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We show that zero level sets of momentum mappings for cotangent actions are coisotropic, even when the momentum mapping is singular. The proof applies to both the finite- and infinite-dimensional cases. We use this result to show that vacuum Yang-Mills theory is indeed first class in the sense of Dirac.

1. INTRODUCTION

Zero level sets of momentum mappings associated to symplectic group actions play a number of important roles in mathematics and physics.[1] For example, in classical field theory such zero levels $J^{-1}(0)$ can often be identified with "constraint sets," that is, spaces of admissible initial data for the evolution equations.[2] In this context J is the momentum map for the gauge group of the system. Other applications are to geometric invariant theory[3] and the motion of a particle in a Yang-Mills field.[4] Moreover, a standard procedure (cf. §4.A of [5]) effectively reduces the case of nonzero level sets to that of zero levels of momentum maps on larger spaces.

When 0 is a (weakly) regular value of an *Ad**-equivariant momentum map J, it follows from Lemma 4.3.2 of [6] that $J^{-1}(0)$ is a coisotropic submanifold of the ambient symplectic space. But in many important applications, such as gravity and Yang-Mills theory, 0 is *not* weakly regular. And when 0 is not weakly regular, $J^{-1}(0)$ need *not* be coisotropic. (For examples, see [7].) This is somewhat disconcerting, since one is accustomed to thinking of such systems as being purely first class in the sense of Dirac.[2,7] The problem is that while the components of the momentum map J in these theories are certainly first class constraints, it does *not* necessarily follow that *every* constraint is first class. In fact, verifying the latter is often a difficult problem in C^∞ real algebraic geometry.[7] To our knowledge, it has never been rigorously proven that either gravity or Yang-Mills theory is strictly first class.

Here we show that zero level sets of momentum maps associated to *cotangent* actions are *always* coisotropic, even when $J^{-1}(0)$ is not a manifold. (In infinite dimensions, some mild technical assumptions are required.) Yang-Mills theory in particular is covered by this result; we explicitly work out the details in §4. The fact that the Yang-Mills constraint set is coisotropic -- and hence first class -- has important

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consequences, several of which are discussed in [2] and [8].

2. SOME LINEAR ALGEBRA

Let V be a topological vector space and let V* be a dual of V. In the case that V is infinite-dimensional, V* may be only a "partial" dual. It must be large enough to separate points of V, but it may not include all linear functionals on V. (For instance, in the Yang-Mills example below, V and V* are C^∞ tensor spaces dual via contraction and integration; the full continuous dual would include distributions.)

For any subset S of V, define the *annihilator* of S in V* to be

$$S^{\vdash} = \{p \in V^* \mid p(q) = 0 \text{ for all } q \in S\}.$$

Similarly for any subset $S^* \subseteq V^*$, define

$$S^{*\vdash} = \{q \in V \mid p(q) = 0 \text{ for all } p \in S^*\}.$$

Note that if S is a subspace of a *finite*-dimensional vector space V, then the double annihilator $S^{\vdash\vdash} = S$. This may not be true more generally, however, even when S is closed in V. (The reason is that V* may only be a partial dual; the result would follow if, e.g., V were locally convex and V* were the full dual.)

Now suppose that (U,Ω) is a symplectic vector space. The *symplectic polar* of a subspace W in U is

$$W^{\perp} = \{u \in U \mid \Omega(u,w) = 0 \text{ for all } w \in W\}.$$

W is *coisotropic* provided $W^{\perp} \subseteq W$ and *Lagrangian* if $W^{\perp} = W$.

In particular, consider $V \oplus V^*$ with its canonical (weak) symplectic form

$$\Omega(q \oplus p, q' \oplus p') = p'(q) - p(q').$$

Let $W \subseteq V$ be a subspace. The main technical result that we require is contained in the following

<u>LEMMA</u>: $W \oplus W^{\vdash} \subset V \oplus V^*$ *is Lagrangian if and only if* $W^{\vdash\vdash} = W$.

