Amenable actions, Lyapunov exponents and an idea of Bowen

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Rufus Bowen, *Hausdorff dimension of quasi-circles*, Publ. Math. IHES 50 (1979), 11–25.

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Let $\tilde{M}$ be the universal covering of M, with geometric boundary $\partial\tilde{M}$. Invariant Borel probability measures on T^1M correspond to $\pi_1(M)$-invariant Radon measures on $\partial M \times \partial M - \Delta$.

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This correspondence is independent of a bounded time-change of the flow.

Any flip-invariant *Gibbs measure*, eg the *Bowen-Margulis measure*, defines a measure μ of *quasi-product type*, ie the measure class of μ equals the class of $\nu \times \nu$ for some $\pi_1(M)$-invariant measure class ν.

Geometric hyperbolicity

Definition

A geodesic metric space is δ-*hyperbolic* if for any geodesic triangle T with sides a, b, c we have

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Simply connected complete Riemannian manifolds of negative sectional curvature are hyperbolic.

Hyperbolic spaces X admit a *Gromov boundary* ∂X, a metrizable $\mathrm{Iso}(X)$-space. They are *non-elementary* if ∂X has at least three points. If X is proper then ∂X is compact.

Definition

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Example

Fundamental groups of closed negatively curved manifolds are hyperbolic.

Definition

An isometric action of a discrete group Γ on a metric space X is *weakly properly discontinuous* if $\forall R > 0 \; \exists \; B(R) > 0, N(R) > 0$: If $d(x, y) \geq B(R)$ then

$$\sharp\{g \in \Gamma \mid d(x, gx) \leq R, d(y, gy) \leq R\} \leq N(R).$$

Definition

A discrete group Γ is *acylindrically hyperbolic* if it admits a non-elementary weakly properly discontinuous action on a separable δ-hyperbolic geodesic metric space.

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Example

1. Hyperbolic groups
2. Fundamental group of finite volume pinched negatively curved manifolds
3. Mapping class groups of surfaces of finite type (Masur-Minsky, Bowditch)
4. $\mathrm{Out}(F_n)$ (Bestvina-Feighn)

But: Lattices in higher rank simple Lie groups are not acylindrically hyperbolic (Burger-Monod, Bestvina-Fujiwara, H)

Definition

A group Γ of homeomorphisms of an infinite compact metric space B is a *convergence group* if, given any sequence of distinct $g_i \in \Gamma$, there are points c and b of B and a subsequence g_{n_i} such that

$$g_{n_i}(z) \to b$$

uniformly for all points z outside compact neighborhoods of c.

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Example

A hyperbolic group Γ acts on its Gromov boundary as a convergence group.

Proposition

Assume that Γ *acts minimally on* B *as a convergence group.*

1. *(Tukia)* $\Gamma \cup B$ *is a compactification of* Γ.
2. *Let* $(g_i) \subset \Gamma$ *be a sequence s.th.* $g_i \to b \in B$*; then* $(g_i)_*\nu \to \delta_b$ *for every non-atomic probability measure* ν *on* B.

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Theorem

(Sun 2016) *A convergence group is acylindrically hyperbolic.*

Let Γ be a discrete group.
The space of $\mathcal{P}(\Gamma)$ of probability measures on Γ is the space of non-negative integrable functions ν of ℓ^1-norm

$$\|\nu\|_1 = \sum_g |\nu(g)| = 1.$$

Definition

The action of the group Γ on a compact uniformly perfect metrizable space B is called *amenable* if the following holds true. There is a sequence $\xi_n : B \to \mathcal{P}(\Gamma)$ with image consisting of finitely supported measures such that

$$\|g\xi_n(x) - \xi_n(gx)\|_1 \to 0$$

uniformly on compact subsets of $B \times \Gamma$.

Amenability also makes sense for actions of countable discrete groups Γ on measure spaces X preserving a measure class μ.

Example

1. A group Γ is amenable iff it admits an amenable action on a point.
2. (Connes-Feldman-Weiss) Assume that Γ acts on (B, μ). The action is amenable iff the orbit equivalence relation is hyperfinite.
3. (Bowen) The stable foliation of a geodesic flow Φ^t on a closed negatively curved manifold M is hyperfinite for any Gibbs measure $\Rightarrow$ the action of $\pi_1(M)$ on $(\partial\tilde{M}, \nu)$ is amenable where ν is the measure class on $\partial\tilde{M}$ induced by ν.

Example

1. The action of a lattice Γ in a simple Lie group G of non-compact type on the Furstenberg boundary G/P of G is amenable.
2. The action of a Gromov hyperbolic group Γ on the Gromov boundary $\partial\Gamma$ is amenable (Adams 1994)
3. The action of a group Γ relatively hyperbolic to a finite collection of amenable subgroups on a geometric boundary is amenable (Ozawa 06)
4. Mapping class groups admit an amenable action on an explicit boundary (Kida, H 09)
5. $\mathrm{Out}(F_n)$ is boundary amenable (Bestvina, Guirardel, Horbez 17)

Definition

A *universal boundary* for Γ is a Γ-space X with the following properties.

