A homology theory for Smale spaces

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Let (X, d) be a compact metric space, φ be a homeomorphism of X such that (X, d, φ) is an irreducible Smale space or the basic set for an Axiom A system.

For $p \geq 1$, let

$$per_p(X, \varphi) = \#\{x \in X \mid \varphi^p(x) = x\}.$$

Theorem 1 (Manning). *For (X, φ) as above, the Artin-Mazur zeta function*

$$\zeta_\varphi(t) = exp\left(\sum_{p=1}^{\infty} \frac{per_p(X, \varphi)}{p} t^p\right)$$

is rational.

Bowen asked if there exist a homology theory for such systems that explains this result. (Problem 7.)

For each $n \geq 0$, $H_n(X, \varphi)$ is a finite-dimensional vector space, non-zero for only finitely many n, automorphisms φ_n of each and

$$\sum_{n=0}^{\infty} (-1)^n \quad Tr[(\varphi_n)^p : H_n(X, \varphi) \quad \to H_n(X, \varphi)]$$

$$= \#\{x \in X \mid \varphi^p(x) = x\},$$

for all $p \geq 1$.

This is an analogue of the Lefschetz formula for smooth maps of manifolds and immediately implies Manning's rationality result.

The point of this talk: Yes.

In fact, there are two H_n^s, H_n^u, $n \in \mathbb{Z}$, and these are finite rank abelian groups. (Use $H_n^s \otimes \mathbb{Q}$ or $H_n^s \otimes \mathbb{R}$ above.)

Smale spaces (D. Ruelle)

(X, d) compact metric space, $\varphi : X \to X$ homeomorphism, $0 < \lambda, \epsilon_0 < 1$,

There is a continuous map

$$[\cdot,\cdot] : \{(x,y) \in X \times X \mid d(x,y) \leq \epsilon_0\} \to X$$

($[x,y]$ is the intersection of the local stable set of x with the local unstable set of y) such that

$$\begin{aligned} [x,x] &= x, \\ [x,[y,z]] &= [x,z]\,, \\ [[x,y]\,,z] &= [x,z]\,, \\ [\varphi(x),\varphi(y)] &= \varphi\,[x,y] \end{aligned}$$

whenever all are defined and

$$[x,y] = y \Rightarrow d(\varphi(x),\varphi(y)) \leq \lambda d(x,y)$$

and

$$[x,y] = x \Rightarrow d(\varphi^{-1}(x),\varphi^{-1}(y)) \leq \lambda d(x,y)$$

We define, for x in X and $0 < \epsilon \leq \epsilon_0$, the local stable and unstable sets by

$$\begin{aligned} X^s(x,\epsilon) &= \{y \in X \mid d(x,y) < \epsilon, [x,y] = y\}, \\ X^u(x,\epsilon) &= \{y \in X \mid d(x,y) < \epsilon, [x,y] = x\}, \end{aligned}$$

and the global stable and unstable sets by

$$\begin{aligned} X^s(x) &= \{y \in X \mid \lim_{n\to+\infty} d(\varphi^n(x), \varphi^n(y)) = 0\}, \\ X^u(x) &= \{y \in X \mid \lim_{n\to+\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0.\} \end{aligned}$$

The map

$$[\cdot,\cdot] : X^u(x,\epsilon) \times X^s(x,\epsilon) \to X$$

is a homeomorphism to a neighbourhood of x with inverse

$$y \to ([y,x],[x,y]).$$

Example 2 (Fried)**.** *Every basic set for an Axiom A system is a Smale space (for some metric).*

- A Smale space does not need to be embedded in a manifold.

- A Smale space can have wandering points.

- The fibred product of two Smale spaces is again a Smale space.

