A THEORY OF HYDRODYNAMIC TURBULENCE BASED ON NON-EQUILIBRIUM STAT. MECH.

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Overview.

- Study intermittency exponents ζ_p such that

$$\langle |\Delta \mathbf{v}|^p \rangle \sim \ell^{\zeta_p}$$

where $\Delta \mathbf{v}$ is contribution to fluid velocity at small scale ℓ.

[Claim:

$$\zeta_p = \frac{p}{3} - \frac{1}{\ln \kappa} \ln \Gamma(\frac{p}{3} + 1)$$

experimentally $(\ln \kappa)^{-1} = 0.32$, i.e., $\kappa \approx 20$ or 25].

- Distribution of radial velocity increment and relation with Kolmogorov-Obukhov.

- Reynolds number ≈ 100 at onset of turbulence.

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1. Obtaining the basic probability distribution.

• Kinetic energy goes down from large spatial scale ℓ_0 to small scales through a cascade of eddies of increasing order n so that

$$\mathbf{v} = \sum_{n \geq 0} \mathbf{v}_n$$

with viscous cutoff.

Eddy of order $n-1$ in ball $R_{(n-1)i}$ decomposes after time $T_{(n-1)i}$ into eddies of order n contained in balls $R_{nj} \subset R_{(n-1)i}$.

Balls R_{nj} form a partition of 3-space into roughly spherical polyhedra of linear size ℓ_{nj}, lifetime T_{nj}.

• Assume that the dynamics of each eddy is universal, up to scaling of space and time, and independent of other eddies.

Conservation of kinetic energy E yields

$$\sum_j \frac{E(R_{nj})}{T_{nj}} = \frac{E(R_{(n-1)i})}{T_{(n-1)i}}$$

Universality of dynamics and inviscid scaling give for initial eddy velocities

$$\frac{\mathbf{v}_n}{\ell_{nj}} = \frac{T_{(n-1)i}}{T_{nj}} \cdot \frac{\mathbf{v}_{n-1}}{\ell_{n-1}}$$

hence

$$\sum_j \int_{R_{nj}} \frac{|\mathbf{v}_n|^3}{\ell_{nj}} = \int_{R_{(n-1)i}} \frac{|\mathbf{v}_{n-1}|^3}{\ell_{(n-1)i}}$$

(implies intermittency).

• For simplicity assume size ℓ_{nj} depends only on n: $\ell_{(n-1)i}/\ell_{nj} = \kappa$. Then

$$\kappa \sum_j \int_{R_{nj}} |\mathbf{v}_n|^3 = \int_{R_{(n-1)i}} |\mathbf{v}_{n-1}|^3$$

• Assume that the distribution of the $\mathbf{v}_n$ between different R_{nj} maximizes entropy: microcanonical distribution $\to$ canonical distribution:

$$\sim \exp[-\beta|\mathbf{v}_n|^3]\, d^3\mathbf{v}_n$$

Integrating over angular variables:

$$\sim \exp[-\beta|\mathbf{v}_n|^3]|\mathbf{v}_n|^2\, d|\mathbf{v}_n| = \frac{1}{3}\exp[-\beta|\mathbf{v}_n|^3]\, d|\mathbf{v}_n|^3$$

hence $V_n = |\mathbf{v}|^3$ has distribution

$$\beta \exp[-\beta V_n]\, dV_n$$

• Finally since the average value β^{-1} of V_n is V_{n-1}/κ, V_n is distributed according to

$$\frac{\kappa}{V_{n-1}} \exp\left[-\frac{\kappa V_n}{V_{n-1}}\right] dV_n$$

Starting from a given value of V_0 the distribution of V_n is given by

$$\frac{\kappa\, dV_1}{V_0} e^{-\kappa V_1/V_0} \cdots \frac{\kappa\, dV_n}{V_{n-1}} e^{-\kappa V_n/V_{n-1}} \qquad (*)$$

The validity of $(*)$ is limited by dissipation due to the viscosity ν: we must have

$$V_n^{1/3} \ell_n > \nu$$

2. Calculating ζ_p.

- To compute the mean value of $|\mathbf{v}_n|^p = V_n^{p/3}$ we note that

$$\frac{\kappa}{V_{n-1}} \int \exp\left[-\frac{\kappa V_n}{V_{n-1}}\right].V_n^{p/3}\, dV_n = \left(\frac{V_{n-1}}{\kappa}\right)^{p/3} \int \exp[-w].w^{p/3}\, dw$$

$$= \kappa^{-p/3} V_{n-1}^{p/3} \Gamma\left(\frac{p}{3}+1\right)$$

hence, using induction and $\ell_n/\ell_0 = \kappa^{-n}$,

$$\langle V_n^{p/3} \rangle = \frac{\kappa}{V_0} \int \exp\left[-\frac{\kappa V_1}{V_0}\right] dV_1 \cdots \frac{\kappa}{V_{n-1}} \int \exp\left[-\frac{\kappa V_n}{V_{n-1}}\right].V_n^{p/3}\, dV_n$$

$$= \kappa^{-np/3} V_0^{p/3} \Gamma\left(\frac{p}{3}+1\right)^n = V_0^{p/3}\left(\frac{\ell_n}{\ell_0}\right)^{p/3} \Gamma\left(\frac{p}{3}+1\right)^n$$