1. X is a compact uniformly perfect Polish space.
2. The action of Γ on X is amenable and minimal.
3. The action of Γ on X is a convergence action.

Groups which admit universal boundaries include groups which are hyperbolic relative to a finite collection of amenable subgroups (eg fundamental groups of finite volume pinched negatively curved manifolds)

Proposition

If B is a universal boundary of a group Γ then $\Gamma \cup B = \overline{\Gamma}$ is a compactification of Γ which is small at infinity.

Means: The right translation action extends continuously to the identity on B

Measures and flows

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Definition

The measure ν is *of quasi-product type* if there exists a Γ-invariant Radon measure on $B \times B - \Delta$ in the measure class of $\nu \times \nu$.

Example

Assume that Γ acts properly and cocompactly on a proper hyperbolic geodesic metric space X. A *Patterson Sullivan measure* ν on ∂X is a limit

$$\lim_{s \searrow \delta} \frac{1}{\sum_g e^{-sd(x,gx)}} \sum_{g \in \Gamma} e^{-sd(x,gx)} \delta_{gx}$$

where $\delta \geq 0$ is the *critical exponent*

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Example

Let M be a closed negatively curved manifold. The *Bowen-Margulis measure* for the geodesic flow Φ^t on the unit tangent bundle T^1M of M projects to a measure of quasi-product type on the boundary $\partial\tilde{M}$ of the universal covering $\tilde{M}$ of M.

Let μ be a *regularized* non-atomic Borel-probability measure on B with Γ-invariant measure class and let ν be a Γ-invariant measure on $B \times B - \Delta$ in the measure class of $\mu \times \mu$.

For $\mu \times \mu$-a.e. $(x, y) \in B \times B$, the Radon Nikodym derivative

$$c^2(x, y) = d\nu(x, y)/d(\mu \times \mu)(x, y)$$

exists. Define

$$\Lambda(g, (x, y, t)) = (gx, gy, s)$$

if $e^{-t}e^{s}c(g(x, y))/c(x, y)$ is the Jacobian of g at x w.r. to μ.

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Proposition

Λ defines a smooth (=countably separated quotient Borel structure) action of Γ on $(B \times B - \Delta) \times \mathbb{R}$ commuting with the action of $\mathbb{R}$ by translations.

Example

If μ is the Patterson Sullivan measure of the fundamental group Γ of a closed negatively curved manifold M then $\Gamma\backslash(B \times B - \Delta) \times \mathbb{R}$ is the unit tangent bundle T^1M of M, with the geodesic flow Φ^t.

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Example

If Γ is a hyperbolic group, with Gromov boundary $\partial\Gamma$, and if μ is the exit measure of a finitely supported symmetric random walk, then μ can be realized as a Borel measure on $\partial\Gamma$. The measure μ is of quasi-product type, and $\Gamma\backslash(B \times B - \Delta) \times \mathbb{R}$ is compact. The translation flow Φ^t is expansive.

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Babillot 02: Relate mixing of this flow to stable lengths of periodic orbits

Applications 1

Definition

A number $r \geq 0$ is contained in the *ratio set* of a measure class preserving action of Γ on (B, ν) if for $A \subset B$ Borel, $\nu(A) > 0$, $\epsilon > 0$ $\exists A' \subset A$, $\nu(A') > 0$, $g \in \Gamma$ s.th.

1. $gA' \subset A$
2. $|\frac{d\nu \circ g}{d\nu} - r| < \epsilon$ for all $b \in A'$

The action is *of type III* if the ratio set is infinite, and of type III_1 if the ratio set is all of $\mathbb{R}^*$.

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The action is *of type III* if the ratio set is infinite, and of type III_1 if the ratio set is all of $\mathbb{R}^*$.

The *Maharan extension* is the action of Γ on $B \times \mathbb{R}$ defined by

$$g(b, t) = (gb, t - \log(\frac{d\nu \circ g}{d\nu}(b)))$$

preserving the measure $\nu \times \theta$, $d\theta = e^t dt$. The action of type III_1 if and only if the action of Γ on $B \times \mathbb{R}$ is ergodic.

Theorem

Let Γ be a hyperbolic group with Gromov boundary $\partial\Gamma$.

