Letters in Mathematical Physics **29**: 271–279, 1993.

On Symplectic Submanifolds of Cotangent Bundles

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(Received: 7 May 1993; revised version: 8 September 1993)

Abstract. Necessary and sufficient conditions are given for a symplectic submanifold of a cotangent bundle to itself be a cotangent bundle.

Mathematics Subject Classifications (1991). 53C15 (primary); 53C80, 70G35 (secondary).

1. Introduction

The phase spaces of most physical systems are (canonical) cotangent bundles and, indeed, much of the apparatus of mechanics and field theory is predicated upon this circumstance [1, 7]. Cotangent bundles have natural polarizations (for quantization), canonical momentum maps (for groups of point transformations), and so forth. They provide a traditional setting for variational principles. Moreover, the identification of initial value constraints with components of the momentum map for the gauge group depends crucially upon the underlying cotangent structure [3].

The question often arises as to when a symplectic submanifold $S \subset T^*M$ is the cotangent bundle of a submanifold $N \subset M$, notably in the context of the 'Dirac bracket' construction [2, 9]. Here one has a submanifold C of T^*M – the 'constraint set' – that, for various purposes, one wishes to be coisotropic. If C is not coisotropic in T^*M, this construction shows that it is possible to find a symplectic submanifold $S \subset T^*M$ containing C relative to which the latter is coisotropic. Regarding the 'Dirac manifold' S as a new phase space, it is therefore important to determine if it is a cotangent bundle. A similar problem arises in the study of mechanical systems with discrete symmetries [5, §8]; in this case the phase space is taken to be the fixed point set of the symmetry group. Physical considerations aside, symplectic submanifolds appear, for instance, when realizing Poisson manifolds [11, §9] and when constructing 'symplectic cross-sections' of Hamiltonian group actions [4, §32].

In this Letter, I give necessary and sufficient conditions for a symplectic submanifold of T^*M to be the canonical cotangent bundle of a submanifold of M or, more generally, of a quotient space of a submanifold of M. I discuss several examples in Section 4.

★Partially supported by NSF grant DMS-9222241.

2. The Case $n = m - k$

Let S be a connected symplectic submanifold of the canonical cotangent bundle (T^*M, ω_M). (By a *canonical cotangent bundle,* I mean a cotangent bundle $\pi_M\colon T^*M \to M$ equipped with its standard symplectic form $\omega_M = -d\theta_M$.) Suppose that $\dim M = m$ and that S has codimension $2k$ in T^*M. Furthermore, assume that $T\pi_M | TS$ has constant rank n. Then $D = \ker T\pi_M \cap TS$ is a rank $2m - 2k - n$ isotropic subbundle of TS, and this is possible only if $2m - 2k - n \leqslant \frac{1}{2}(2m - 2k)$, that is, $n \geqslant m - k$. Examples show that n can take on any value $m - k \leqslant n \leqslant m$. The case $n = m - k$ is special, and leads to the following characterization of S.

THEOREM 1. *Suppose that $n = m - k$. Then there exists a submanifold N of M, a surjective submersion $\pi\colon S \to N$ and a symplectic local diffeomorphism ϕ of S with the canonical cotangent bundle T^*N such that the following diagram commutes:*

$$\begin{array}{ccc} S & \xrightarrow{\ \phi\ } & T^*N \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_N} & \\ & N & \end{array}$$

Proof. Take $N = \pi_M(S)$. Then π_M restricts to a submersion π of S onto $N \subseteq M$, and it follows that N is an n-dimensional submanifold of M. Let i be the inclusion $N \hookrightarrow M$. The restriction of $i^*\colon T^*_N M \to T^*N$ to S induces a map $\phi\colon S \to T^*N$ covering the identity on N. I will prove that ϕ is symplectic.

Let $j\colon S \hookrightarrow T^*M$ be the inclusion, so that $\phi = i^* \circ j$. The task is to show that $\phi^*\omega_N = j^*\omega_M$; I claim that, in fact, $\phi^*\theta_N = j^*\theta_M$. Indeed, for each $v \in T_\alpha S$, the definition of the canonical 1-form yields

$$\begin{aligned} &\langle (\phi^*\theta_N)(\alpha), v\rangle \\ &\quad = \langle \theta_N(\phi(\alpha)), \phi_*(v)\rangle = \langle \phi(\alpha), (\pi_N \circ \phi)_*(v)\rangle \\ &\quad = \langle (i^* \circ j)(\alpha), (\pi_N \circ i^* \circ j)_*(v)\rangle = \langle j(\alpha), (i \circ \pi_N \circ i^*)_* j_*(v)\rangle \\ &\quad = \langle j(\alpha), \pi_{M*} j_*(v)\rangle = \langle \theta_M(j(\alpha)), j_*(v)\rangle \\ &\quad = \langle (j^*\theta_M)(\alpha), v\rangle, \end{aligned}$$

where I have used the relation $i \circ \pi_N \circ i^* = \pi_M$.

