Exponential decay of correlations for volume-preserving Anosov flows in dimension 3

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Aim of this talk

Explanation about my (not so) recent result

Theorem (T. arXiv:1601.00063)

Almost all volume-preserving Anosov flows in dimension 3 are exponentially mixing.

The main idea behind this result is

Analysis of the "geometry" of stable and unstable foliations.

- This enables us to state a quantitative condition on non-integrability between the stable and unstable foliation.
- Note that the stable and unstable foliations are generically non-smooth.

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Mixing of Anosov flows

Consider a volume-preserving Anosov flow $F^t : M \to M$ on a closed R-manifold M, with the normalized R-volume m.

Definition

The flow F^t is mixing if

$$\left| \int \varphi \cdot \psi \circ F^t dm - \int \varphi dm \int \psi dm \right| \xrightarrow{t\to\infty} 0 \qquad \forall \varphi, \psi \in L^2(M)$$

and is exponential mixing if $\exists C, c > 0$ s.t.

$$\left| \int \varphi \cdot \psi \circ F^t dm - \int \varphi dm \int \psi dm \right| \leq Ce^{-ct} \cdot \|\varphi\|_{C^1} \cdot \|\psi\|_{C^1}$$

for any $\varphi, \psi \in C^1(M)$.

Remark: For exponential mixing, we need to consider φ, ψ with some smoothness, but the degree of smoothness is not important.

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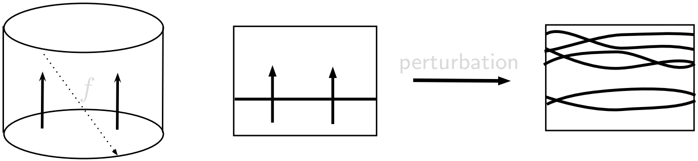
A non-example

Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be an vol-pres Anosov diffeomorphism and consider its suspension flow with a constant roof function:

$$F^t : X \to X, \quad F^t(x,s) = (x, s+t)$$
$$X = \{(x,s) \in \mathbb{T}^2 \times [0,1]\}/(x,1) \sim (f(x),0)$$

This Anosov flow F^t is not mixing (but its time-changes will be).



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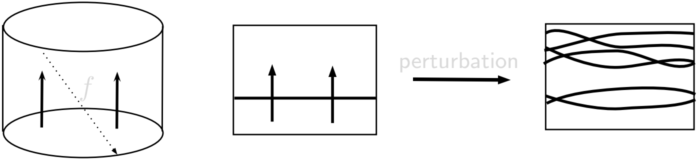

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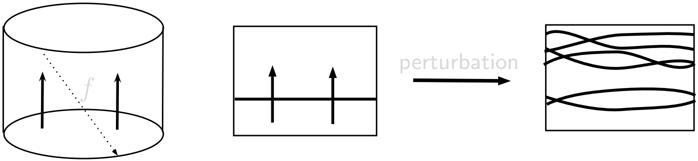

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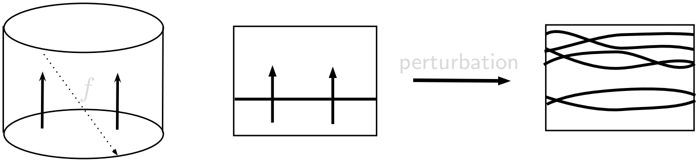

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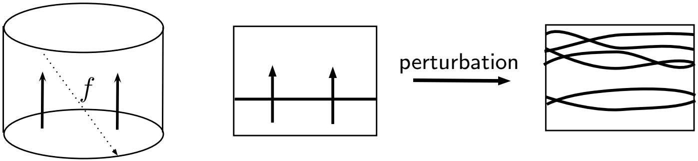

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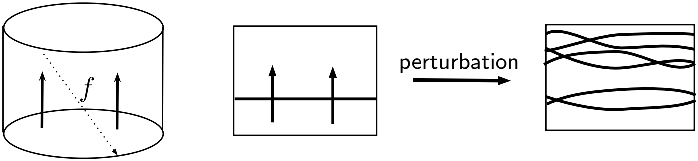

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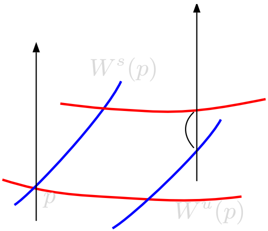
Bowen-Ruelle conjecture

$\mathcal{F}^s = \{W^s(p)\}$, $\mathcal{F}^u = \{W^u(p)\}$: the stable (unstable) foliations.