  Given $Y \xrightarrow{\pi} X \xleftarrow{\rho} Z$

  the fibred product is

$$\{(y, z) \in Y \times Z \mid \pi(y) = \rho(z)\}$$

Example 3. *Every shift of finite type (SFT) is a Smale space. These are exactly the totally disconnected Smale spaces.*

Example 4. *q/p-solenoid*

Let $p < q$ be primes.

$$X = \mathbb{Q}_p \times \mathbb{R} \times \mathbb{Q}_q / \mathbb{Z}\left[(pq)^{-1}\right],$$

with

$$\varphi\,[x, y, z] = \left[p^{-1}qx, p^{-1}qy, p^{-1}qz\right]$$

Expanding coordinates $\mathbb{Q}_p \times \mathbb{R} \times \{z\}$ and contracting coordinates $\{x\} \times \{y\} \times \mathbb{Q}_q$.

Example 5. *Nekrashevych: construction from actions of self-similar groups.*

Example 6. *R.F. Williams' solenoids, expanding attractors*

Example 7. *S. Wieler's solenoids.*

To find a homology theory for Smale spaces.

Step 1: Find the invariant for shifts of finite type: Wolfgang Krieger (1980). (There is also another by Bowen and Franks.)

Step 2: Extend it to all Smale spaces.

Going from 1. to 2. involves Markov partitions. But ordinary Markov partitions will not do, we need *better* Markov partitions.

Krieger's invariants for SFT's

Motivation: Cech cohomology

For any compact space X, its Cech cohomology is computed by considering a finite, open cover $U_1, \ldots, U_N$ and the nerve of the cover; that is, the data of the non-empty intersections of these sets.

If X is totally disconnected, there are open covers which simplify this calculation: partitions into clopen sets, so that the intersections are all trivial.

Ultimately, the Cech cohomology (in dimension zero) is the abelian group generated by the clopen sets with relations

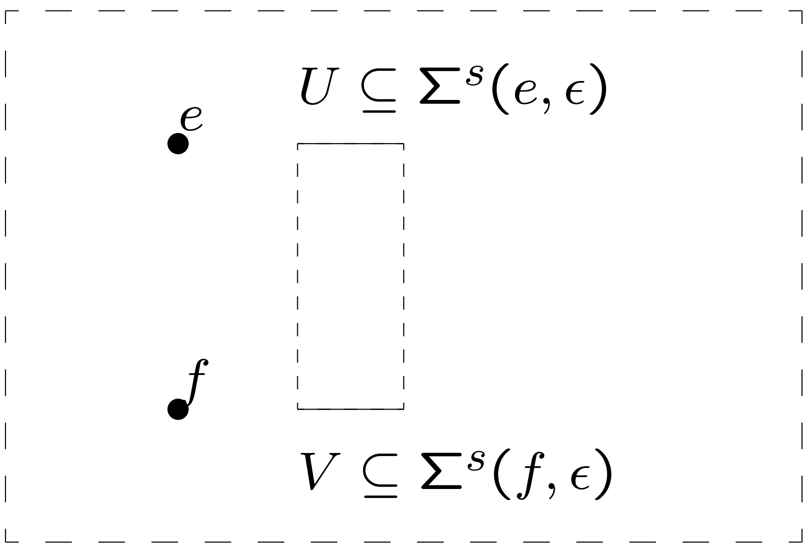
$$U + V = U \cup V,$$

whenever U, V are disjoint.

Krieger's idea: look at

$$U \subseteq \Sigma^s(e, \epsilon),$$

clopen in the relative topology. Equivalence relation $\sim$: if we have



then $U \sim V$.

And $U \sim V \Leftrightarrow \sigma(U) \sim \sigma(V)$.

The invariant $D^s(\Sigma, \sigma)$ is an abelian group generated by the equivalence classes of relatively clopen sets with relations $[U \cup V] = [U] + [V], U \cap V = \emptyset$ and has a natural automorphism induced by σ.

An obvious question about Krieger's invariant: can it be computed?

Theorem 8 (Krieger)**.** *If G is a finite directed graph and (Σ_G, σ_G) is the associated SFT, then*

$$D^s(\Sigma_G, \sigma_G) \cong \lim \mathbb{Z}^N \xrightarrow{A_G} \mathbb{Z}^N \xrightarrow{A_G} \cdots$$

where

$$N = \#G^0, A_G = \text{ adjacency matrix of } G.$$

The automorphism σ_*^{-1} is multiplication by A_G.