<u>Proof</u>: First assume that $W^{\vdash\vdash} = W$. Then we must show that

$$(W \oplus W^{\vdash})^{\perp} = W \oplus W^{\vdash}$$

in $V \oplus V^*$. Fix $q \oplus p \in V \oplus V^*$ and let $q' \oplus p' \in W \oplus W^{\vdash}$. Setting $q' = 0$ and $p' = 0$ in turn, it is clear from the expression above for Ω that

$$\Omega(q \oplus p, q' \oplus p') = 0$$

if and only if $p'(q) = 0$ and $p(q') = 0$ for all $p' \in W^{\vdash}$ and $q' \in W$. Equivalently, $q \in W^{\vdash\vdash} = W$ and $p \in W^{\vdash}$. Thus $q \oplus p \in (W \oplus W^{\vdash})^{\perp}$ iff $q \oplus p \in W \oplus W^{\vdash}$.

Conversely, suppose that $q \in W^{\vdash\vdash}$ with $q \notin W$. Then $q \oplus 0 \in (W \oplus W^{\vdash})^{\perp}$ but $q \oplus 0 \notin W \oplus W^{\vdash}$. ■

Thus when V is finite-dimensional, $W \oplus W^{\vdash} \subset V \oplus V^*$ is always Lagrangian.

Now let $\mathcal{M}$ be a manifold modelled on V. (If V* is a partial dual, the atlas on $\mathcal{M}$ must be chosen so that the transitions between overlapping charts preserve V* in the full dual.) Then $T^*\mathcal{M}$ is symplectically modelled on $(V \oplus V^*, \Omega)$. If $\mathcal{N}$ is a submanifold of $\mathcal{M}$, define

$$T\mathcal{N}^{\vdash} = \coprod_{n \in \mathcal{N}} T_n\mathcal{N}^{\vdash};$$

then $T\mathcal{N}^{\vdash}$ is a subbundle of $T^*\mathcal{M}|\mathcal{N}$. Note that if $\mathcal{N}$ is modelled on the (closed) subspace $\mathbf{W}$ of $\mathbf{V}$, then $T\mathcal{N}^{\vdash}$ is modelled on $\mathbf{W} \oplus \mathbf{W}^{\vdash} \subset \mathbf{V} \oplus \mathbf{V}^*$. Thus if $T\mathcal{N}^{\vdash\vdash} = T\mathcal{N}$ in $T\mathcal{M}|\mathcal{N}$, the Lemma implies that $T\mathcal{N}^{\vdash}$ is a Lagrangian submanifold of $T^*\mathcal{M}$ (that is, $[T_\alpha(T\mathcal{N}^{\vdash})]^{\perp} = T_\alpha(T\mathcal{N}^{\vdash})$ for all $\alpha \in T\mathcal{N}^{\vdash}$).

3. ZERO LEVELS FOR COTANGENT ACTIONS

Let $\mathcal{G}$ be a Lie group, and let Φ: $\mathcal{G} \times \mathcal{M} \to \mathcal{M}$ be a smooth action of $\mathcal{G}$ on a manifold $\mathcal{M}$. Recall that, in finite dimensions, the orbit $\mathcal{O}_m$ of m is locally an embedded submanifold. That is, for each $m \in \mathcal{M}$, there exists an open set $\mathcal{U}$ containing the identity in $\mathcal{G}$ such that $\Phi_m(\mathcal{U}) \subseteq \mathcal{O}_m$ is an embedded submanifold of $\mathcal{M}$ (cf. Corollary 4.1.22 of [6]). However, we allow both $\mathcal{G}$ and $\mathcal{M}$ to be infinite-dimensional, in which case this characterization of the orbit must be explicitly checked.

For each $\xi \in \mathfrak{g}$, the Lie algebra of $\mathcal{G}$, let $\xi_{\mathcal{M}}$ be the vector field on $\mathcal{M}$ which generates the action of the one-parameter subgroup determined by ξ.

The action Φ lifts to an action of $\mathcal{G}$ on $T^*\mathcal{M}$ by pullback. This *cotangent action* has a canonical Ad^*-equivariant momentum map J: $T^*\mathcal{M} \to \mathfrak{g}^*$ given by

$$J(\alpha)\cdot\xi = \alpha(\xi_{\mathcal{M}}(\pi(\alpha)))$$

for $\xi \in \mathfrak{g}$, where π: $T^*\mathcal{M} \to \mathcal{M}$ is the projection. See Chapter 4 of [6] for background and details on momentum maps.