Theorem (Anosov-Sinai)

A volume-preserving Anosov flow F^t is mixing if $\mathcal{F}^s$ and $\mathcal{F}^u$ are not locally jointly integrable (an open dense property).



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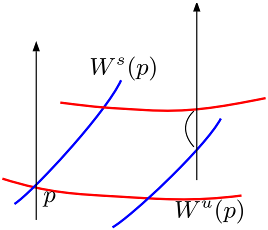

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Mixing will imply exponential mixing for (vol-pres.) Anosov flows.

Question

How often is a volume-preserving Anosov flow mixing?

Need to find a "quantitative" condition on non-integrability between $\mathcal{F}^s$ and $\mathcal{F}^u$.

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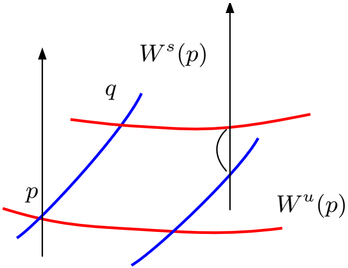
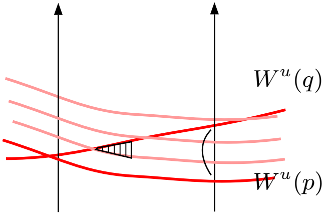
Theorem (Dolgopyat 1998)

If the stable and unstable foliations are C^1 and not jointly integrable, then the flow F^t is exponential mixing.





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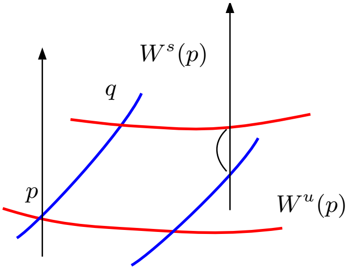

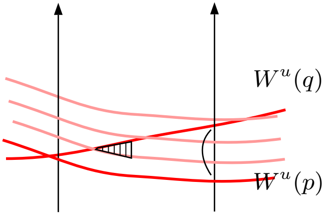

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Main Result

Theorem (T. arXiv:1601.00063)

Exponential mixing is a C^r generic ($r \geq 3$, open-dense) property for volume-preserving Anosov flows on 3-dimensional manifolds.

Remark

- In the setting above, $\mathcal{F}^u$ and $\mathcal{F}^s$ are not smooth generically.
- We perturb flows by "time-changes" and prove the conclusion.
- We can also show prevalence ("almost every" in measure theoretical sense).
- We show "spectral gap" for the transfer operator on some "anisotropic" Sobolev space.
- For the moment, the result (and proof) is limited to 3-dim.
- For super polyonomial decay, there is a result by Fields-Melbourne-Török in much more general setting.

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Structure of the proof

The proof of the theorem consists of two part:

1. Formulation of "quantitative non-integrability condition" between $\mathcal{F}^u$ and $\mathcal{F}^s$.
2. Proof that the "quantitative non-integrability condition" implies exponential mixing.

The former is the main novelty behind the theorem and based on geometric structure of the foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ (which are not smooth!).

The latter is an application of (the idea in) Dolgopyat argument, which is not simple but discussed in many places.