Corollary 9.

$$per_p(\Sigma_G, \sigma_G) = Tr(A_G^p) = Tr((\sigma_G)_*^{-p})$$

Another obvious question about Krieger's invariant: how did he think of it?

A superficial look at the definition - clopen set modulo unstable equivalence - makes it look like we are computing $\check{H}^0(\Sigma/\Sigma^u)$. We are not and that is fortunate since, for mixing SFT's, Σ/Σ^u is uncountable and indiscrete.

One of the principles of noncommutative topology is that when one finds such a quotient space, one should have built a C^*-algebra instead. Krieger saw this C^*-algebra quite explicitly and could compute its K_0-group. That is the invariant.

Recall the problem: find a homology theory for Smale spaces.

Step 1: Find the invariant for shifts of finite type: Wolfgang Krieger (1980).

Step 2: Extend it to all Smale spaces.

For the second step, we look to the proof of Manning's Theorem ...

Bowen's Theorem

(Also, Adler-Weiss, Sinai, etc.)

Theorem 10 (Bowen)**.** *For a (non-wandering) Smale space, (X, φ), there exists a SFT (Σ, σ) and*

$$\pi : (\Sigma, \sigma) \to (X, \varphi),$$

with $\pi \circ \sigma = \varphi \circ \pi$, continuous, surjective and finite-to-one.

The proof is the existence of Markov partitions.

(Σ, σ) is not unique.

Manning's proof: keep track of when π is N-to-1, for various values of N.

For $N \geq 0$, define

$$\begin{aligned}\Sigma_N(\pi) &= \{(e_0, e_1, \ldots, e_N) \mid \\ &\quad \pi(e_n) = \pi(e_0), \\ &\quad 0 \leq n \leq N\}.\end{aligned}$$

For all $N \geq 0$, $(\Sigma_N(\pi), \sigma)$ is also a shift of finite type and S_{N+1} acts on $\Sigma_N(\pi)$.

We can form $D^s(\Sigma_N(\pi), \sigma))^{alt}$.

This is a good candidate for $H_N(X, \varphi)$ *except* that it depends on (Σ, σ) and π.

Manning used the periodic point data from the sequence $\Sigma_N(\pi)$ (with the action of S_{N+1}) to compute $per_n(X, \varphi)$.

This is extremely reminiscent of using the nerve of an open cover to compute homology of a compact manifold.

Topology	Dynamics
'good' open cover $U_1, \ldots, U_I$	Bowen's Theorem $\pi : (\Sigma, \sigma) \to (X, \varphi)$
multiplicities $U_{i_0} \cap \cdots \cap U_{i_N} \neq \emptyset$	multiplicities $\Sigma_N(\pi)$
groups C^N generated by $U_{i_0} \cap \cdots \cap U_{i_N} \neq \emptyset$	groups $D^s(\Sigma_N(\pi))^{alt}$
boundary maps $\partial^1(U_i \cap U_j) = U_j - U_i$	boundary maps ? ?

The problem:

For $0 \leq n \leq N$, let $\delta_n : \Sigma_N(\pi) \to \Sigma_{N-1}(\pi)$ be the map which deletes entry n. This is a nice map between the dynamical systems.

Unfortunately, a map

$$\rho : (\Sigma, \sigma) \to (\Sigma', \sigma)$$

between shifts of finite type does *not* always induce a group homomorphism

$$\rho_* : D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma)$$

between Krieger's invariants.

But this problem is well-understood in symbolic dynamics ...

Definition 11. *A factor map* $\pi : (Y, \psi) \to (X, \varphi)$ *between Smale spaces is* s-bijective *if, for all* y *in* Y

$$\pi : Y^s(y) \to X^s(\pi(y))$$

is bijective.

It is a consequence that, for any y, $\epsilon > 0$, there is $\delta > 0$ such that $\pi(Y^s(y, \delta))$ is an open subset of $X^s(\pi(y), \epsilon)$ and π is a homeomorphism from $Y^s(y, \delta)$ to its image.

