

On the Groenewold–Van Hove problem for \mathbf{R}^{2n}

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(Received 28 September 1998; accepted for publication 16 December 1998)

We discuss the Groenewold–Van Hove problem for \mathbf{R}^{2n} , and completely solve it when $n=1$. We rigorously show that there exists an obstruction to quantizing the Poisson algebra of polynomials on \mathbf{R}^{2n} , thereby filling a gap in Groenewold’s original proof. Moreover, when $n=1$ we determine the largest Lie subalgebras of polynomials which can be consistently quantized, and explicitly construct all their possible quantizations. © 1999 American Institute of Physics. [S0022-2488(99)00304-7]

I. INTRODUCTION

In 1946 Groenewold¹ presented a remarkable result which states that one cannot consistently quantize the Poisson algebra of all polynomials in the positions q^i and momenta p_i on \mathbf{R}^{2n} as symmetric operators on some Hilbert space \mathcal{H} , subject to the requirement that the q^i and p_i be irreducibly represented. Van Hove² subsequently refined Groenewold’s result. Thus it is *in principle* impossible to quantize—by *any* means—every classical observable on \mathbf{R}^{2n} , or even every polynomial observable, in a way consistent with Schrödinger quantization (which, according to the Stone–Von Neumann theorem, is the import of the irreducibility requirement on the q^i and p_i). At most one can consistently quantize certain Lie subalgebras of observables, for instance polynomials which are at most quadratic, or observables which are affine functions of the momenta.

This is not quite the end of the story, however; there are two loose ends which need to be tied up. The first is that there is a technical gap in Groenewold’s proof.³ This gap has been filled in Ref. 2 (see also Ref. 4) by means of a certain functional analytic assumption. Although “small,” this gap is nevertheless vexing, and its elimination in this manner is not entirely satisfactory. Second, in the absence of such a polynomial quantization, it is important to determine the largest Lie subalgebras of polynomials that can be consistently quantized along with their quantizations. While some results are known along these lines, this program has not yet been fully carried out.

In this paper we consider the Groenewold–Van Hove problem for \mathbf{R}^{2n} . We present two variants of Groenewold’s theorem (“strong” and “weak”); the weak one is the version that Groenewold actually proved, while the strong one is the result referred to above. We then show that the strong version follows from the weak one *without* introducing extra hypotheses. Thus we fill the gap in Groenewold’s proof. Moreover, when $n=1$ we determine the largest quantizable Lie subalgebras of polynomials and explicitly construct all their possible quantizations.

To make the presentation self-contained, we include a detailed discussion of previous work on the Groenewold–Van Hove problem.

II. BACKGROUND

Let $P(2n)$ denote the Poisson algebra of polynomials on \mathbf{R}^{2n} with Poisson bracket

$$\{f, g\} = \sum_{k=1}^n \left[\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q^k} - \frac{\partial g}{\partial p_k} \frac{\partial f}{\partial q^k} \right].$$

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$P(2n)$ contains several distinguished Lie subalgebras: The *Heisenberg algebra*

$$\mathfrak{h}(2n) = \text{span}\{1, q^i, p_i \mid i = 1, \dots, n\};$$

the *symplectic algebra*

$$\text{sp}(2n, \mathbf{R}) = \text{span}\{q^i q^j, q^i p_j, p_i p_j \mid i, j = 1, \dots, n\};$$

the *extended (or inhomogeneous) symplectic algebra*

$$\text{hsp}(2n, \mathbf{R}) = \text{span}\{1, q^i, p_j, q^i q^j, q^i p_j, p_i p_j \mid i, j = 1, \dots, n\},$$

which is the semidirect product of $\mathfrak{h}(2n)$ with $\text{sp}(2n, \mathbf{R})$; and the *coordinate (or position) algebra*

$$C(2n) = \left\{ \sum_{i=1}^n f^i(q) p_i + g(q) \right\},$$

where f^i and g are polynomials. All of these will play an important role in our development.

Let $P^k(2n)$ denote the subspace of polynomials of degree at most k , and $P_k(2n)$ the subspace of homogeneous polynomials of degree k . Then $P^1(2n) = \mathfrak{h}(2n)$, $P_2(2n) = \text{sp}(2n, \mathbf{R})$, and $P^2(2n) = \text{hsp}(2n, \mathbf{R})$. When n is fixed, we simply write $P = P(2n)$, etc.