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The "linearizing" coordinates along unstable manifolds

For each $p \in M$, consider the 2-dim vector bundle over $W^u(p)$:

$$N(p) := T_{W^u(p)}M/E_u \quad (\simeq \text{Normal bundle of } W^u(p)).$$

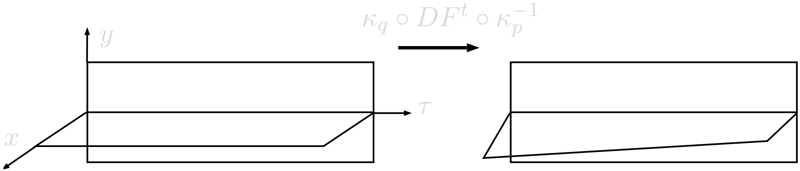
Then introduce "dynamical" and "linearizing" coordinates on it

$$\kappa_p : N(p) \to \mathbb{R}_\tau \times \mathbb{R}^2_{(x,y)}$$

for all $p \in M$ simultaneously and C^∞ bounded uniformly s.t.

$$\kappa_q \circ DF^t \circ \kappa_p^{-1}(\tau, x, y) = (\ \lambda_u \tau + c,\ \lambda_s x,\ y + x \cdot (\alpha\tau + \beta)\ )$$

if $q \in F^t(W^u(p))$ where $\lambda_\sigma, c, \alpha, \beta$ depending only on p, q and t.



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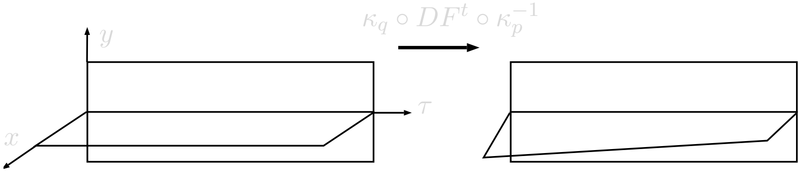

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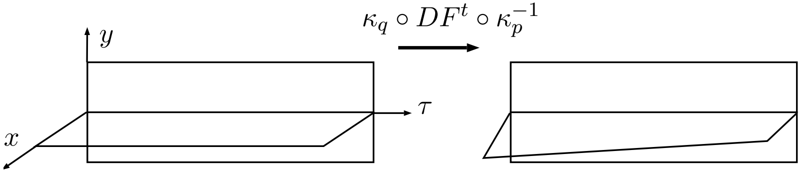


Geometry of unstable foliation $\mathcal{F}^u$

Identify $N(p)$ with the normal bundle of $W^u(p)$ and then consider

$$\Psi_p := \exp \circ \kappa_p^{-1} : \mathbb{R} \times \mathbb{R}^2 \to M \quad \text{on a nbd of } [-1,1] \times \mathbf{0}$$

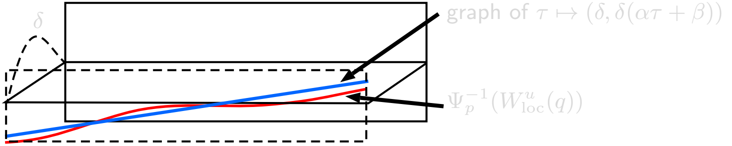
as a local chart around $p \in M$. If $d(q,p) \ll 1$, $W^u_{\mathrm{loc}}(q)$ is represented as a graph of a section $\gamma_{p,q} : [-1,1] \to \mathbb{R}^2$.

Lemma (The geometric property of $\mathcal{F}^u$)

There exists $\theta > 0$ such that, if $d(q,p) \ll 1$,

$$|\gamma_{p,q}(\tau) - (\tau, \delta, \delta(\alpha\tau + \beta))| < C \cdot \delta^{1+\theta}$$

for some α, β and $\delta \lesssim d(p,q)$.



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$$\Psi_p := \exp \circ \kappa_p^{-1} : \mathbb{R} \times \mathbb{R}^2 \to M \quad \text{on a nbd of } [-1,1] \times \mathbf{0}$$

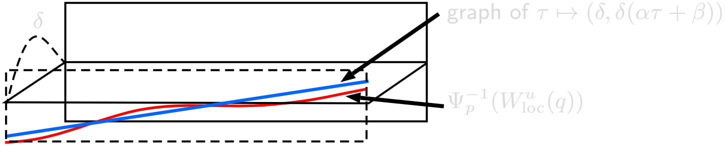
as a local chart around $p \in M$. If $d(q,p) \ll 1$, $W^u_{\mathrm{loc}}(q)$ is represented as a graph of a section $\gamma_{p,q} : [-1,1] \to \mathbb{R}^2$.

