.

Mild Formulation of the Cauchy Problem

- Define the solution along it's trajectories

$$f^{\#}(t,x,p) = f\left(t, x + t\hat{p}, p\right)$$

- The Mild Formulation of the Cauchy Problem

$$f^{\#}(t,x,p) = f_0(x,p) + \int_0^t ds \ \mathcal{Q}^{\#}(f,f)$$

- with Collision Operator

$$\mathcal{Q}^{\#}(f,f) = \int_{\mathbb{R}^N} dq \int_{S^{N-1}} d\omega \ v_c \ \sigma \ f(t, x+t\hat{p}, p')f(t, x+t\hat{p}, q')$$

$$- \int_{\mathbb{R}^N} dq \int_{S^{N-1}} d\omega \ v_c \ \sigma \ f(t, x+t\hat{p}, q)f(t, x+t\hat{p}, p)$$

- Recall we are in the Center - of - Momentum Variables (seems to be crucial)
- This is the mild form of: $\partial_t f + \hat{p} \cdot \nabla_x f = \mathcal{Q}(f,f)$

Global Existence Theorem uniform for $c \geq 1$

Theorem

Consider initial values $0 \leq f_{0,c}(x,p) \in C^0(\mathbb{R}^3_x \times \mathbb{R}^3_p)$ *and additionally*

$$\frac{f_{0,c}(x,p)}{\rho_c(x,p)} \leq b.$$

There exists a positive number b_0*, which is independent of the speed of light* $c \geq 1$*, with the property that if* $b \leq b_0$ *then there exists a unique non-negative global solution* $f_c(t,x,p)$ *to the mild form of the Cauchy problem.*

This solution satifies the estimates

$$\|f^{\#}_c\|_c = \sup_{t,x,p} \frac{f^{\#}_c(t,x,p)}{\rho_c(x,p)} \leq b_1,$$

The constant $b_1 = b_1(b_0)$ *is explicit and does not depend upon* $c \geq 1$*.*

Newtonian Limit $c \to \infty$

Theorem (Newtonian Limit)

Suppose that for any $c_n, c_m \geq c \geq 1$ and $\epsilon > 0$ we have a collection of initial data satisfying the estimates

$$\|f_{0,c_n} - f_{0,c_m}\|_{L^1_p L^\infty_x} \leq A/c^{1+\epsilon}.$$

For some uniform constant $A > 0$ which is independent of c, c_n, c_m. Further suppose

$$\|\nabla_x f_{0,c_n}\|_{L^\infty_x L^1_p} + \|\nabla_p f_{0,c_n}\|_{L^\infty_x L^1_p} \leq A_1 < \infty$$

uniformly in c_n. Then for any fixed $T > 0$ (which is allowed to be large) the solution corresponding to these initial data satisfy

$$\|f_{c_n}(t) - f_{c_m}(t)\|_{L^1_p L^\infty_x} \leq A(T)/c, \quad c \to \infty.$$

These solutions thereby converge to a solution of the Newtonian Boltzmann equation.

- We do not assume the existence of any solution to the limit equation, instead we recover global existence in the limit.
- Even though convergence is in the weak space $L^1_p L^\infty_x$, we still recover

$$f_{c_m} \to f_\infty$$

with

$$f_\infty(t,x,p) \leq b_1 e^{-\alpha|x|^2} e^{-\beta|p|^2}$$

(In the Classical Case you would have $p = v$.)

Relativistic Vlasov-Maxwell-Boltzmann System

- Relativistic Vlasov-Maxwell-Boltzmann System:

$$\partial_t F_+ + \frac{p}{p_0} \cdot \nabla_x F_+ + \left(E + \frac{p}{p_0} \times B \right) \cdot \nabla_p F_+ = \mathcal{C}(F_+, F_+) + \mathcal{C}(F_+, F_-)$$

$$\partial_t F_- + \frac{p}{p_0} \cdot \nabla_x F_- - \left(E + \frac{p}{p_0} \times B \right) \cdot \nabla_p F_- = \mathcal{C}(F_-, F_-) + \mathcal{C}(F_-, F_+)$$