We now state what it means to “quantize” a Lie algebra of polynomials on \mathbf{R}^{2n} . Throughout, the Heisenberg algebra $\mathfrak{h}(2n)$ is regarded as a “basic algebra of observables.”⁵

Definition 1: Let \mathcal{O} be a Lie subalgebra of $P(2n)$ containing the Heisenberg algebra $\mathfrak{h}(2n)$. A *quantization* of \mathcal{O} is a linear map \mathcal{Q} from \mathcal{O} to the linear space $\text{Op}(D)$ of symmetric operators which preserve a fixed dense domain D in some separable Hilbert space \mathcal{H} , such that for all $f, g \in \mathcal{O}$,

(Q1) $\mathcal{Q}(\{f, g\}) = i/\hbar [\mathcal{Q}(f), \mathcal{Q}(g)],$

(Q2) $\mathcal{Q}(1) = I,$

(Q3) If the Hamiltonian vector field X_f of f is complete, then $\mathcal{Q}(f)$ is essentially self-adjoint on D ,

(Q4) \mathcal{Q} represents $\mathfrak{h}(2n)$ irreducibly, and

(Q5) D contains a dense set of separately analytic vectors for the standard basis of $\mathcal{Q}(\mathfrak{h}(2n))$.

We briefly comment on these conditions; a full exposition along with detailed motivation is given in Ref. 5.

Condition (Q1) is Dirac’s famous “Poisson bracket→commutator” rule; here \hbar is Planck’s reduced constant. The second condition reflects the fact that if an observable f is a constant c , then the probability of measuring $f=c$ is one regardless of which quantum state the system is in. Regarding (Q3), we remark that in contradistinction with Van Hove,² we do not confine our considerations to only those classical observables whose Hamiltonian vector fields are complete. Rather than taking the point of view that “incomplete” classical observables cannot be quantized, we simply do not demand that the corresponding quantum operators be essentially self-adjoint (“e.s.a.”).

(Q4) and (Q5) emphasize the fundamental role of the Heisenberg algebra. The technical condition (Q5) guarantees the integrability of the Lie algebra representation $\mathcal{Q}(\mathfrak{h}(2n))$ on D .⁶ [There do exist nonintegrable representations of the Heisenberg algebra;⁷ however, none of them seem to have physical significance. (Q5) thus serves to eliminate these “spurious” representations.] By virtue of the Stone–Von Neumann theorem, (Q5) along with the irreducibility criterion (Q4) imply that $\mathcal{Q}(\mathfrak{h}(2n))$ is unitarily equivalent to a restriction of the *Schrödinger quantization* $d\Pi$:

$$q^i \mapsto q^i, \quad p_j \mapsto -i\hbar \partial/\partial q^j, \quad \text{and} \quad 1 \mapsto I \tag{1}$$

on the Schwartz space $\mathcal{S}(\mathbf{R}^n) \subset L^2(\mathbf{R}^n)$. Indeed, by (Q5) $\mathcal{Q}(\mathfrak{h}(2n))$ can be integrated to a representation τ of the Heisenberg group $H(2n)$ which, according to (Q4), is irreducible. The Stone–Von Neumann theorem then states that this representation of $H(2n)$ is unitarily equivalent to the Schrödinger representation Π , and hence $\tau = U\Pi U^{-1}$ for some unitary map $U: L^2(\mathbf{R}^n) \rightarrow \mathcal{H}$. Consequently,⁸ $\mathcal{Q}(f) = U\overline{d\Pi(f)}U^{-1}|_D$ for all $f \in \mathfrak{h}(2n)$, where the bar denotes closure. It now follows from (1), the invariance of the domain D , and Sobolev’s lemma that $U^{-1}D \subseteq \mathcal{S}(\mathbf{R}^n)$, so that $U^{-1}\mathcal{Q}U$ is the restriction of $d\Pi$ to $U^{-1}D$.

Finally, there is a sixth criterion that a quantization must satisfy in general, viz., that \mathcal{Q} be faithful when restricted to the given basic algebra of observables.⁵ In the case of the Heisenberg algebra, however, this follows automatically by virtue of (Q1) and (Q2).

III. THE WEAK NO-GO THEOREM

In the next two sections we argue that there are no quantizations of $P(2n)$. Extensive discussions can be found in Refs. 1–4 and 9–13. We shall state the main results for \mathbf{R}^{2n} but, for convenience, usually prove them only for $n = 1$. The proofs for higher dimensions are immediate generalizations of these.

We begin by observing that there *does* exist a quantization $d\varpi$ of $\mathfrak{hsp}(2n, \mathbf{R})$. It is given by the familiar formulas

$$d\varpi(q^i) = q^i, \quad d\varpi(1) = I, \quad d\varpi(p_j) = -i\hbar \frac{\partial}{\partial q^j},$$

$$d\varpi(q^i q^j) = q^i q^j, \quad d\varpi(p_i p_j) = -\hbar^2 \frac{\partial^2}{\partial q^i \partial q^j}, \tag{2}$$

$$d\varpi(q^i p_j) = -i\hbar \left(q^i \frac{\partial}{\partial q^j} + \frac{1}{2} \delta_j^i \right), \tag{3}$$

on the domain $\mathcal{S}(\mathbf{R}^n) \subset L^2(\mathbf{R}^n)$. Properties (Q1)–(Q3) are readily verified. (Q4) follows automatically since the restriction of $d\varpi$ to P^1 is just the Schrödinger representation. For (Q5) we recall that the Hermite functions $h_{k_1 \dots k_n}(q^1, \dots, q^n) = h_{k_1}(q^1) \cdots h_{k_n}(q^n)$, where