Theorem 12. *Let* $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ *be a factor map between SFT's.*

If π *is* s*-bijective, then there is a map*

$$\pi^s : D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma).$$

If π *is* u*-bijective, then there is a map*

$$\pi^{s*} : D^s(\Sigma', \sigma) \to D^s(\Sigma, \sigma).$$

A better Bowen's Theorem

Let (X, φ) be a Smale space. We look for a Smale space (Y, ψ) and a factor map

$$\pi_s : (Y, \psi) \to (X, \varphi)$$

satisfying:

1. π_s is s-bijective,

2. $Y^u(y, \epsilon)$ totally disconnected.

That is, $Y^u(y, \epsilon)$ is totally disconnected, while $Y^s(y, \epsilon)$ is homeomorphic to $X^s(\pi_s(y), \epsilon)$.

This is a "one-coordinate" version of Bowen's Theorem.

Similarly, we look for a Smale space (Z, ζ) and a u-bijective factor map $\pi_u : (Z, \zeta) \to (X, \varphi)$ with $Z^s(z, \epsilon)$ totally disconnected.

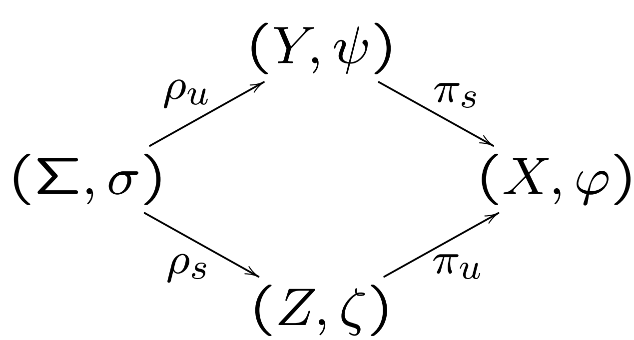
We call $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ a *s/u-bijective pair* for (X, φ).

Theorem 13 (Better Bowen). *If (X, φ) is a non-wandering Smale space, then there exists an s/u-bijective pair, $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$.*

Like the SFT in Bowen's Theorem, this is not unique.

The fibred product is a SFT:

$$\Sigma = \{(y, z) \in Y \times Z \mid \pi_s(y) = \pi_u(z)\}.$$



For $L, M \geq 0$, we define

$$\begin{aligned}\Sigma_{L,M}(\pi) &= \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid \\ &\quad y_l \in Y, z_m \in Z, \\ &\quad \pi_s(y_l) = \pi_u(z_m)\}.\end{aligned}$$

Each of these is a SFT.

Moreover, the maps

$$\begin{aligned}\delta_{l,} : \ \Sigma_{L,M} \to \ \Sigma_{L-1,M}, \\ \delta_{,m} : \ \Sigma_{L,M} \to \ \Sigma_{L,M-1}\end{aligned}$$

which delete y_l and z_m are s-bijective and u-bijective, respectively.

This is the key point! These maps *do* induce maps on Krieger's invariant and we can use them to make boundary maps.

We get a double complex:

$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,2})^{alt} & \longleftarrow & D^s(\Sigma_{1,2})^{alt} & \longleftarrow & D^s(\Sigma_{2,2})^{alt} & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,1})^{alt} & \longleftarrow & D^s(\Sigma_{1,1})^{alt} & \longleftarrow & D^s(\Sigma_{2,1})^{alt} & \longleftarrow & \\
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D^s(\Sigma_{0,0})^{alt} & \longleftarrow & D^s(\Sigma_{1,0})^{alt} & \longleftarrow & D^s(\Sigma_{2,0})^{alt} & \longleftarrow &
\end{array}$$

$$\begin{aligned}
\partial^s_N : \quad & \oplus_{L-M=N} D^s(\Sigma_{L,M})^{alt} \\
\to \quad & \oplus_{L-M=N-1} D^s(\Sigma_{L,M})^{alt}
\end{aligned}$$

$$\partial^s_N = \sum_{l=0}^{L}(-1)^l \delta^s_{l,} + \sum_{m=0}^{M+1}(-1)^{m+M}\delta^{s*}_{,m}$$

$$H^s_N(\pi) = \ker(\partial^s_N)/Im(\partial^s_{N+1}).$$

Theorem 14. *The groups $H^s_N(\pi)$ depend on (X, φ), but not the choice of s/u-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$.*

From now on, we write $H^s_N(X, \varphi)$.