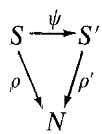
Since S and T^*N have the same dimension $2(m - k)$, the symplecticity of ϕ implies that it is a local diffeomorphism. □

The map $\phi\colon S \to T^*N$ need be neither injective nor surjective, in which cases S is not quite a cotangent bundle, but rather a 'covering' of an open submanifold of one. However, it is possible to determine when S will be a cotangent bundle exactly.

To this end, observe that when $n = m - k$, the (involutive) subbundle $D = \ker T\pi_M \cap TS$ is actually Lagrangian in TS. Thus, $\pi\colon S \to N$ is a Lagrangian submersion, and it is well known that the fibers of π are then affine manifolds. The following is a consequence of [10] and [12, §4.7].

PROPOSITION 2. *Let S be an exact symplectic manifold, and let $S \to N$ be a locally trivial Lagrangian fibration. Then $S \to N$ is equivalent to the canonical cotangent bundle $T^*N \to N$ iff the fibers of $S \to N$ are connected, simply connected and complete as affine manifolds.*

Here, two Lagrangian submersions $\rho: S \to N$ and $\rho': S' \to N$ are *equivalent* if there exists a symplectomorphism $\psi: S \to S'$ such that the diagram



commutes.

This result provides an intrinsic means of determining whether S is equivalent to a cotangent bundle, in that it does not rely upon S being embedded in a cotangent bundle. *A priori*, Proposition 2 might yield a symplectomorphism ψ of S with T^*N even when ϕ is badly behaved. But it turns out this is impossible:

THEOREM 3. *$\pi: S \to N$ is equivalent to the canonical cotangent bundle $\pi_N: T^*N \to N$ iff $\phi: S \to T^*N$ is a symplectomorphism.*

Proof. The converse follows immediately from Theorem 1.

For the necessity, suppose $\pi: S \to N$ is equivalent to $\pi_N: T^*N \to N$. (In this regard, note that the (pullback) symplectic form $\omega = -d\theta$ on S is exact.) I will construct an explicit diffeomorphism $\psi: T^*N \to S$ and show that ψ is the inverse of ϕ.

Given $\eta \in T_q^*N$, define a vector field V_η on the fiber $S_q = \pi^{-1}(q)$ by $i(V_\eta)\omega = \pi^*\eta$. Each V_η is parallel with respect to the affine structure on S_q (by the very definition of the latter), and by Proposition 2 its flow Ψ_η^t is complete. Now fix a section s of π (which exists since $S \to N$ is assumed equivalent to a vector bundle) and define $\psi_q: T_q^*N \to S_q$ by

$$\psi_q(\eta) = \Psi^1_{\eta - (s^*\theta)(q)}(s(q)).$$

The map ψ_q is, in fact, independent of the choice of s, as may be verified by examining its coordinate representation. Then by the topological conditions in Proposition 2, ψ_q is a diffeomorphism, and as q varies the ψ_q amalgamate to produce the required diffeomorphism ψ.

Now let $\alpha_q \in S_q$, and extend α_q to a section α of π. Then $\phi \circ \alpha = (\phi \circ \alpha)^*\theta_N = \alpha^*\phi^*\theta_N = \alpha^*j^*\theta_M = \alpha^*\theta$ by the proof of Theorem 1, and so

$$\psi(\phi(\alpha_q)) = \Psi^1_{(\alpha^*\theta)(q) - (s^*\theta)(q)}(s(q)).$$

Since ψ does not depend upon the choice of section, one can take $s = \alpha$, whence it is apparent that $\psi(\phi(\alpha_q)) = \alpha_q$. Consequently, $\psi = \phi^{-1}$. □

Thus if $\pi: S \to N$ is a canonical cotangent bundle then, up to equivalence, the symplectomorphism must be given by ϕ. *Caveat:* Even if $\pi: S \to N$ is not (equivalent to) a cotangent bundle, it is still possible for S to be symplectomorphic to T^*N. The catch is that the cotangent bundle structure is then defined relative to a Lagrangian submersion $S \to N$ *other than* π.