$$h_k(q) = e^{q^2/2} \frac{d^k}{dq^k} e^{-q^2}$$

for $k = 0, 1, 2, \dots$, form a dense set of separately analytic vectors for $d\varpi(P^1)$. As these functions are also separately analytic vectors for $d\varpi(P_2)$,¹⁴ the operator algebra $d\varpi(P^2)$ is integrable to a unique representation ϖ of the universal cover $\widetilde{\text{HSp}}(2n, \mathbf{R})$ of the extended (or inhomogeneous) symplectic group $\text{HSp}(2n, \mathbf{R})$ (thereby justifying our notation “ $d\varpi$ ”).¹⁵ ϖ is known as the “extended metaplectic representation;” detailed discussions of it may be found in Refs. 10 and 13.

We call $d\varpi$ the “extended metaplectic quantization.” It has the following crucial property.

Proposition 1: The extended metaplectic quantization is the *unique* quantization of $\mathfrak{hsp}(2n, \mathbf{R})$ which exponentiates to a unitary representation of $\widetilde{\text{HSp}}(2n, \mathbf{R})$.

By “unique,” we mean up to unitary equivalence and restriction of representations.

Proof: Suppose \mathcal{Q} were another such quantization of $\mathfrak{hsp}(2n, \mathbf{R})$ on some domain D in a Hilbert space \mathcal{H} . Then $\mathcal{Q}(\mathfrak{hsp}(2n, \mathbf{R}))$ can be integrated to a representation τ of $\widetilde{\text{HSp}}(2n, \mathbf{R})$, and (Q4) implies that τ , when restricted to $H(2n) \subset \widetilde{\text{HSp}}(2n, \mathbf{R})$, is irreducible. The Stone–Von Neumann theorem then states that this representation of $H(2n)$ is unitarily equivalent to the Schrödinger representation, and hence $\tau = U\varpi U^{-1}$ for some unitary map $U: L^2(\mathbf{R}^n) \rightarrow \mathcal{H}$.¹⁶ Conse-

quently, $\mathcal{Q}(f) = \overline{Ud\overline{\omega}(f)U^{-1}}|_D$ for all $f \in \text{hsp}(2n, \mathbf{R})$. Arguing as in the discussion following Definition 1, we see that $U^{-1}\mathcal{Q}U$ is in fact the restriction of $d\overline{\omega}$ to $U^{-1}D \subseteq \mathcal{S}(\mathbf{R}^n)$. \square

The existence of an obstruction to quantization now follows from Theorem 2.

Theorem 2 (Weak No-Go Theorem): The extended metaplectic quantization of P^2 cannot be extended beyond P^2 in P .

Since P^2 is a maximal Lie subalgebra of P ,¹⁷ (Q1) implies that any quantization which extends $d\overline{\omega}$ must be defined on all of P . Thus we may restate this as: *There exists no quantization of P which reduces to the extended metaplectic quantization on P^2 .*

Proof: Let \mathcal{Q} be a quantization of P which extends the metaplectic quantization of P^2 . As noted previously, we may assume that the domain $D \subseteq \mathcal{S}(\mathbf{R}^n)$. We will show that a contradiction arises when cubic polynomials are considered.

Take $n = 1$. By inspection of (1)–(3) we see that the ‘‘Von Neumann rules’’

$$\mathcal{Q}(q^2) = \mathcal{Q}(q)^2, \quad \mathcal{Q}(p^2) = \mathcal{Q}(p)^2, \tag{4}$$

$$\mathcal{Q}(qp) = \frac{1}{2}(\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)), \tag{5}$$

hold. These in turn lead to higher degree Von Neumann rules.^{9,10}

Lemma 1: For all real-valued polynomials r ,

$$\mathcal{Q}(r(q)) = r(\mathcal{Q}(q)), \quad \mathcal{Q}(r(p)) = r(\mathcal{Q}(p)),$$

$$\mathcal{Q}(r(q)p) = \frac{1}{2}[r(\mathcal{Q}(q))\mathcal{Q}(p) + \mathcal{Q}(p)r(\mathcal{Q}(q))],$$

$$\mathcal{Q}(qr(p)) = \frac{1}{2}[\mathcal{Q}(q)r(\mathcal{Q}(p)) + r(\mathcal{Q}(p))\mathcal{Q}(q)].$$