Lemma (The geometric property of $\mathcal{F}^u$)

There exists $\theta > 0$ such that, if $d(q,p) \ll 1$,

$$|\gamma_{p,q}(\tau) - (\tau, \delta, \delta(\alpha\tau + \beta))| < C \cdot \delta^{1+\theta}$$

for some α, β and $\delta \lesssim d(p,q)$.



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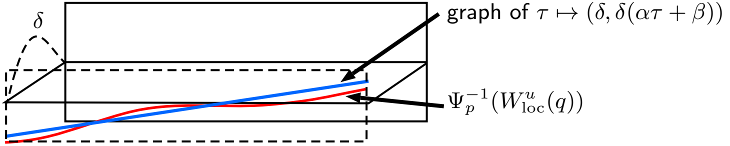


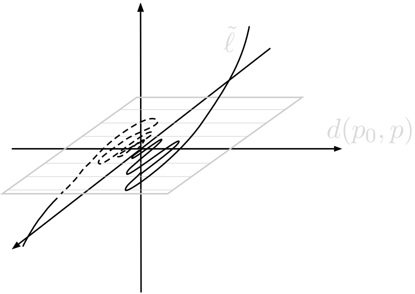
Geometry of unstable foliation $\mathcal{F}^u$ (continued)

The content of the last lemma can be interpreted as follows.

Let $\ell : (-\varepsilon, \varepsilon) \to M$ be a smooth curve s.t. $\ell(0) = p$ and $\ell'(0)$ is in the (nearly) stable direction. Then we consider the curve in the infinite dimensional space of curves:

$$\tilde{\ell} : (-\varepsilon, \varepsilon) \ni s \mapsto W^u_{\mathrm{loc}}(\ell(s)) \in \text{“space of curves on } M\text{”}$$

The last lemma tells that the curve $\tilde{\ell}$ tangents to a 3-dim subspace at each point on it (even though it is not differentiable!).



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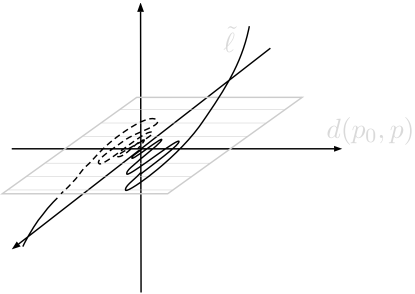

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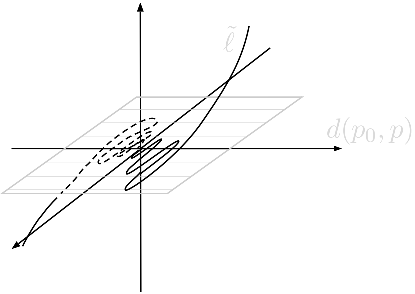

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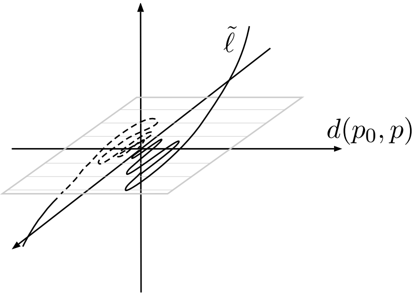


The "quantitative" condition on non-integrability

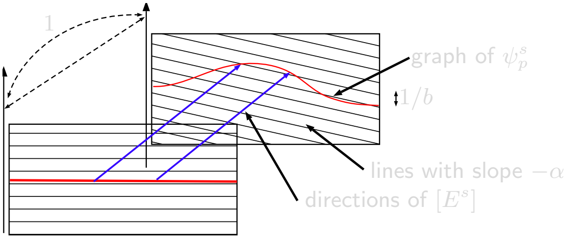
Let $\psi_p^s : [-1,1] \to \mathbf{R}$ be a continuous function "representing the direction of E^s" in the sense that

$$\kappa_p^{-1}(\tau, 1, \psi_p^s(\tau)) \in [E^s] \subset N(p) = T_{W^u(p)}M/E^u.$$

Definition (The non-integrability condition $(NI)_\rho$ for $\rho > 0$)

$$\left| \int_{-1}^{1} \exp\left(ib\left(\psi_p^s(\tau) + \alpha\tau\right)\right) d\tau \right| < b^{-\rho}$$

for sufficiently large $b > 0$ and all $\alpha \in \mathbb{R}$ and $p \in M$.