- coupled with Maxwell's Equations:

$$\partial_t E - \nabla_x \times B = -\mathcal{J}, \quad \partial_t B + \nabla_x \times E = 0$$

- And constraints: $\nabla_x \cdot B = 0, \quad \nabla_x \cdot E = \rho$
- Non-Linear coupling

$$\mathcal{J} = \int_{\mathbb{R}^3} \frac{p}{p_0} (F_+ - F_-) \, dp, \quad \rho = \int_{\mathbb{R}^3} (F_+ - F_-) \, dp$$

- Relativistic Boltzmann Collision Operator

$$\mathcal{C}(F_+, F_-) = \int_{\mathbb{R}^3 \times S^2} \frac{s}{p_0 q_0} B[F_+(p')F_-(q') - F_+(p)F_-(q)] d\omega dq$$

- with kernel

$$B = B(p,q,\omega) \equiv c \frac{(p_0+q_0)^2 p_0 q_0 \left|\omega \cdot \left(\frac{p}{p_0} - \frac{q}{q_0}\right)\right|}{\left[(p_0+q_0)^2 - (\omega \cdot [p+q])^2\right]^2}.$$

- Post-Collisional Momentum:

$$p' = p + a(p,q,\omega)\omega, \quad q' = q - a(p,q,\omega)\omega,$$

- where: $a(p,q,\omega) = \frac{2(p_0+q_0)p_0 q_0 \left\{\omega \cdot \left(\frac{q}{q_0} - \frac{p}{p_0}\right)\right\}}{(p_0+q_0)^2 - \{\omega \cdot (p+q)\}^2}$

Conservation Laws

The conservation of mass, total momentum and total energy for solutions as

$$\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} m_+F_+(t) = \frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} m_-F_-(t) = 0,$$

$$\frac{d}{dt}\left\{\int_{\mathbb{T}^3\times\mathbb{R}^3} p(m_+F_+(t) + m_-F_-(t)) + \frac{1}{4\pi}\int_{\mathbb{T}^3} E(t)\times B(t)\right\} = 0,$$

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3}(m_+p_0^+F_+(t) + m_-p_0^-F_-(t)) + \cdots\right\} = 0.$$

$$\cdots + \frac{d}{dt}\left\{\frac{1}{8\pi}\int_{\mathbb{T}^3}|E(t)|^2 + |B(t)|^2\right\} = 0.$$

The entropy increasing

$$-\frac{d}{dt}\int\{F_+\ln F_+ + F_-\ln F_-\}\,dxdp \geq 0.$$

This is Boltzmann's H-Theorem

Asymptotic Stability of Relativistic Vlasov-Maxwell-Boltzmann System

Theorem (Guo-S, 2009)

Fix $N \geq 4$, $x \in \mathbb{T}^3$. *Let* $F_0(x,p) = \mu_{rel} + \sqrt{\mu_{rel}} f_0(x,p) \geq 0$, *where* $\mu_{rel} = e^{-p_0}$. *Assume*

$$Conservation\ Laws(F_0, E_0, B_0) = Conservation\ Laws(\mu_{rel}, 0, \bar{B})$$

Then $\exists C_N > 0$, $\epsilon_N > 0$ *small enough such that if*

$$\mathcal{E}_N(f_0) \leq \epsilon_N$$

Then there exists a unique positive global smooth solution with

$$\frac{d}{dt}\mathcal{E}_N(t) + \mathcal{D}_N(t) \leq 0$$

$\mathcal{D}_N(t)$ measures the dissipation of the linearized collision operator.

Key Difficulty

- **Problem:** derivatives of the collisional map grow

$$\left|\nabla_p q_i'\right| + \left|\nabla_p p_i'\right| \leq C q_0^5 \left(1 + |p \cdot \omega|^{1/2} \mathbf{1}_{\{|p\cdot\omega| > |p\times\omega|^{3/2}\}}\right).$$

- There is no place to put these moments...
- We develop a long series of non-linear changes of coordinates to facilitate integration by parts and to move these weights onto lower order derivatives.

Thanks

THANK YOU!