1. *If ν is a Patterson Sullivan measure of a word metric then the action is of type III (L. Bowen 14), but it is not of type III_1.*
2. *If ν is the exit measure of a finitely supported symmetric random walk and if $\partial\Gamma$ is connected then the action is of type III_1.*

Generalizations

Theorem

1. *(Sun 16) If Γ acts on B as a convergence group then the Gromov boundary of an acylindrically hyperbolic Γ-space admits a Γ-equivariant embedding into B.*
2. *(Maher-Tiozzo 15) If Γ admits an acylindrically action on a hyperbolic space X then the exit measure of any finitely supported symmetric random walk can be realized as a measure on the Gromov boundary of X.*

The action of Γ is of type III for measures on B of quasi-product type, and it is of type III_1 for exit measures of finitely supported random walks if B is connected.

Applications 2

Let Γ be a hyperbolic group and ν the generating measure of a random walk, finite and symmetric. Then ν defines a *Green's metric* d on Γ.
For $R > 0$ let $N(R)$ be the number of conjugacy classes of elements of Γ of translation length $\leq R$ for d.

Theorem
There is $h > 0$ s.th. $N(R) \sim \frac{e^{hR}}{hR}$ as $R \to \infty$.

For word metrics one gets (Coorneart-Knieper 02, H)

Theorem
There are $h > 0, 0 < a < b$ s.th. $N(R) \in [a\frac{e^{hR}}{hR}, b\frac{e^{hR}}{hR}]$ for large R.

Lyapunov exponents

Oseledets multiplicative ergodic theorem: Let Φ^t be a flow on a space X preserving a Borel probability measure μ, $c : X \times \mathbb{R} \to SL(n, \mathbb{R})$ a bounded cocycle for Φ^t; then there is a measurable filtration $X \times \mathbb{R}^n = A_1 \subset \cdots \subset A_s$, $A_i \to X$ a measurable bundle, and numbers $\kappa_1 < \cdots < \kappa_s$ s.th. for $v \in A_i - A_{i-1}$, a.e. $x \in X$, $\lim_{t\to\infty} \log \|\Phi^t v\| = \kappa_i$.

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Let Γ be a finitely generated group with universal boundary B, $\rho : \Gamma \to SL(n,\mathbb{R})$ a homomorphism, $\tilde{X}$ a geodesic metric space with cocompact action of Γ, $X = \tilde{X}/\Gamma$
$\Rightarrow \tilde{X} \times \mathbb{R}^n/\Gamma \to X$ is a flat vector bundle with canonical connection which defines a cocycle over any flow Φ^t on X. Then the Lyapunov exponents are defined for a Φ^t-invariant Borel probability measure.

The *Lyapunov spectrum is simple* if $\dim(A_i) - \dim(A_{i-1}) = 1$ for all i. This is invariant under a bounded time change of the flow.

Example

Let Φ^t be the geodesic flow on a closed negatively curved manifold M. Let μ be the invariant Gibbs measure defined by the exit measure of a finitely supported random walk on $\pi_1(M)$. Let $\rho : \pi_1(M) \to SL(n, \mathbb{R})$ be a homomorphism with Zariski dense image. Then the Lyapunov spectrum of the cocycle is simple (Guivarch-Raugy, Margulis)

Theorem

Let ν be a measure of product type on a universal boundary B of Γ. Let $\rho : \Gamma \to SL(n, \mathbb{R})$ be a homomorphism with Zariski dense image. The for ν-a.e. $x \in B$, and a geodesic $\gamma : [0, \infty) \to \Gamma$ ending at x, ex. a filtration $A_1 \subset \cdots \subset A_{n-1} \subset \mathbb{R}^n$ s.th. for all $x \in A_i - A_{i-1}, y \in A_{i-1}$ have

$$\liminf_{t \to \infty} \log \frac{1}{t} (\log \| c(\gamma(0), t) x \| - \log \| c(\gamma(0), t) y \|) > \epsilon(x) > 0.$$

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$$\lim \inf_{t \to \infty} \log \frac{1}{t} (\log \|c(\gamma(0), t)x\| - \log \|c(\gamma(0), t)y\|) > \epsilon(x) > 0.$$

Application: If μ is a Φ^t-invariant Borel probability measure for the geodesic flow on M which is of quasi-product type and if $\rho : \pi_1(M) \to SL(n, \mathbb{R})$ is a homomorphism with Zariski dense image then the Lyapunov spectrum of the corresponding cocycle is simple.

Fact: Let Γ be a discrete group acting amenably on a compact space, with invariant measure class μ. Assume that Γ acts on the compact space X. Then $\exists$ equivariant measurable map $(B, \mu) \to \mathcal{P}(X)$ *Furstenberg map* where $\mathcal{P}(X)$ is the space of Borel probability measures on X.

Show: Image consists of point masses (use convergence action, can also be done using a *strip condition* in the sense of Kaimanovich).

Apply to action of Γ on the *flag variety*; deduce proximality of the action.

Use *regularity* of a measure of product type to deduce information on Lyapunov exponents.