I now describe several situations in which ϕ is guaranteed to be a symplectomorphism.

COROLLARY 4. *If S is an affine subbundle of T_N^*M then $\phi: S \to T^*N$ is a symplectomorphism.*

Proof. Apply Proposition 2 and Theorem 3, using the fact that the symplectic form on S is exact.

Alternately, one can proceed directly. Since S projects onto N, it suffices to show that ϕ restricts to a bijection $S_q \to T_q^*N$ for every $q \in N$. By assumption, each fiber S_q is an affine subspace of T_q^*M, and so the desired result follows from the equidimensionality of S_q and T_q^*N and the fact that ϕ is locally injective. □

Although the converse of this corollary does not hold, in applications it often happens that S is an affine (or even a vector) subbundle of T^*M. As an illustration, consider a mechanical system which is constrained to move in a submanifold N of the configuration space M. Let $TN^\perp$ be any complement of TN in T_NM and set $S = (TN^\perp)^\vdash$, where '$\vdash$' denotes annihilator. Then S is symplectic and since it is a vector subbundle of T_N^*M, S must be symplectomorphic to T^*N. Thus S defines an explicit embedding of the constrained phase space T^*N in the original phase space T^*M. (For more about holonomic constraints, see [1, 7].) Another example is provided by fixed point sets of symplectic group actions, cf. Section 4.

Typically symplectic submanifolds S are given as the zero sets of cotangent observables, as evidenced by the examples in Section 4. (A *cotangent observable* is a function on T^*M of the form $P_X + \pi_M^* f$, where P_X is the momentum of the vector field X on M and f is a function on M.) If nonempty, each fiber S_q is then an affine subspace of T_q^*M, and Corollary 4 yields

COROLLARY 5. *If S is defined globally by the vanishing of cotangent observables and $n = m - k$, then S is symplectomorphic to a cotangent bundle.*

One may relax the requirement that S be defined *globally* in this fashion provided the fibers S_q are connected.

An important instance in which Corollary 5 applies is when a Lie group G acts on M. Suppose that the isotropy groups of G on M are all conjugate, say to the subgroup H of G. Let $J: T^*M \to \mathfrak{g}^*$ be the momentum map for the lifted action of G on T^*M, where $\mathfrak{g}$ is the Lie algebra of G. Then one recovers the following version of the cotangent bundle reduction theorem [6] when the reducing manifold is symplectic.

COROLLARY 6. *Suppose that* (*i*) *$J^{-1}(\mu)$ is a symplectic submanifold with codimension* 2*k*, *and* (*ii*) dim $G/H = k$. *Then $\phi: J^{-1}(\mu) \to T^*N$ is a symplectomorphism, where $N = \pi_M(J^{-1}(\mu))$.*

Proof. By definition $\alpha \in J^{-1}(\mu)$ iff $\langle \alpha, \xi_M(q) \rangle = \langle \mu, \xi \rangle$ for all $\xi \in \mathfrak{g}$, where $q = \pi_M(\alpha)$ and ξ_M is the infinitesimal generator on M associated to ξ. Thus $J^{-1}(\mu) \cap T_q^*M = \{\alpha\} + T_q(G \cdot q)^{\perp}$, where $G \cdot q$ is the orbit of q. Assumption (*ii*) then implies that $n = m - k$, and Corollary 6 now follows from Corollary 5 upon recalling that the components of the momentum map for a lifted action are cotangent observables. □

Remarks (1) When $\mu \in \mathfrak{g}^*$ is a weakly regular value of J, $J^{-1}(\mu)$ is symplectic iff $\dim \mathfrak{g} - \dim \mathfrak{g}_\mu = 2k$, where $\mathfrak{g}_\mu$ is the isotropy algebra of μ.

(2) Assumptions (*i*) and (*ii*) impose stringent conditions on the action of G on M. Indeed, the rank of TJ is at most the dimension of G, so if μ is a weakly regular value of J, then

$$\dim G \geqslant \operatorname{rk} TJ = \operatorname{codim} J^{-1}(\mu) = 2k.$$

Thus, G cannot act freely, for then $\dim G = k$. Furthermore, H cannot be normal in G.