Proof: We illustrate this for $r(q) = q^3$. The other rules follow similarly using induction. Now $\{q^3, q\} = 0$ whence by (Q1) we have $[\mathcal{Q}(q^3), \mathcal{Q}(q)] = 0$. Since also $[\mathcal{Q}(q)^3, \mathcal{Q}(q)] = 0$, we may write $\mathcal{Q}(q^3) = \mathcal{Q}(q)^3 + T$ for some operator T which (weakly) commutes with $\mathcal{Q}(q)$. We likewise have using (4)

$$[\mathcal{Q}(q^3), \mathcal{Q}(p)] = -i\hbar \mathcal{Q}(\{q^3, p\}) = 3i\hbar \mathcal{Q}(q^2) = 3i\hbar \mathcal{Q}(q^2) = [\mathcal{Q}(q)^3, \mathcal{Q}(p)]$$

from which we see that T commutes with $\mathcal{Q}(p)$ as well. Consequently, T also commutes with $\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)$. But then from (5),

$$\begin{aligned} \mathcal{Q}(q^3) &= \frac{1}{3}\mathcal{Q}(\{pq, q^3\}) = \frac{i}{3\hbar}[\mathcal{Q}(pq), \mathcal{Q}(q^3)] \\ &= \frac{i}{3\hbar} \left[\frac{1}{2}(\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)), \mathcal{Q}(q)^3 + T \right] \\ &= \frac{i}{6\hbar}[\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q), \mathcal{Q}(q)^3] = \mathcal{Q}(q)^3. \quad \nabla \end{aligned}$$

With this lemma in hand, it is now a simple matter to prove the no-go theorem. Consider the classical equality

$$\frac{1}{9}\{q^3, p^3\} = \frac{1}{3}\{q^2p, p^2q\}.$$

Quantizing and then simplifying this, the formulas in Lemma 1 give

$$\mathcal{Q}(q)^2\mathcal{Q}(p)^2 - 2i\hbar \mathcal{Q}(q)\mathcal{Q}(p) - \frac{2}{3}\hbar^2 I$$

for the lhs, and

$$\mathcal{Q}(q)^2 \mathcal{Q}(p)^2 - 2i\hbar \mathcal{Q}(q) \mathcal{Q}(p) - \frac{1}{3}\hbar^2 I$$

for the rhs, which is a contradiction. □

IV. THE STRONG NO-GO THEOREM

In Groenewold’s paper¹ a stronger result was claimed: His assertion was that there is no quantization of P , period. This is *not* what Theorem 2 states. For if \mathcal{Q} is a quantization of P , then while of course $\mathcal{Q}(P^1)$ must coincide with Schrödinger quantization, it is not obvious that \mathcal{Q} need be the extended metaplectic quantization when restricted to P^2 . Referring to Proposition 1, the problem is that $\mathcal{Q}(P^2)$ is not *a priori* integrable; (Q5) only guarantees that $\mathcal{Q}(P^1)$ can be integrated. From a different point of view, the problem lies in *deducing* the relations (2) and (3) or, equivalently, the Von Neumann rules (4) and (5) from the quantization axioms (Q1)–(Q5) and the properties of the extended symplectic algebra alone.

Van Hove² supplied an extra assumption guaranteeing the integrability of $\mathcal{Q}(P^2)$, which in particular implies: If the Hamiltonian vector fields of f, g are complete and $\{f, g\} = 0$, then $\mathcal{Q}(f)$ and $\mathcal{Q}(g)$ *strongly* commute.¹⁸ This assumption is used to derive relations (2) and (3) in Refs. 4 and 9. It is also possible to enforce the integrability of $\mathcal{Q}(P^2)$ in a more direct manner.³

We now show that Van Hove’s assumption is unnecessary; in fact, we may establish the integrability of $\mathcal{Q}(P^2)$ directly, via the following generalization of Proposition 1.

Proposition 3: Let \mathcal{Q} be a quantization of P^2 on a dense invariant domain D in a Hilbert space \mathcal{H} . Then there is a unitary transformation $U: L^2(\mathbf{R}^n) \rightarrow \mathcal{H}$ such that $\mathcal{Q}(f) = Ud\varpi(f)U^{-1}|_D$ for all $f \in P^2$.

Thus, up to unitary equivalence, \mathcal{Q} must be either $d\varpi$ or a restriction thereof. As such, $\mathcal{Q}(P^2)$ must be integrable and, consequently, Van Hove’s strong commutativity assumption holds for elements of P^2 .

Before giving the proof, we establish two technical lemmas.

Lemma 2: Let A be an e.s.a. operator on a Hilbert space, and B a closable operator, both of which have a common dense invariant domain D . Suppose that D consists of analytic vectors for A , and that A (weakly) commutes with B . Then $\exp(i\bar{A})$ (weakly) commutes with \bar{B} on D .

