Ivan. Niven’s slick proof of 1946 is here presented in a more leisurely fashion. Its crux is:

$$\int_0^\pi x^n(\pi - x)^n \sin(x)dx \in n! \langle \pi^n, \ldots, \pi \rangle,$$

where $\langle \ldots \rangle$ means “group generated by”. This is proved by using (a) integration by parts and (b) the connection between the $k$-th coefficient $a_k$ of a polynomial and the constant term of its $k$-th derivative, namely $f^{(k)}(0) = k! a_k$

1. Integration. In the beginning was integration by parts. Since any polynomial $f(x)$ will finally run out of non-zero higher derivatives, we have

$$\int f(x) \sin(x)dx = -f(x) \cos(x) + \int f'(x) \cos(x)dx$$

$$= -f(x) \cos(x) + f'(x) \sin(x) + \int f''(x) \sin(x)dx$$

$$= F(x) \sin(x) - G(x) \cos(x).$$

where $F(x)$ and $G(x)$ are alternating sums of higher derivatives of $f(x)$ of odd and even order, respectively.

The context does not matter; any derivation $D$ will do, as long as $D^N f = 0$, and $\cos = D \sin$ with $(D^2 + 1) \sin = 0$. If also $\sin(0) = \sin(\psi) = 0$, we get

$$\int_0^\psi f(x) \sin(x)dx = G(0) + G(\psi).$$

Again this can be understood formally: the “definite integral” being defined as just the difference between the values at the “limits” of any anti-derivative.

2. Symmetry. As to the nature of $G(0)$, we note: if $f(x) = a_n x^n + a_{n+1} x^{n+1} + \cdots + a_m x^m$, we have $f^{(k)}(0) = k! a_k$, whence

$$G(0) \in n! \langle a_n, \ldots, a_m \rangle.$$

To get $G(\psi)$ into the same set, we shall need a suitably symmetric $f(x)$.

Since the reflection $\rho : x \mapsto (\psi - x)$ around $\psi/2$ has order 2, we can make a $\rho$-invariant function $f(x)$ from any $g(x)$ by setting $f(x) = g(x) g(\psi - x)$. It is entirely symmetric around $\psi/2$. Hence all its even higher derivatives at $x = 0$ are the same as at $x = \psi$. In the situation above, we now have $G(0) = G(\psi)$. Doing this for $g(x) = x^n$, we get $f(x) = x^n (\psi - x)^n$ whose coefficients are $\{ \pm \psi^k | 0 \leq k \leq n \}$ each multiplied by some binomial coefficient, whence (†) as promised at the outset, with $\psi$ replacing $\pi$.

3. Irrationality. If now $\pi$ were rational, say $\pi = a/b$, this would yield

$$\frac{b^n}{n!} \int_0^\pi f(x) \sin(x)dx \in Z.$$

We finally have to admit that we are working in the real number system, and that our definite integral is positive. Choosing $n$ large enough makes (‡) impossible, as the integrand is positive but less than $(\pi/2)^{2n} = (\pi/4)^n \cdot \pi^n$. 

The Irrationality of Pi According to Ivan.
4. A **Variant.** The following simpler variant might serve as an appetizer. By simply differentiating $f(x) \exp(x)$, we get

$$\int f(x) \exp(x) \, dx = f(x) \exp(x) - \int f'(x) \exp(x) \, dx = \cdots = F(x) \exp(x),$$

where $F(x)$ is an alternating sum of higher derivatives of $f(x)$. Again using $f(x) = x^n (\psi - x)^n$, but choosing $\psi = \log 7$, we get $F(0) = F(\psi) \in n! \langle \psi^n, \ldots, \psi \rangle$. If we had $\psi = a/b$, we could conclude

$$\frac{b^n}{n!} \int_0^\psi f(x) \exp(x) \, dx = \frac{b^n}{n!} \cdot 6F(0) \in \mathbb{Z},$$

and get into the same trouble as before. Of course, any natural number could take the place of 7.