Theorem 15. *The functor $H^s_*(X, \varphi)$ is covariant for s-bijective factor maps, contravariant for u-bijective factor maps.*

Theorem 16. *The groups $H^s_N(X, \varphi)$ are all finite rank and non-zero for only finitely many $N \in \mathbb{Z}$.*

Theorem 17 (Lefschetz Formula). *Let* (X, φ) *be any non-wandering Smale space and let* $p \geq 1$.

$$\begin{aligned} \sum_{N \in \mathbb{Z}} (-1)^N \quad Tr[(\varphi^s)^{-p} : \quad & H_N^s(X, \varphi) \otimes \mathbb{Q} \\ \rightarrow \qquad & H_N^s(X, \varphi) \otimes \mathbb{Q}] \\ = \qquad & \#\{x \in X \mid \varphi^p(x) = x\} \end{aligned}$$

Example 18. *Shifts of finite type*

If $(X, \varphi) = (\Sigma, \sigma)$, then $Y = \Sigma = Z$ is an s/u-bijective pair. In the double complex, only the lower left group is non-zero and

$$\begin{aligned} H^s_0(\Sigma, \sigma) &= D^s(\Sigma, \sigma), \\ H^s_N(\Sigma, \sigma) &= 0, N \neq 0. \end{aligned}$$

Remark 19. *In any Smale space with totally disconnected stable sets, we may choose $Z = X$. Then the double complex is only non-zero in its bottom row.*

Example 20. $\frac{q}{p}$*-solenoid[N. Burke-P.]*

Let $p < q$ be primes and (X, φ) the $\frac{q}{p}$-solenoid.

$$\begin{aligned} H^s_0(X, \varphi) &\cong \mathbb{Z}[1/q] \\ H^s_1(X, \varphi) &\cong \mathbb{Z}[1/p] \\ H^s_N(X, \varphi) &= 0, N \neq 0, 1. \end{aligned}$$