Proof: Recall that as $\psi \in D$ is analytic for A ,

$$e^{i\bar{A}}\psi = \sum_{k=0}^{\infty} \frac{1}{k!} (iA)^k \psi =: \phi.$$

Define the partial sums

$$\phi_K = \sum_{k=0}^K \frac{1}{k!} (iA)^k \psi \in D;$$

then using the (weak) commutativity of A and B ,

$$B\phi_K = \sum_{k=0}^K \frac{1}{k!} (iA)^k B\psi \in D.$$

Since $B\psi \in D$ is analytic for A , the sequence $B\phi_K$ converges:

$$\chi := \lim_{K \rightarrow \infty} B\phi_K = e^{i\bar{A}}B\psi = e^{i\bar{A}}\bar{B}\psi.$$

But \bar{B} is closed, hence $\phi = \lim_{K \rightarrow \infty} \phi_K$ is in the domain of \bar{B} and $\chi = \bar{B}\phi$, i.e.,

$$e^{i\bar{A}}\bar{B}\psi = \bar{B}e^{i\bar{A}}\psi$$

for all $\psi \in D$. ▽

Lemma 3: Let B be a closable operator. If a bounded operator T (weakly) commutes with \bar{B} on $D(B)$, then they also commute on $D(\bar{B})$.

Proof: If $\psi \in D(\bar{B})$, then from the definition of closure there exists a sequence $\{\psi_k\}$ in $D(B)$ with $\psi_k \rightarrow \psi$ such that $B\psi_k \rightarrow \bar{B}\psi$. Because the operator T is continuous,

$$T\bar{B}\psi = T \lim_{k \rightarrow \infty} B\psi_k = \lim_{k \rightarrow \infty} TB\psi_k = \lim_{k \rightarrow \infty} \bar{B}T\psi_k$$

as T commutes with \bar{B} on $D(B)$. Again applying the definition of closure to the sequence $\{T\psi_k\}$ in $D(\bar{B})$, we get that $\lim_{k \rightarrow \infty} T\psi_k = T\psi \in D(\bar{B})$ and

$$\bar{B}T\psi = \lim_{k \rightarrow \infty} \bar{B}T\psi_k = T\bar{B}\psi$$

for every $\psi \in D(\bar{B})$. ▽

Proof of Proposition 3: Let \mathcal{Q} be a quantization of P^2 . As discussed earlier, we may assume that $\mathcal{Q}(P^1)$ is the Schrödinger representation (1) on $L^2(\mathbf{R}^n)$, and that the domain $D \subseteq \mathcal{S}(\mathbf{R}^n)$. Again taking $n = 1$, we will prove by brute force that the Von Neumann rules (4) and (5) hold.

We begin by determining $\mathcal{Q}(q^2)$. Set $\Delta = \mathcal{Q}(q^2) - \mathcal{Q}(q)^2$. We readily verify that $[\Delta, \mathcal{Q}(q)] = 0$ and $[\Delta, \mathcal{Q}(p)] = 0$ on D . Now let $D_\omega \subseteq D$ be the space of separately analytic vectors for $\mathcal{Q}(q)$ and $\mathcal{Q}(p)$; by (Q5) we have that D_ω is dense. According to Proposition 1 of Ref. 6, Δ leaves D_ω invariant. By Corollary 2 in Sec. X.6 of Ref. 7, $\mathcal{Q}(q)|_{D_\omega}$ is e.s.a.; moreover, $\Delta_\omega := \Delta|_{D_\omega}$ is symmetric and hence closable. Upon taking $A = \mathcal{Q}(q)|_{D_\omega}$ and $B = \Delta_\omega$ in Lemma 2, it follows that $\exp(i\mathcal{Q}(q)|_{D_\omega}) = \exp(i\mathcal{Q}(q))$ and Δ_ω commute on D_ω . Lemma 3 then shows that $\exp(i\mathcal{Q}(q))$ and Δ_ω commute on $D(\Delta_\omega)$. Likewise $\exp(i\mathcal{Q}(p))$ and Δ_ω commute on $D(\Delta_\omega)$. But now the unbounded version of Schur's lemma¹⁹ implies that $\Delta_\omega = EI$ for some real constant E on $D(\Delta_\omega) = L^2(\mathbf{R})$. Since $\bar{\Delta}_\omega$ is the smallest closed extension of Δ_ω and $\Delta_\omega \subset \Delta \subset \bar{\Delta}$, it follows that $\bar{\Delta} = EI$, whence Δ itself is a constant multiple of the identity on D . Thus $\mathcal{Q}(q^2) = \mathcal{Q}(q)^2 + EI$ on D .

An identical argument yields $\mathcal{Q}(p^2) = \mathcal{Q}(p)^2 + FI$ on D . Quantizing the relation $4pq = \{p^2, q^2\}$ and using these formulas then gives

$$\mathcal{Q}(pq) = \frac{1}{2}(\mathcal{Q}(p)\mathcal{Q}(q) + \mathcal{Q}(q)\mathcal{Q}(p))$$

on D . But upon quantizing $2q^2 = \{pq, q^2\}$ we find that $E = 0$. Similarly $F = 0$. It follows from (1)–(3) that $\mathcal{Q} = d\varpi|_D$. □

Thus, up to unitary equivalence and restriction of representations, we may as well suppose that $D = \mathcal{S}(\mathbf{R}^n)$. If we were to take this as our starting point, then we could reverse our constructions and derive (4) and (5) in a simpler fashion, cf. Sec. 5.1 of Ref. 5.