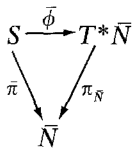
Since level sets $J^{-1}(\mu)$ are symplectically orthogonal to G-orbits, one might expect there to be an analogous result for orbits. So suppose that $\mathcal{O}$ is a symplectic G-orbit in T^*M of codimension $2\tilde{k} = 2m - 2k$, where $k = \dim G/H$. Then $\bar{\mathcal{O}} = \pi_M(\mathcal{O})$ is an orbit on M and $n = m - \tilde{k}$. Thus Theorem 1 gives a symplectic local diffeomorphism $\mathcal{O} \to T^*\bar{\mathcal{O}}$ covering the identity on $\bar{\mathcal{O}}$.

3. The Case $n > m - k$

I mentioned earlier that although there are no symplectic submanifolds S of T^*M with $n < m - k$, it is certainly possible to find Ss with $m - k \leqslant n \leqslant m$. None of these submanifolds are *obviously* cotangent bundles of submanifolds of the base M unless $n = m - k$. (Indeed, if S is to be the cotangent bundle of $N \subset M$, the only apparent candidate is $N = \pi_M(S)$, so that one must have $n = m - k$.) But this is not to say that an S with $n > m - k$ cannot ultimately be of just this form. In fact, this is exactly what happens for the Proca field (see Section 4) and Palatini gravity (cf. [3]). Such occurrences might seem impossible to predict without additional data, such as (in the Proca case) a second cotangent bundle structure on T^*M which is transversal to the given one. However, further study of the Proca example suggests that it may be more useful to think of S not as the cotangent bundle of a submanifold of the base, but rather as the cotangent bundle of a *quotient space* of $N = \pi_M(S)$. I now formalize this observation, assuming henceforth that $n > m - k$.

Suppose the isotropic subbundle D of TS can be extended to a Lagrangian subbundle L satisfying $i(L)\theta = 0$, where $\theta = j^*\theta_M$. (Note that $i(D)\theta = 0$.) Then L is involutive; let $\bar{N}$ denote the corresponding quotient space. Assume that $\bar{N}$ is a manifold and that the projection $\bar{\pi}: S \to \bar{N}$ is a submersion. Furthermore, assume that the fibers of $\pi: S \to N$ are connected. Then $\bar{\pi}$ factors through π and hence $\bar{N}$ may be regarded as the quotient manifold of N generated by the involutive distribution $\bar{L} = \pi_* L$.

THEOREM 7. *Under the above assumptions, there exists a quotient manifold $\bar{N}$ of N, a surjective submersion $\bar{\pi}$: $S \to \bar{N}$ and a symplectic local diffeomorphism $\bar{\phi}$ of S with the canonical cotangent bundle $T^*\bar{N}$ such that the following diagram commutes:*



Proof. The distribution $\bar{L}$ has a 'momentum map' $\bar{J}$ on T^*N defined in the usual way [3]: $\bar{J}(\beta) = \beta \restriction \bar{L}$. I claim that $\phi(S) \subset \bar{J}^{-1}(0)$, where ϕ: $S \to T^*N$ is defined as before. (Although now dim $T^*N >$ dim S, ϕ is still a symplectic immersion.) To see this, use the definitions of ϕ and θ and compute:

$$\begin{aligned} \langle \phi(\alpha), \bar{L} \rangle &= \langle j(\alpha), i_* \bar{L} \rangle = \langle j(\alpha), i_* \pi_* L \rangle = \langle j(\alpha), \pi_{M*} j_* L \rangle \\ &= \langle \theta_M(j(\alpha)), j_* L \rangle = \langle \theta(\alpha), L \rangle = 0 \end{aligned}$$

by the defining property of L.

According to the cotangent bundle reduction theorem for foliations [3], the reduction of T^*N by $\bar{J}^{-1}(0)$ is symplectomorphic to $(T^*\bar{N}, \omega_{\bar{N}})$. Since $\phi(S) \subset \bar{J}^{-1}(0)$, reduction thus induces a symplectic map $\bar{\phi}$: $S \to T^*\bar{N}$. Now $S \to \bar{N}$ is a *Lagrangian* submersion, so by dimensions the map $\bar{\phi}$ is a local diffeomorphism. □

The case when the fibers of π: $S \to N$ are not connected can be handled with only minor alterations. Suppose π is a fibration, so that the leaf space S/D covers N. Then $\bar{N} = S/L$ is a quotient manifold of S/D, rather than of N. The above proof still goes through, provided N is replaced by S/D and ϕ is composed with the pullback of the covering map $S/D \to N$.