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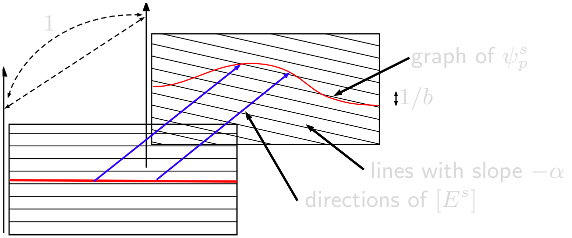

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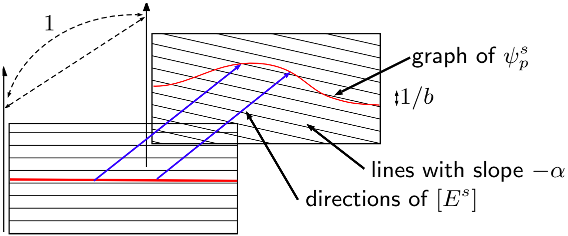

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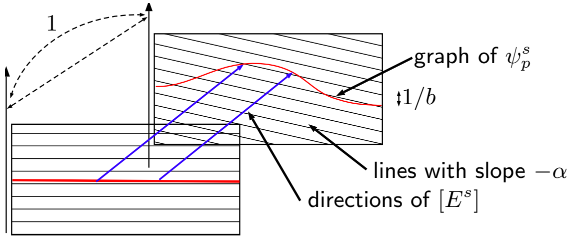

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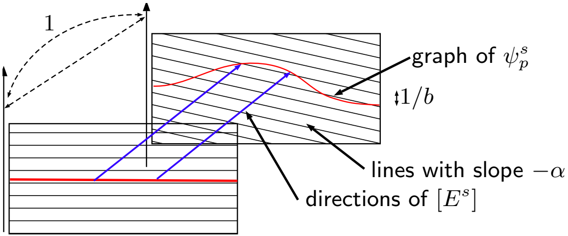

Genericity of the non-integrability condition

Genericity of $(NI)_\rho$ for some small $\rho > 0$ is not difficult to prove. Roughly we use the facts that

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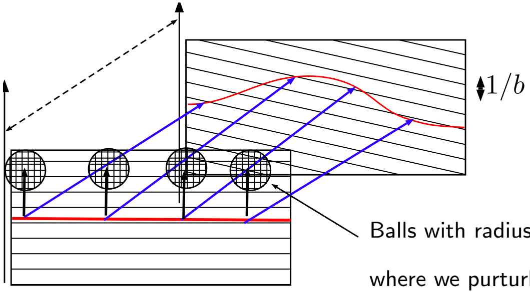

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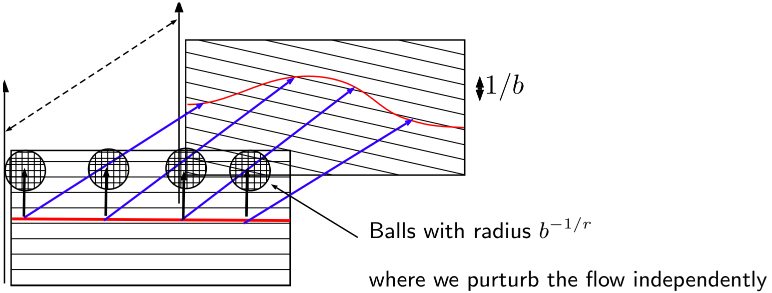