Example 21. 1*-dimensional solenoids of R.F. Williams [Amini, P., Saeidi]*

If (X, φ) is an orientable 1-d solenoid, then

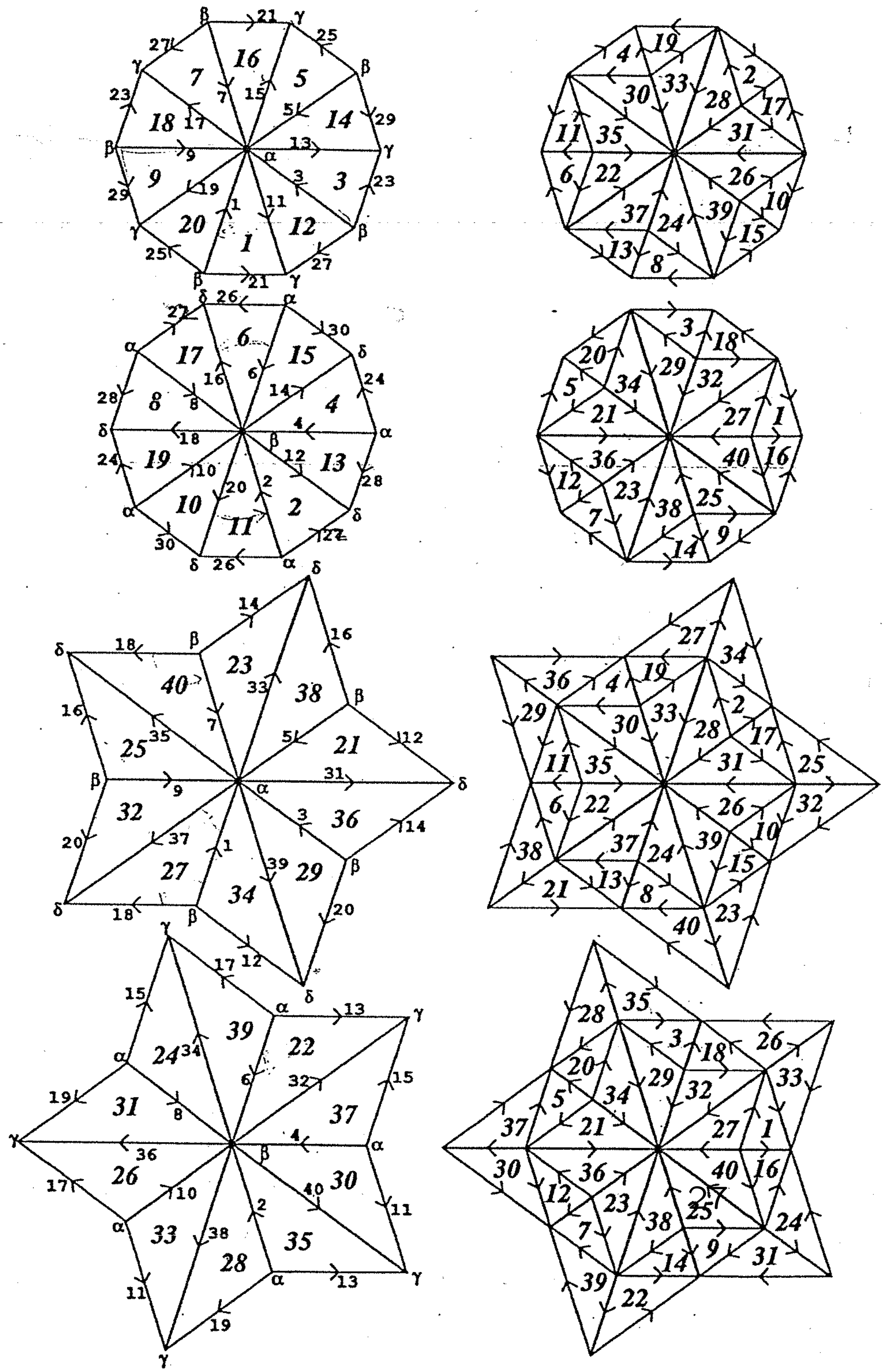
$$\begin{aligned} H^s_0(X,\varphi) &\cong \check{H}^1(X) \\ H^s_1(X,\varphi) &\cong \check{H}^0(X) \cong \mathbb{Z}, \\ H^s_N(X,\varphi) &= 0, N \neq 0, 1 \end{aligned}$$

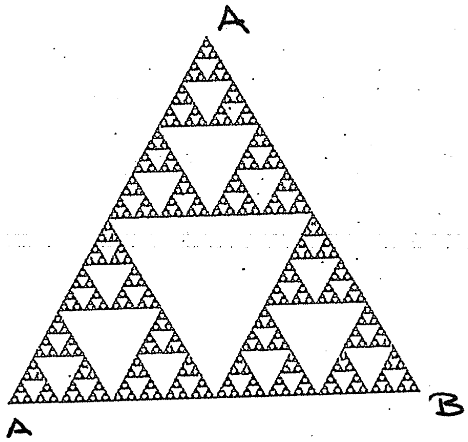
If it is non-orientable, then these do not hold (in general) and $H^s_i(X, \varphi)$ has torsion.

Example 22. *Full* 2*-shift x non-orientable* 1*solenoid*

Deeley-Killough-Whittaker

This has the same homology as a SFT, but is not a SFT.