- Therefore

$$\ln\langle|\mathbf{v}_n|^p\rangle = \ln\langle V_n^{p/3}\rangle = \ln V_0^{p/3} + \frac{p}{3}\ln\left(\frac{\ell_n}{\ell_0}\right) - \frac{\ln(\ell_n/\ell_0)}{\ln\kappa}\ln\Gamma\Big(\frac{p}{3}+1\Big)$$

$$= \ln V_0^{p/3} + \ln\left(\frac{\ell_n}{\ell_0}\right).\left[\frac{p}{3} - \frac{1}{\ln\kappa}\ln\Gamma\Big(\frac{p}{3}+1\Big)\right] = \ln\left[V_0^{p/3}\Big(\frac{\ell_n}{\ell_0}\Big)^{\zeta_p}\right]$$

where

$$\zeta_p = \frac{p}{3} - \frac{1}{\ln\kappa}\ln\Gamma\Big(\frac{p}{3}+1\Big)$$

or

$$\langle|\mathbf{v}_n|^p\rangle = V_0^{p/3}\Big(\frac{\ell_n}{\ell_0}\Big)^{\zeta_p} \sim \ell_n^{\zeta_p}$$

as announced.

3. Estimating the probability distribution $F(u)$ of the radial velocity increment u. Relation with Kolmogorov-Obukhov.

- If $r \approx \ell_n$ we have $u \approx u_n \approx$ radial component of $\mathbf{v}_n$
$\Rightarrow$ rough estimate of the probability distribution of u:

$$F(u) = \Big(\prod_{k=1}^{n} \int_0^\infty \frac{\kappa \, dV_k}{V_{k-1}} e^{-\kappa V_k / V_{k-1}} \Big) \frac{1}{2 V_n^{1/3}} \chi_{[-V_n^{1/3}, V_n^{1/3}]}(u)$$

$$= \frac{1}{2} \big(\frac{\kappa^n}{V_0} \big)^{1/3} \int \cdots \int_{w_1 \cdots w_n > (\kappa^n / V_0)|u|^3} \prod_{k=1}^{n} \frac{dw_k \, e^{-w_k}}{w_k^{1/3}}$$

- The distribution $G_n(y)$ of $y = (\kappa^n / V_0)^{1/3} |u|$ is given by

$$G_n(y) = \int \cdots \int_{w_1 \cdots w_n > y^3} \prod_{k=1}^{n} \frac{dw_k \, e^{-w_k}}{w_k^{1/3}}$$

• This satisfies

$$e^t G_n(e^t) = (\phi^{*(n-1)} * \psi)(t) \qquad (**)$$

with

$$\phi(t) = 3\exp(3t - e^{3t}) \quad , \quad \psi(t) = e^t \int_t^\infty e^{-s}\phi(s)\, ds$$

[$\Rightarrow$ $G_n(y)$ is a decreasing function of y].

• For small u, G_n gives a good description of the distribution of u, with normalized $\langle |u|^2 \rangle$ (see Schumacher et al.).

• $(**)$ suggests a lognormal distribution with respect to u in agreement with Kolmogorov-Obukhov, but this fails because ϕ, ψ tend to 0 only exponentially at $-\infty$.

4. The onset of turbulence.

• We may estimate the Reynolds number $\mathcal{R}e = |\mathbf{v_0}|\ell_0/\nu$ for the onset of turbulence by taking

$$1 \approx \left\langle \frac{\nu}{|\mathbf{v}_1|\ell_1} \right\rangle = \left\langle \frac{\nu}{V_1^{1/3}\kappa_{-1}\ell_0} \right\rangle = \mathcal{R}e^{-1}\left\langle \kappa^{4/3}\left(\frac{V_0}{\kappa V_1}\right)^{1/3} \right\rangle$$

[Relation to dissipation is dictated by dimensional arguments] $\Rightarrow$

$$\mathcal{R}e \approx \kappa^{4/3}\int_0^\infty \left(\frac{\kappa V_1}{V_0}\right)^{-1/3} \frac{\kappa\, dV_1}{V_0}\, e^{-\kappa V_1/V_0}$$

$$= \kappa^{4/3}\int_0^\infty \alpha^{-1/3}\, d\alpha\, e^{-\alpha} = \kappa^{4/3}\Gamma\left(\frac{2}{3}\right)$$

Taking $1/\ln\kappa = .32$ hence $\kappa^{4/3} = 64.5$, with $\Gamma(2/3) \approx 1.354$ gives $\mathcal{R}e \approx 87$ agreeing with $\mathcal{R}e \approx 100$ as found in Schumacher et al.