Simpson meets Archimedes.

The heart of “Simpson’s Rule” is a lemma which might well have been known to Archimedes. In modern terminology and notation, it says:

*The area under the graph of a quadratic function \( q(x) \) on the interval \(-h \leq x \leq h\) equals*

\[
\frac{h}{3} \cdot \left( q(-h) + 4q(0) + q(h) \right).
\]

Indeed let \( a, b, c \) denote those three values of \( q \). To have a fixed picture in mind, assume that they are non-negative, and that the graph is concave downward, i.e. \( b > (a + c)/2 \). Subtracting a linear function whose graph includes \((-h, a)\) and \((h, c)\) diminishes the area in question by the trapezoidal \( h \cdot (a + c) \), and produces another quadratic function \( p(x) \), whose graph includes \((\pm h, 0)\) and \((0, m)\), where \( m = b - (a + c)/2 \). As Archimedes had shown, the region under this new graph makes up \( 2/3 \) of the rectangle \(-h \leq x \leq h, 0 \leq y \leq m \). Its area is therefore \( 4mh/3 = (4b - 2a - 2c) \cdot h/3 \). Adding this to the trapezoidal \( (3a + 3c) \cdot h/3 \) yields the desired result.

If we connect the three points \((-h, a), (0, b), (h, c)\) by two straight lines instead of a parabolic arc, the resulting area is

\[
\frac{h}{2} \cdot (a + 2b + c).
\]

It seems like such a good idea: instead of doing numerical integration by a series of trapezoids, use vertical strips bounded by the \( x \)-axis and a parabolic arc. But if the number of subdivisions is large, it does not seem to yield a great advantage. At least, I have not yet found a function where it does.