Random perturbations of predominantly hyperbolic systems

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joint work with Jinxin Xue and Lai-Sang Young

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Current Trends in Dynamical Systems
and the Mathematical Legacy of Rufus Bowen

Lyapunov exponents and nonuniform hyperbolicity

Let $F : M \to M$ be a smooth diffeo. of a compact manifold M, $\dim M \geq 2$.

Question

When is

$$\lambda_1(p) := \limsup_{n\to\infty} \frac{1}{n} \log \| dF_p^n \| > 0$$

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- Positive Lyapunov exponent $\Rightarrow$ sensitivity to initial conditions
- **Nonuniform hyperbolicity**: first step towards ergodic components, mixing properties, limit laws, etc...

Challenges

Estimating LE is a **delicate** cancellation problem:

- Growing vectors 'twisted' into contracting directions
- Dissipative: presence of sinks of high period
- Conservative: elliptic islands

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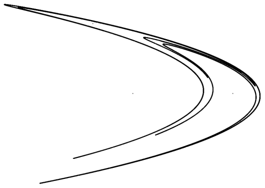
- Dissipative: presence of sinks of high period
- Conservative: elliptic islands

Obstructions are real:

- **Dissipative:** coexistence of wild hyperbolic sets and infinitely many sinks (Newhouse 74)
- **Conservative:** For Chirikov standard map, proliferation of elliptic islands for large set of L (Duarte 95)

Existing positive results



Dissipative:

- Dynamics of Hénon map in (Benedicks & Carleson 91)
- One direction of instability (Wang & Young 01, 08)

Results entail **intensive** parameter exclusion to rule out bad behavior, e.g., formation of sinks.

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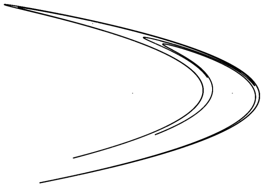

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Conservative:

- (Gorodetski 12) Chirikov standard map: $\lambda_1 > 0$ on set of Hausdorff dimension 2 (zero volume)

Our goal:

Natural way of making problem tractable: small random perturbations “unlock” hyperbolicity

- Seek broad applicability: use only ‘rough’ geometry of hyperbolicity
- Look for checkable conditions: verifiable from **finitely** many iterates

Not unrealistic: real world is inherently noisy

The Model: maps with "Hénon flavor"

Write $\mathcal{C} = \mathbb{S}^1 \times \mathbb{R}$ for the cylinder.

- Let $\psi : \mathbb{S}^1 \to \mathbb{R}$ be C^3;
- let $a \in \mathbb{S}^1$, $b \in (0, 1]$, $L > 1$.

Define $F : \mathcal{C} \to \mathcal{C}$ by

$$F_{\psi,L,a,b}(x, y) := (f_{\psi,L,a}(x) - y, bx) .$$

where $f_{\psi,L,a} := L\psi(x) + a$.

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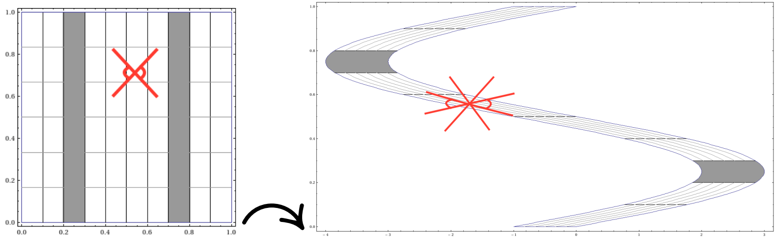
Note: F discontinuous along $\mathcal{D} = \{x = 0\}$ if $b < 1$.

Properties of F



F is *predominantly hyperbolic* when $L \gg 1$:

- Everywhere $\det dF \equiv b$.
- Outside *critical strips* (shaded), dF expands in horizontal cone to order L.
- Width of critical strips is $O(L^{-1})$.

Introducing the random model

Introduce IID random perturbations:

$$F_\omega(x, y) = F(x + \omega, y)$$
$$F^n_{\underline{\omega}} := F_{\omega_n} \circ \cdots \circ F_{\omega_1}$$

Here $\underline{\omega} = (\omega_1, \omega_2, \cdots)$, where $\omega_i \sim \mathrm{Unif}[-\epsilon, \epsilon]$ are IID.

- Heuristically: randomness helps avoid obstructions by "smearing" away
- Not so unnatural: real world is inherently noisy!

Formulation of problem

"Large" perturbations: when $L \gg 1$, $\epsilon \approx 1$, simple exercise to show

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Question:

For a given (possibly large) L, how small can ϵ be for randomness to 'unlock' hyperbolicity of F?