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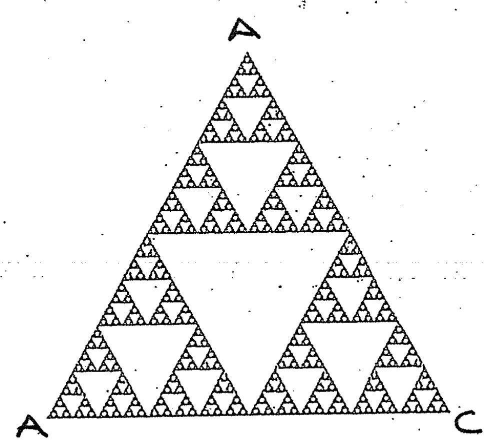
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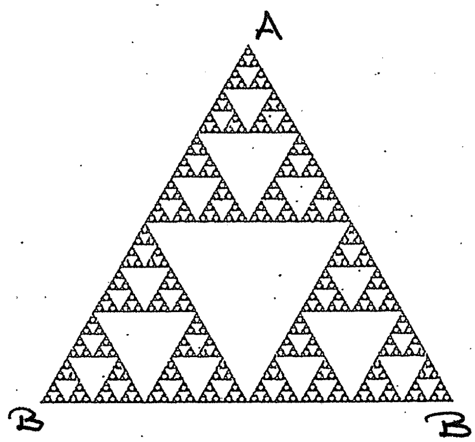
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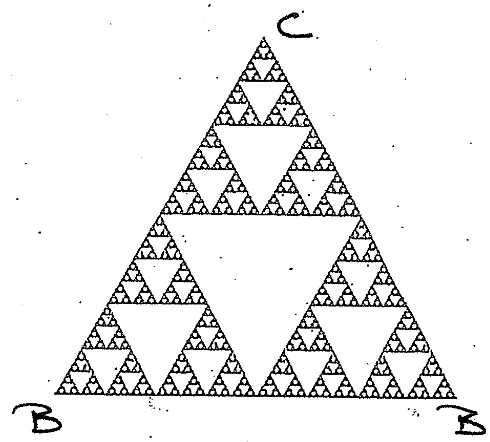
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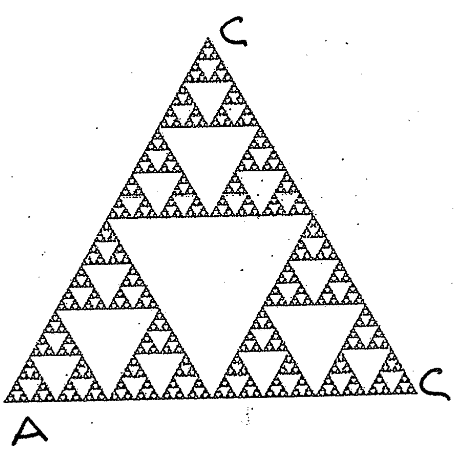
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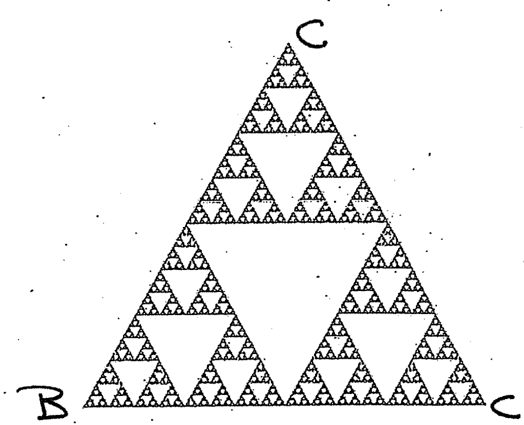
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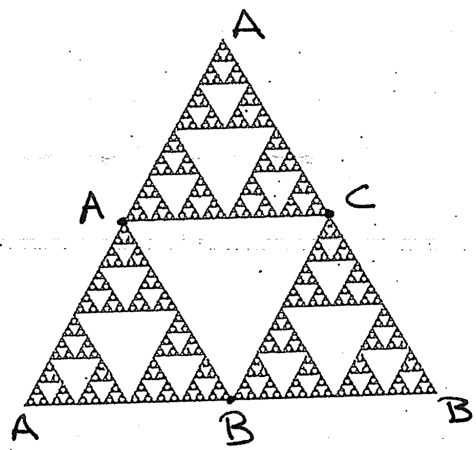
