Non-uniform specification properties

Ronnie Pavlov

University of Denver
www.math.du.edu/∼rpavlov

Current Trends in Dynamical Systems & the Mathematical Legacy of Rufus Bowen
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Outline

- Intro to topological pressure/equilibrium states

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- Some ideas from the proof

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  - If $\phi(A) = \phi(B)$, <u>all</u> invariant measures are equilibrium states

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  - **Corollary:** all Hölder potentials are Bowen for subshifts

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- Let's start with mixing

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- Informally: both gap function for (X, T) and partial sum growth rate for ϕ need not be bounded, just sublogarithmic

Applications

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  - **Conclusion:** $O(\log n)$ is the correct "tipping point" for f, g

Some words about the proof

- Bowen's approach was to create MME μ from limit of equidistributed periodic points and then prove μ has partial mixing, i.e. $\liminf_n \frac{\mu(A \cap T^n B)}{\mu(A)\mu(B)} > 0$

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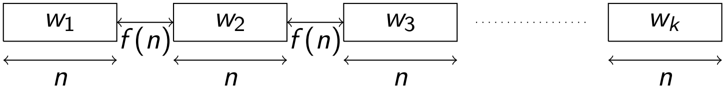

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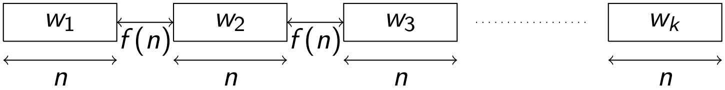

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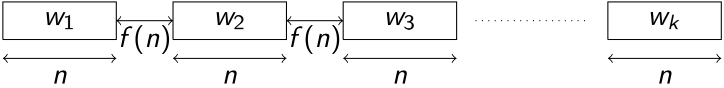

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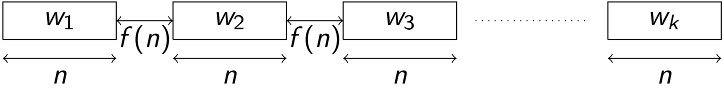

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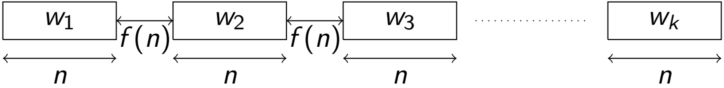

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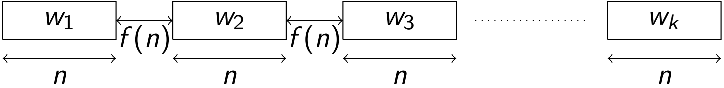

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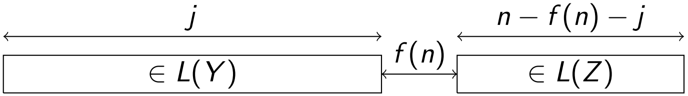

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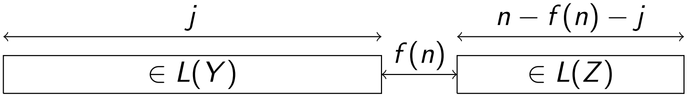

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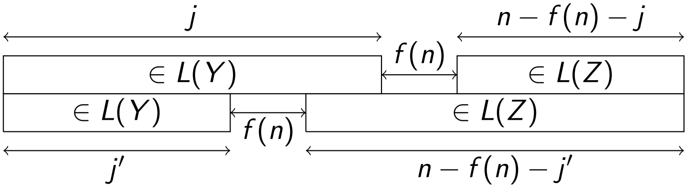

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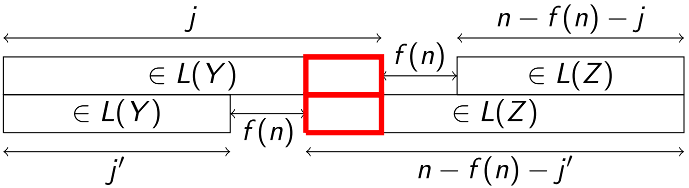

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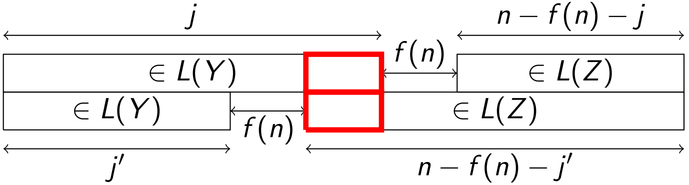

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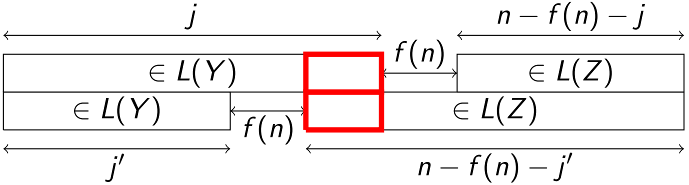

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- Then $|L_n(X)| \gtrapprox \frac{n}{N+f(n)} e^{(n-f(n))h(X)} = e^{nh(X)} n^{1-o(1)}$, contradicting Step 1

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  - Main new idea: use <u>maximal</u> ergodic theorem!

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- **Maximal ergodic theorem** tells us that
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- So, get "large" $W_n \subset L_n(X)$ of words where every prefix has proportion $\leq 2\mu(S) < 2\epsilon$ of visits to S

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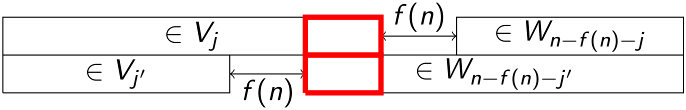

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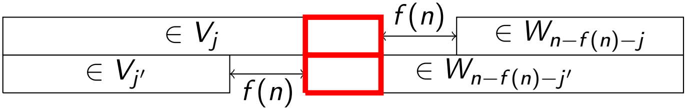

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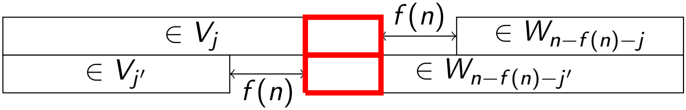

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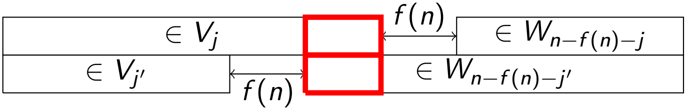

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- General proof (adding ϕ, expansive (X, T), etc.) similar

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- Does non-uniform specification ever imply existence of periodic points?
- Can "$o(\log n)$" replace "bounded" in other proofs using mixing and/or regularity hypotheses?

Thanks for listening!