As in Section 2, $\bar{\pi}: S \to \bar{N}$ will be equivalent to the canonical cotangent bundle $\pi_{\bar{N}}$: $T^*\bar{N} \to \bar{N}$ iff $\bar{\phi}$: $S \to T^*\bar{N}$ is a symplectomorphism.

4. Examples

4.1. MASSIVE SCALAR PARTICLES

Here is a way to construct the phase space for a spinless relativistic particle with mass $m > 0$ which does not involve reduction. The ambient phase space is the cotangent bundle of Minkowski spacetime M, parametrized by 4-positions q and 4-momenta p. The 'elementary system' (in the terminology of Woodhouse [12]) for such a particle can be explicitly realized as a subcotangent bundle of T^*M by imposing the mass constraint $p_\mu p^\mu = -m^2$ and restricting to a spacelike hyperplane $N \subset M$. The symplectic submanifold S singled out by these conditions has two components, each of which may be identified with T^*N. It is amusing to note that this construction fails when $m^2 \leqslant 0$: in the case of a photon ($m = 0$) S is singular, while for a tachyon ($m^2 < 0$) S, although nonsingular, is not everywhere symplectic. For further discussion of these matters, see [12].

4.2. FIXED-POINT SETS OF SYMPLECTIC GROUP ACTIONS

Suppose G is a (not necessarily connected) Lie group which acts properly on a manifold M and, thence, properly on T^*M by cotangent lift. Consider the fixed point set $(T^*M)_G$. Then $(T^*M)_G$ is a symplectic submanifold [4, Theorem 27.2]. Furthermore, $(T^*M)_G$ fibers over M_G: that $\pi_M((T^*M)_G) \subseteq M_G$ is obvious, and the reverse inclusion follows from the fact that $(T^*M)_G$ contains the zero section over M_G. Since the action of G is linear along the fibers $\pi^{-1}(q) = (T_q^*M)_G$, $(T^*M)_G$ is a vector subbundle of T^*M. Thus, $\dim (T^*M)_G = 2 \dim M_G$ and Corollary 4 yields $(T^*M)_G \approx T^*(M_G)$. Fixed point sets are of interest in the context of mechanical systems which possess discrete symmetries, cf. [5, §8].

I remark that in the literature these results presuppose that G is compact (to show that M_G and $(T^*M)_G$ are smooth, and then that $(T^*M)_G$ is symplectic). But in actuality one only needs the existence of invariant Riemannian metrics, and for this it suffices that the action be proper [8, Theorem 4.3.1].

4.3. DIRAC MANIFOLDS FOR ANGULAR MOMENTUM

Consider a particle moving in $M = \mathbb{R}^3$ with constant angular momentum $\mathbf{l} \neq 0$. Setting $J(\mathbf{q}, \mathbf{p}) = \mathbf{q} \times \mathbf{p}$, it is straightforward to check that $J^{-1}(\mathbf{l})$ is a three-dimensional submanifold which is not coisotropic. (If $\mathbf{l} = \mathbf{0}$, the constraint set is four-dimensional away from the origin and is already coisotropic in $T^*\mathbb{R}^3$.) Therefore it is of interest to construct a Dirac manifold in this situation (cf. the Introduction). To this end, set $S = J^{-1}([\mathbf{l}])$, where $[\mathbf{l}]$ denotes the *open* ray through $\mathbf{l}$ and the origin in so(3)*. Then S is a symplectic four-dimensional submanifold of $T^*\mathbb{R}^3$ containing $J^{-1}(\mathbf{l})$ relative to which $J^{-1}(\mathbf{l})$ is coisotropic. One verifies that $n = 2$ and that $N = \{\mathbf{l}\}^{\perp} \backslash \{\mathbf{0}\}$. In this case $\phi: S \to T^*N$ is not onto, because of the condition that $[\mathbf{l}]$ be open in the definition of S. (Without this restriction, D would not have constant rank 2, and S would not be globally symplectic.) But this condition merely removes a closed half-plane from each fiber, and so the Dirac manifold S is symplectomorphic to an open subbundle of T^*N.

One can similarly construct a Dirac manifold for a rigid body which is rotating with constant angular momentum $\mathbf{l} \neq \mathbf{0}$. The main difference is that whereas the particle fell into the case $n = m - k$, the rigid body has $n > m - k$.

