The Irrationality of Pi According to Ivan.

Niven's slick proof of 1946 is here presented in a more leisurely fashion. Its crux is:

$$\int_0^\pi x^n(\pi - x)^n \sin(x)dx \in n!\,\langle \pi^n, \ldots, \pi\rangle\,, \tag{†}$$

where $\langle \ldots \rangle$ means "group generated by". This is proved by using (a) integration by parts and (b) the connection between the k-th coefficient a_k of a polynomial and the constant term of its k-th derivative, namely $f^{(k)}(0) = k!\,a_k$

1. Integration. In the beginning was integration by parts. Since any polynomial $fx)$ will finally run out of non-zero higher derivatives, we have

$$\begin{aligned}\int f(x)\sin(x)dx &= -f(x)\cos(x) + \int f'(x)\cos(x)dx \\ &= -f(x)\cos(x) + f'(x)\sin(x) + \int f''(x)\sin(x)dx \\ &= F(x)\sin(x) - G(x)\cos(x)\,.\end{aligned}$$

where $F(x)$ and $G(x)$ are alternating sums of higher derivatives of $f(x)$ of odd and even order, respectively.

The context does not matter; any derivation D will do, as long as $D^N f = 0$, and $\cos = D\sin$ with $(D^2+1)\sin = 0$. If also $\sin(0) = \sin(\psi) = 0$, we get

$$\int_0^\psi f(x)\sin(x)dx = G(0) + G(\psi)\,. \tag{$*$}$$

Again this can be understood formally: the "definite integral" being defined as just the difference between the values at the "limits" of any anti-derivative.

2. Symmetry. As to the nature of $G(0)$, we note: if $f(x) = a_n x^n + a_{n+1}x^{n+1} + \cdots + a_m x^m$, we have $f^{(k)}(0) = k!\,a_k$, whence

$$G(0) \in n!\,\langle a_n, \ldots, a_m\rangle\,. \tag{$**$}$$

To get $G(\psi)$ into the same set, we shall need a suitably *symmetric* $f(x)$.

Since the reflection $\rho : x \mapsto (\psi - x)$ around $\psi/2$ has order 2, we can make a ρ-invariant function $f(x)$ from any $g(x)$ by setting $f(x) = g(x)g(\psi - x)$. It is entirely symmetric around $\psi/2$. Hence all its even higher derivatives at $x = 0$ are the same as at $x = \psi$. In the situation above, we now have $G(0) = G(\psi)$. Doing this for $g(x) = x^n$, we get $f(x) = x^n(\psi - x)^n$ whose coefficients are $\{\pm\psi^k \mid 0 \le k \le n\}$ each multiplied by some binomial coefficient, whence (†) as promised at the outset, with ψ replacing π.

3. Irrationality. If now π were rational, say $\pi = a/b$, this would yield

$$\frac{b^n}{n!}\int_0^\pi f(x)\sin(x)dx \in \mathbf{Z}\,. \tag{‡}$$

We finally have to admit that we are working in the real number system, and that our definite integral is *positive*. Choosing n large enough makes (‡) impossible, as the integrand is positive but less than $(\pi/2)^{2n} = (\pi/4)^n \cdot \pi^n$.

4. A Variant. The following simpler variant might serve as an appetizer. By simply differentiating $f(x)\exp(x)$, we get

$$\int f(x)\exp(x)dx = f(x)\exp(x) - \int f'(x)\exp(x)dx = \cdots = F(x)\exp(x)\,,$$

where $F(x)$ is an alternating sum of higher derivatives of $f(x)$. Again using $f(x) = x^n(\psi - x)^n$, but choosing $\psi = \log 7$, we get $F(0) = F(\psi) \in n!\,\langle \psi^n, \dots, \psi\rangle$. If we had $\psi = a/b$, we could conclude

$$\frac{b^n}{n!}\int_0^{\psi} f(x)\exp(x)dx = \frac{b^n}{n!}\cdot 6\,F(0) \;\in\; \mathbf{Z}\,,$$

and get into the same trouble as before. Of course, any natural number could take the place of 7.