Results: volume-preserving ($b = 1$)

Theorem (Joint with JX, LSY; Ann. Math.)

Assume ψ satisfies some (checkable) nondegeneracy conditions. Then there exists $L_0, c > 0$ such that for any $L \ge L_0$ and

$$\epsilon > L^{-cL^{9/10}},$$

the top Lyapunov exponent $\lambda_1^\epsilon(p) = \lim_{n\to\infty} \frac{1}{n} \log \|(dF^n_{\underline{\omega}})_p\|$ exists, is almost surely constant over $p, \underline{\omega}$, and satisfies

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Corollary

Theorem applies to Chirikov standard map

$$F(x, y) = (L \sin(2\pi x) + 2x - y, x) .$$

Comments on Theorem

- No assumptions made on detailed dynamics of F:
  - Elliptic fixed points and periodic points allowed.
  - Typical length T of sojourn to vicinity of elliptic fixed point:

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- By precluding elliptic orbits of period ≤ 3, we can allow

$$\epsilon > L^{-cL^{19/10}} .$$

  - Consistent with parameter exclusion ideas in [BC], [WY].

LE and decay of correlations ($b \leq 1$)

Theorem (Joint with JX and LSY; accepted to CMP)

Let ψ satisfy same nondegeneracy conditions. Let $b \in (0, 1]$. Then there exists $L_0 = L_0(\psi, b) > 0$ such that for any $L \geq L_0$ and $\epsilon \geq L^{-9/10}$, we have

- *the top Lyapunov exponent λ_1^ϵ exists almost surely at all points of $\mathcal{C}$, and satisfies $\lambda_1^\epsilon \geq \frac{9}{10} \log L$; and*

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- *the top Lyapunov exponent λ_1^ϵ exists almost surely at all points of $\mathcal{C}$, and satisfies $\lambda_1^\epsilon \geq \frac{9}{10} \log L$; and*
- *There exists $K_0 \in \mathbb{N}, \sigma = \sigma(\psi)$ such that*

$$\left| \int \phi d(\mu_1 P^n) - \int \phi d(\mu_2 P^n) \right| \leq L^{-\sigma(n - K_0)} .$$

for all $\phi \in L^\infty(\mathcal{C})$, μ_1, μ_2 Borel probabilities on $\mathcal{C}$, $n \geq K_0$.

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- No assumptions on detailed dynamics of $F-$ sinks could exist!
  - Sinks have basins of size $O(L^{-1})$; perturbations are just large enough **to escape with high probability**
- Precluding sinks of period ≤ 3 permits us to take $\epsilon \geq L^{-19/10}$ instead.

Proof sketch: dissipative case

Argument for ergodicity:
Let $\delta \in (0,1)$ and assume $\epsilon \geq L^{-1+\delta}$. Will show there exists $K = K(\delta)$ so that for any $(X_0, Y_0) \in \mathcal{C}$, X_K is distributed with positive density on $[0,1]$.

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- For $\text{dist}(x, \{f' = 0\}) \geq L^{-1+\delta/2}$, have $|f'(x)| \geq L^{\delta/2}$.
  - Pick component $\check{\gamma}_0$ of $\gamma_0 \setminus \{\text{dist}(x, \{f' = 0\}) < L^{-1+\delta/2}\}$ and map forward
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- Repeating, curves $\gamma_2, \gamma_3, \cdots$ have successively longer length until $K = K(\delta)$, when γ_K crosses $\mathcal{C}$ horizontally.

Proof sketch: dissipative case

Elaboration on this argument implies:

Key Lemma

For any (X_0, Y_0), have that X_K is distributed like

$$(1 - L^{-\delta/4}) \, \mathrm{Leb}_{[0,1)} + O(L^{-\delta/4}) \, .$$

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Estimate LE: suffices to bound

$$\int \log \|(dF_{\omega_{K+2}})_{(X_{K+1}, Y_{K+1})} u_{K+1}\| d\mathbb{P}(\omega_1, \cdots, \omega_{K+2}) \approx \log L$$

for arbitrary $(X_0, Y_0, u_0) \in P(\mathcal{C})$ (projective bundle).

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- Combine: $|f'_{\omega_{K+2}}(X_{K+1})| \approx L$ and u_{K+1} roughly horizontal with high probability.

Conclusion

- Small random perturbations simplify estimate of Lyapunov exponents
- Methods rely only on (checkable) rough geometry of the maps, not on detailed infinite-time dynamics
  - Amenable to broad generalization (e.g. higher dimension)
- Not so unnatural from modeling standpoint: the real world is inherently noisy!

Thank you!