For the rigid body with one point fixed $M = \mathrm{SO}(3)$, with SO(3) acting on itself by left translations. Work in space coordinates, so that $T^*\mathrm{SO}(3) \approx \mathrm{SO}(3) \times \mathrm{so}(3)^*$. The momentum map for the lifted action on $T^*\mathrm{SO}(3)$ is just $J(g, \mathbf{j}) = \mathbf{j}$ and hence $J^{-1}(\mathbf{l}) = \mathrm{SO}(3) \times \{\mathbf{l}\}$. Again define $S = J^{-1}([\mathbf{l}])$; then S is a Dirac manifold for $J^{-1}(\mathbf{l})$

with $n = 3$. (As before, the zero section must be deleted if S is to be globally symplectic.) To apply Theorem 7, it is necessary to construct a Lagrangian subbundle L of TS satisfying $i(L)\theta = 0$. For this, fix any nonzero vector $\mathbf{k}$ in $\mathrm{so}(3)^* \approx \mathrm{so}(3)$ perpendicular to $\mathbf{l}$ and let K be the distribution on SO(3) spanned by its infinitesimal generator. For $\alpha = (g, \mathbf{j}) \in S$, set $L_\alpha = K_g \oplus [\mathbf{j}]$; it is straightforward to check that L is the required subbundle. Again ϕ is not onto, with the consequence that S is symplectomorphic to the *punctured* cotangent bundle of $S/L \approx \mathrm{SO}(3)/\mathrm{SO}(2)_{\mathbf{k}} \approx S^2$, where $\mathrm{SO}(2)_{\mathbf{k}}$ is the subgroup of rotations about the axis $\mathbf{k}$.

Under favorable conditions, it is possible to generalize this construction of Dirac manifolds; Theorem 26.7 of [4] gives the local theory. A similar construction is involved in the problem of realizing Lie-Poisson manifolds [11, §9].

4.4. THE PROCA FIELD

This is the standard example of a field theory with second class constraints in the sense of Dirac [2, 9]. The configuration space M is the space of 1-forms on Minkowski spacetime restricted to a constant time hyperplane Σ. Decompose elements A of M as $(A_0, \mathbf{A})$ and their canonical momenta E as $(E^0, \mathbf{E})$. The Proca constraints are:

$$E^0 = 0, \qquad \nabla \cdot \mathbf{E} + \mu^2 A_0 = 0,$$

where $\mu > 0$ is the mass. These define a symplectic submanifold S of T^*M; thus $k = 1 \times \infty^3$. But $n = m = 4 \times \infty^3$, so Theorem 1 and Corollary 5 would not seem to be pertinent. Nonetheless, by solving the second constraint for A_0, it is clear that S is symplectomorphic to T^*Q, where $Q \subset M$ is the subspace of 1-forms $\mathbf{A}$ *on* Σ.

One reason this example 'works' is that M in this instance is a (reflexive) linear space, so that $T^*M = M \times M^* \approx T^*M^*$. If one views S as residing in T^*M^*, then the dimension of $\ker T\rho \cap TS$ is $3 \times \infty^3$ and so now Corollary 5 applies (at least formally) to the projection $\rho\colon T^*M^* \to M^*$. Thus one obtains $S \approx T^*Q^* \approx Q^* \times Q = T^*Q$, with $Q^* \subset M^*$ being the subspace of all 'electric' 1-forms $\mathbf{E}$ on Σ.

However, it is somewhat more natural to regard Q as a quotient space of M than as a subspace. Proceeding as in Section 3, then, let L be the distribution on S spanned by the vector fields

$$\mathbf{f} \cdot \frac{\delta}{\delta \mathbf{E}}, \qquad \mu^2 \mathbf{g} \cdot \frac{\delta}{\delta \mathbf{E}} - (\nabla \cdot \mathbf{g}) \frac{\delta}{\delta A_0},$$

where $\mathbf{f}$, $\mathbf{g}$ are arbitrary vector functions on Σ with $\nabla \cdot \mathbf{f} = 0$. L is Lagrangian and clearly annihilates $\theta = \mathbf{E} \cdot d\mathbf{A}$. Thus Theorem 7 applies and since the constraints are linear, S is symplectomorphic to the cotangent bundle of $\bar{N} = S/L \approx N/\bar{L} \approx Q$.

A similar, but vastly more complicated phenomenon occurs in the Palatini formulation of Einstein gravity, cf. [3].

Acknowledgements

I would like to thank Jędrzej Śniatycki and Jim Isenberg for helpful discussions, and the referee for some useful suggestions.

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