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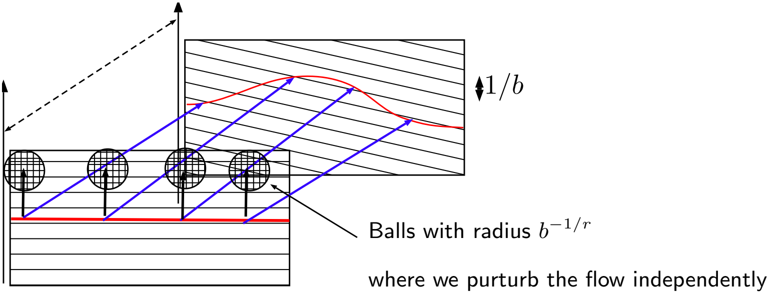
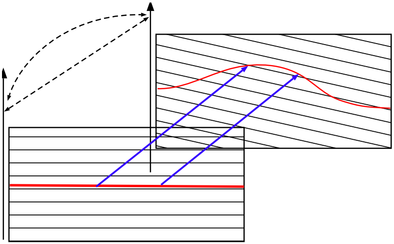


Dolgopyat argument (very briefly)

Non-integrability condition $\Longrightarrow$ Exponential mixing.

- As the transfer operator $\mathcal{L}^t : \psi \mapsto \psi \circ F^{-t}$ (virtually) preserves the frequency ω in the flow direction, we decompose the functions on M with respect to it and consider the components. (The limit $\omega \to \infty$ is important.)
- We prove that $A \circ \mathcal{L}^t$ is contracting the (L^2) norm, where A is an averaging along (small intervals in) the stable foliation. The non-integrability condition works in this proof.



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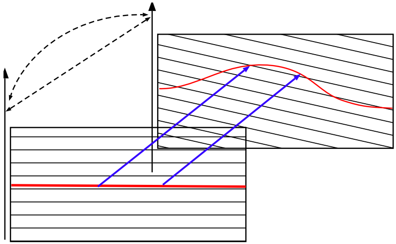

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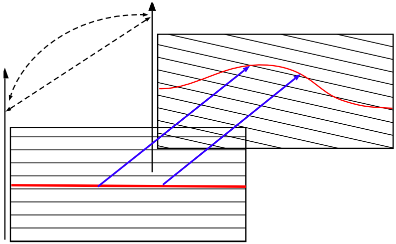
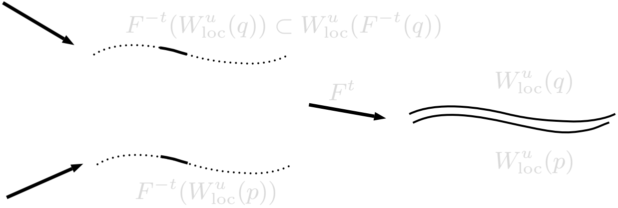


Origin of the "finite dimensional property" of $\mathcal{F}^u$

Suppose $d(p,q) \ll 1$ and take $t > 0$ s.t. $d(F^{-t}(p), F^{-t}(q)) \sim 1$.

Observation

The curves $F^{-t}(W^u_{\mathrm{loc}}(q))$ and $F^{-t}(W^u_{\mathrm{loc}}(p))$ are very short and their derivatives are bounded. Therefore they are approximated very precisely by their Taylor expansion of (some) order r , which is valid even after applying F^t.



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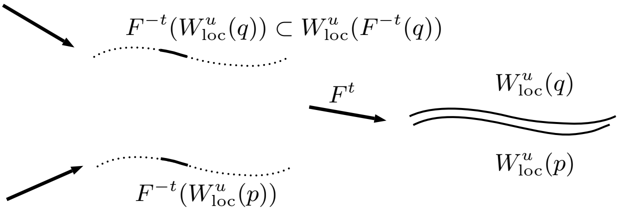

Discussion about generalizations

- For 3-dimensional C^∞ partially hyperbolic diffeomorphisms, our argument is valid (partly at least) and give the "Lemma" with $\alpha\tau + \beta$ replaced by some polynomial of order r (depending on the pinching rate.)

- In higher dim. case where some of E^σ is of dim ≥ 2, the situation is much more complicated, by pinching inside E^σ.

  Quesion: How does the (non-smooth) correspondence $q \mapsto W^u_{\mathrm{loc}}(q)$ look like in higher dimensional setting?

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