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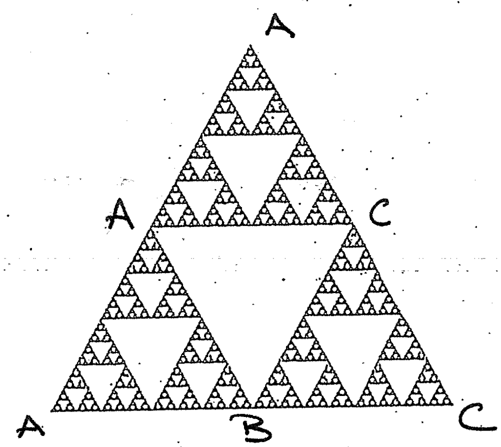
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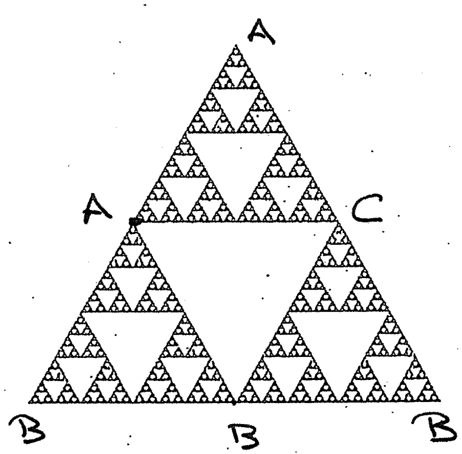
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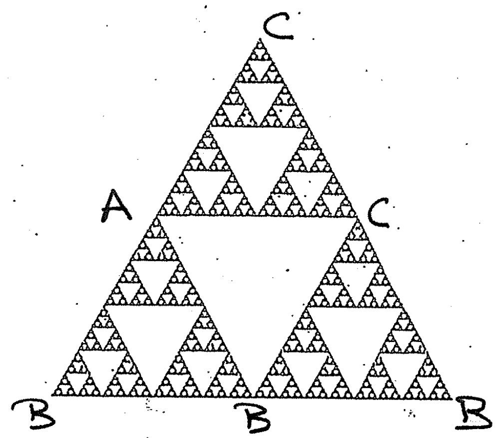
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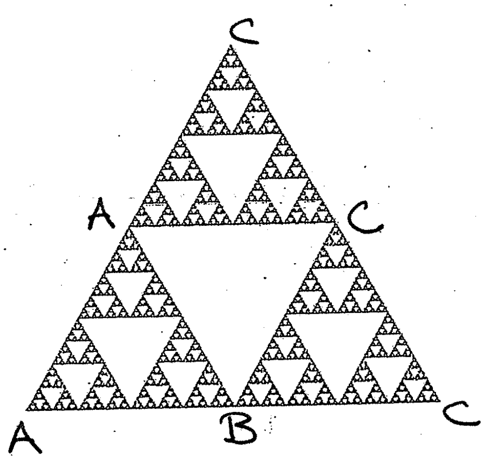
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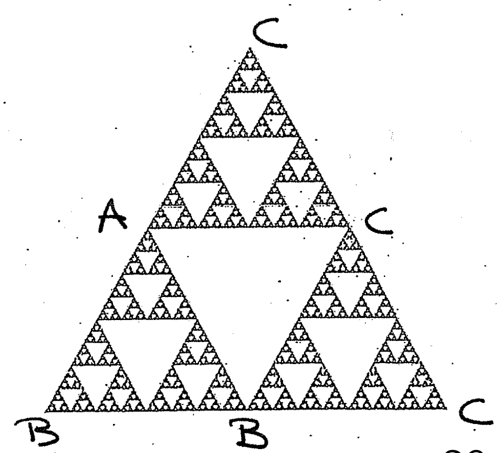
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