If \mathcal{Q} were a quantization of P , $\mathcal{Q}(P^2)$ must therefore be unitarily equivalent to (a restriction of) the extended metaplectic quantization, and this contradicts Theorem 2. Thus we have proven our main result.

Theorem 4 (Strong No-Go Theorem): There is no quantization of P .

Van Hove² gave a slightly different analysis using only those observables $f \in C^\infty(\mathbf{R}^{2n})$ with complete Hamiltonian vector fields, and still obtained an obstruction [but now to quantizing all of $C^\infty(\mathbf{R}^{2n})$]. Yet other variants of Groenewold's theorem are presented in Refs. 12 and 20. Related results can be found in Refs. 21 and 22.

V. QUANTIZABLE LIE SUBALGEBRAS OF POLYNOMIALS

We hasten to add that there are Lie subalgebras of $P(2n)$ other than $P^2(2n)$ which can be quantized. For example, consider the coordinate subalgebra $C(2n)$. It is straightforward to verify that for each $\eta \in \mathbf{R}$, the map $\sigma_\eta: C(2n) \rightarrow \text{Op}(\mathcal{S}(\mathbf{R}^n))$ given by

$$\sigma_\eta \left(\sum_{i=1}^n f^i(q) p_i + g(q) \right) = -i\hbar \sum_{i=1}^n \left(f^i(q) \frac{\partial}{\partial q^i} + \left[\frac{1}{2} + i\eta \right] \frac{\partial f^i(q)}{\partial q^i} \right) + g(q) \tag{6}$$

is a quantization of $C(2n)$. σ_0 is the familiar ‘‘position’’ or ‘‘coordinate representation.’’ The significance of the parameter η is explained in Ref. 23 and 24 (see also Ref. 25). There it is shown that while the quantizations σ_η and $\sigma_{\eta'}$ are unitarily inequivalent if $\eta \neq \eta'$, they are related by a *nonlinear* norm-preserving isomorphism.

Proposition 5: C is a maximal Lie subalgebra of P .

Proof: We take $n = 1$. Suppose that V were a Lie subalgebra of P strictly containing C . V must contain a polynomial h of the form

$$h(q, p) = f(q)p^k + \text{terms of degree at most } k-1 \text{ in } p$$

for some $k > 1$ and some polynomial $f \neq 0$ of degree l . Now both $q, p \in V$, and so by bracketing h with q ($k-2$)-times, we get

$$\frac{k!}{2} f(q)p^2 + \text{terms of degree at most degree 1 in } p \in V.$$

Since $C \subset V$ this implies that $f(q)p^2 \in V$. By bracketing this expression with p l -times, we conclude that $p^2 \in V$. Now both $q^2, qp \in V$, so $P^2 \subset V$. The maximality of P^2 implies that $V = P$, whence C is maximal. \square

As a consequence, any quantization which extends σ_η must be defined on all of P . Thus Theorem 4 yields

Corollary 6: The quantizations σ_η of C cannot be extended beyond C in P .

Furthermore a variant of Proposition 3 (see also Theorem 8 in Ref. 25) yields ‘‘uniqueness:’’

Proposition 7: Let \mathcal{Q} be a quantization of C on a dense invariant domain D in a Hilbert space \mathcal{H} . Then there is an $\eta \in \mathbf{R}$ and a unitary transformation $U : L^2(\mathbf{R}^n) \rightarrow \mathcal{H}$ such that $\mathcal{Q}(f) = U\sigma_\eta(f)U^{-1}|_D$ for all $f \in C$.

Proof: Again set $n = 1$. As in the proof of Proposition 3, we may assume that $\mathcal{Q}(P^1)$ is given by (1) on $L^2(\mathbf{R})$ and that $D \subseteq \mathcal{S}(\mathbf{R})$.

Just as before, we first compute that $\mathcal{Q}(q^2) = \mathcal{Q}(q)^2 + EI$ on D for some real constant E .

Now consider $\mathcal{Q}(q, p)$. Set

$$\Delta = \mathcal{Q}(qp) - \frac{1}{2}(\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)).$$

It is straightforward to verify that Δ commutes with both $\mathcal{Q}(q)$ and $\mathcal{Q}(p)$. The same argument based on Lemmas 2 and 3 and the unbounded Schur’s lemma that was used in the proof of Proposition 3 can be applied *mutatis mutandis* to give $\Delta = GI$ on D for some real constant G . Thus

$$\mathcal{Q}(qp) = \frac{1}{2}(\mathcal{Q}(q)\mathcal{Q}(p) + \mathcal{Q}(p)\mathcal{Q}(q)) + GI \tag{7}$$

on D . By quantizing the Poisson bracket relation $\{qp, q^2\} = 2q^2$ we find that $E = 0$. Arguing as in the proof of Lemma 1, we then find that on D

$$\mathcal{Q}(q^k) = \mathcal{Q}(q)^k. \tag{8}$$

Next, quantizing the Poisson bracket relations $\{q^k p, q\} = q^k$ and $\{q^k p, p\} = -kq^{k-1}p$ yields

$$[\mathcal{Q}(q^k p), \mathcal{Q}(q)] = -i\hbar \mathcal{Q}(q^k) \quad \text{and} \quad [\mathcal{Q}(q^k p), \mathcal{Q}(p)] = i\hbar k \mathcal{Q}(q^{k-1} p), \tag{9}$$

respectively. Now consider the classical relation $(1-k)q^k p = \{q^k p, qp\}$. Quantizing this and simplifying by means of (7), (9), and (8) produces the recursion relation

$$\mathcal{Q}(q^k p) = \frac{1}{1-k} (\mathcal{Q}(q^k) \mathcal{Q}(p) - k \mathcal{Q}(q) \mathcal{Q}(q^{k-1} p)).$$

Iterating this computation $(k-1)$ -times gives

$$\mathcal{Q}(q^k p) = (1-k) \mathcal{Q}(q^k) \mathcal{Q}(p) + k \mathcal{Q}(q)^{k-1} \mathcal{Q}(qp).$$

Again using (7) and simplifying, we finally get

$$\mathcal{Q}(q^k p) = \mathcal{Q}(q^k) \mathcal{Q}(p) + k \left(G - \frac{i\hbar}{2} \right) \mathcal{Q}(q)^{k-1}.$$

Recalling (1) and (8), this can be rewritten

$$\mathcal{Q}(q^k p) = -i\hbar \left[q^k \frac{d}{dq} + \left(\frac{1}{2} + \frac{iG}{\hbar} \right) \frac{dq^k}{dq} \right]$$

on D . Consolidating this with (8), we obtain (6) where $\eta = G/\hbar$. We thus have $\mathcal{Q} = \sigma_\eta \upharpoonright D$, as claimed. \square

Notice that unlike in the proof of Proposition 3, we cannot quantize the Poisson bracket relation $\{q^2, p^2\} = -4qp$ to obtain $G=0$ since $p^2 \notin C$. The fact that G remains arbitrary is mirrored by the presence of the parameter η in (6).

Thus far we have encountered two maximal Lie subalgebras of P containing P^1 : P^2 and C . When $n=1$, it turns out that these are essentially the *only* such subalgebras.

Theorem 8: ($n=1$) Up to isomorphism, P^2 and C are the only maximal Lie subalgebras of P which contain P^1 .

Proof: Suppose that W were a maximal Lie subalgebra of P containing P^1 , distinct from P^2 . We will show that W must be isomorphic to C . Denote $W^k = W \cap P^k$, etc.

Since $W \neq P^2$ there must exist a polynomial of degree $k, k > 2$, in W . By bracketing this polynomial $(k-2)$ times with an appropriate number of $p, q \in W$, we obtain a nonzero polynomial $h \in W^2$. Since $P^1 \subset W$, we may subtract off terms of degree one or less, so we may assume that h is homogeneous quadratic. By means of a rotation we may diagonalize h ; thus we may further suppose that canonical coordinates have been chosen so that $h(q, p) = ap^2 + cq^2$. Now $\dim W_2 \neq 3$, for otherwise $P^2 \subset W$, and then the maximality of P^2 implies that $W = P$. We break the argument into parts, depending on whether $\dim W_2 = 1$ or 2 .

(i) $\dim W_2 = 1$: Then W_2 is spanned by h . We first claim that either $W^3 = W^2$ or $W^3 \subset C^3$. Indeed, if $f \in W^3$, then the quadratic terms of both $\{p, f\}, \{f, q\} \in W^2$ must be proportional to h : $\{p, f\} = rh + \text{l.d.t.}$ and $\{f, q\} = sh + \text{l.d.t.}$, where ‘‘l.d.t.’’ means lower degree terms. The particular form of h then implies that

$$f(q, p) = \frac{1}{3}(sap^3 + rcq^3) + \text{l.d.t.},$$

along with $sc = 0$ and $ra = 0$. Since $h \neq 0$, both a, c cannot vanish. If both $r, s = 0$, then $f \in W^2$ and so $W^3 = W^2$. If both $s, a = 0$, then h is proportional to q^2 and f must be of the form

$$\frac{1}{3}rcq^3 + \alpha q^2 + \beta qp + \gamma p^2 + \text{l.d.t.}$$

But then $\{f, h\} \propto 2\beta q^2 + 4\gamma qp \in W^2$, which forces $\gamma = 0$. Thus $W^3 \subset C^3$. The canonical transformation $q \mapsto p, p \mapsto -q$ reduces the subcase with $r, c = 0$ to the previous one.

If $W^3 = W^2$ then $W = W^2 \subset P^2$, which contradicts the assumed maximality of W .

If $W^3 \subset C^3$, then a similar argument shows that $W^4 \subset C^4$, and so on. Thus $W \subset C$, which again contradicts the maximality of W .

(ii) $\dim W_2 = 2$: Now we may suppose that h, g form a basis for W_2 , where h is as above and

$$g(q, p) = rp^2 + spq + tq^2.$$

If $s=0$ then, as h, g are linearly independent, both $p^2, q^2 \in W_2$. But then $\{p^2, q^2\} = 4pq \in W$, so that $\dim W_2 = 3$. Without loss of generality, we may thus assume that $s=1$.

Now $\{h, g\} \in W_2$, and a computation shows that $h, g, \{h, g\}$ are linearly dependent iff

$$ac + (at - cr)^2 = 0. \tag{10}$$

Again we consider various subcases. If $a=0$ then (10) gives $r=0$, and it follows from the above expressions for h, g that $W_2 = C_2$. As in case (i), the subcase $c=0$ can be reduced to that of $a=0$ by means of a linear canonical transformation. It remains to consider the subcase $ac \neq 0$. We may suppose that $a=1$; (10) then implies that $c < 0$. Setting $\beta = t - rc$, we may thus take

$$h = p^2 - \beta^2 q^2 \quad \text{and} \quad \{g, h\} = 2(p^2 + 2\beta pq + \beta^2 q^2).$$

as a basis for W_2 . But now the canonical transformation

$$p \mapsto \frac{1}{\sqrt{2\beta}}(p - \beta q), \quad q \mapsto \frac{1}{\sqrt{2}}(p + \beta q)$$

reduces this subcase to that of $a=0$. Thus up to isomorphism we have $W_2 = C_2$.

Similarly we have $W_k \subseteq C_k$, and so $W \subseteq C$. The maximality of W now implies that $W = C$. \square

In particular, the subalgebras $\{f(\mu p + q)(p - \mu q) + g(\mu p + q)\}$, where f, g are polynomials and $\mu \in \mathbf{R}$, are all maximal Lie subalgebras of $P(2)$ containing $P^1(2)$ isomorphic to $C(2)$. [These are the normalizers in $P(2)$ of the polarizations $\{g(\mu p + q)\}$.] So is the ‘‘momentum algebra’’ consisting of polynomials which are at most affine in the position.

As both $P^2(2)$ and $C(2)$ are quantizable, it follows from Theorem 2 and Corollary 6 that the following corollary is true.

Corollary 9: Up to isomorphism, $P^2(2)$ and $C(2)$ are the largest quantizable subalgebras of $P(2)$ containing $P^1(2)$.

Unfortunately, neither Theorem 8 nor Corollary 9 hold in higher dimensions. To see this, take $n=2$ and consider the Lie algebra

$$\{f(q^1)p_1 + g(q^1, q^2, p_2)\},$$

where f, g are polynomials. This subalgebra is maximal, but not isomorphic to either $C(4)$ or $P^2(4)$. It is also not quantizable—if it were, we would obtain a quantization of the polynomial algebra in q^2, p_2 , contrary to the Strong No-Go Theorem. Furthermore, the subalgebra thereof for which g is at most quadratic in q^2, p_2 is maximal quantizable, but also not isomorphic to either $C(4)$ or $P^2(4)$.

VI. DISCUSSION

We have thus completely solved the Groenewold–Van Hove problem for \mathbf{R}^2 in that we have identified (the isomorphism classes of) the largest quantizable Lie subalgebras of $P(2)$ [viz. $P^2(2)$ and $C(2)$] and explicitly constructed all their possible quantizations [given by (1)–(3) and (6), respectively]. It remains to carry out this program in higher dimensions; the key missing ingredient is a classification of the maximal Lie subalgebras of $P(2n)$ containing $P^1(2n)$. Unfortunately, this appears to be a difficult problem. We emphasize, however, that all the results of this paper other than Theorem 8 and Corollary 9 hold for arbitrary n .

Of course, Groenewold’s classical result is valid only for \mathbf{R}^{2n} . Similar obstructions appear when trying to quantize certain other phase spaces, e.g., S^2 and T^*S^1 . Complete solutions of the corresponding Groenewold–Van Hove problems in these two examples are given in Refs. 26 and 25, respectively. On the other hand, in some instances there are *no* obstructions to quantization, such as T^2 and $T^*\mathbf{R}_+$, cf. Refs. 27 and 12, respectively. (Although probably not of physical interest, it is amusing to wonder what happens for \mathbf{R}^{2n} , $n > 1$, with an exotic symplectic structure.)

It is important, therefore, to understand the mechanisms which are responsible for these divergent outcomes. Already some results have been established along these lines, to the effect that under certain circumstances there are obstructions to quantizing both compact and noncompact phase spaces.^{12,28–30} We refer the reader to Ref. 5 for an up-to-date summary.

ACKNOWLEDGMENTS

We wish to thank H. Grundling for valuable discussions, including his help in pointing out and correcting an error in a previous version of this manuscript. This research was supported in part by NSF Grant No. DMS 96-23083.

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- ¹⁶See Ref. 10, Prop. 4.58.
- ¹⁷See Ref. 13, Sec. 16.
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