Now consider the zero level $J^{-1}(0)$. In many interesting situations this fails to be a manifold. Nevertheless, at each point $\alpha \in J^{-1}(0)$ we may define the tangent space $T_\alpha(J^{-1}(0))$ to be the linear span of the (one-sided) tangent vectors at α to smooth curves in $\mathcal{M}$ lying in $J^{-1}(0)$. (See §2 of [7] for further discussion.)

The main result is:

<u>THEOREM 1</u>: *Let J be the canonical momentum map for the cotangent action of a Lie group $\mathcal{G}$ on $T^*\mathcal{M}$. For every $m \in \mathcal{M}$ suppose that* (i) *the orbit $\mathcal{O}_m$ is locally an embedded submanifold of $\mathcal{M}$, and that* (ii) $(T_m\mathcal{O}_m)^{\vdash\vdash} = T_m\mathcal{O}_m$. *Then the zero level $J^{-1}(0)$ is coisotropic.*

<u>Proof</u>: It suffices to find at each $\alpha \in J^{-1}(0)$ a Lagrangian submanifold $\mathcal{L}$ of $T^*\mathcal{M}$ with $\alpha \in \mathcal{L} \subseteq J^{-1}(0)$. For then

$$[T_\alpha(J^{-1}(0))]^{\perp} \subseteq [T_\alpha\mathcal{L}]^{\perp}$$

$$= T_\alpha\mathcal{L} \subseteq T_\alpha(J^{-1}(0)).$$

For any $\alpha \in J^{-1}(0)$, let $m = \pi(\alpha)$. By (i) there exists a neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$ such that $\mathcal{N} := \Phi_m(\mathcal{U}) \subseteq \mathcal{O}_m$ is an embedded submanifold of $\mathcal{M}$ containing m. Set $\mathcal{L} = T\mathcal{N}^{\vdash} \subset T^*\mathcal{M}$. By (ii) and the remarks following the Lemma, $\mathcal{L}$ is Lagrangian.

It remains to show that $\alpha \in \mathcal{L}$ with $\mathcal{L} \subseteq J^{-1}(0)$. Since $T_n\mathcal{N} = \{\xi_{\mathcal{M}}(n) \mid \xi \in \mathfrak{g}\}$ we have that, for any $\beta \in \pi^{-1}(\mathcal{N})$,

$$\beta \in T\mathcal{N}^{\vdash}$$

$$\Leftrightarrow \beta(\xi_{\mathcal{M}}(\pi(\beta))) = 0 \text{ for all } \xi \in \mathfrak{g}$$

$$\Leftrightarrow J(\beta) = 0.$$

Thus $\alpha \in T\mathcal{N}^{\vdash} = \mathcal{L} \subseteq J^{-1}(0)$. ■

<u>COROLLARY 1</u>: *If the model space of $\mathcal{M}$ is reflexive, then $J^{-1}(0)$ is a first class variety in the sense of Dirac.*[2,7]

In other words, the ideal $I(J^{-1}(0))$

in $C^\infty(T^*\mathcal{M})$ generated by the zero level set consists *entirely* of first class constraints.

Proof: Since $\mathcal{M}$ is modelled on a reflexive space, the standard symplectic structure on $T^*\mathcal{M}$ is strongly nondegenerate. Now apply Proposition 2.2 of [7], which states that a coisotropic subvariety of a strong symplectic manifold is first class. ■

Remarks: (1) In finite dimensions, the hypotheses of both Theorem 1 and Corollary 1 are always satisfied. In this circumstance, then, $J^{-1}(0)$ *is necessarily coisotropic as well as first class.*

(2) In any case, these assumptions usually are straightforward to verify in practice (see, e.g., the details for Yang-Mills theory below).

(3) We emphasize that the conclusion of Theorem 1 may *not* hold for more general actions, even in finite dimensions.[7] This shows that cotangent actions are really rather special.

4. YANG-MILLS FIELDS

We now use these results to show that the vacuum Yang-Mills constraint set is coisotropic and thus first class. This requires setting up a certain amount of machinery; we refer the reader to [9] and [10] for more detailed discussions. Throughout, we work in the C^∞ category.

In the (3+1)-Hamiltonian formulation, the gauge field evolves on a Riemannian 3-manifold X. For technical reasons we assume that X is compact. (Asymptotically Euclidean may also suffice provided suitable fall-off conditions are placed on all fields and gauge transformations.)

The configuration space for the Yang-Mills system is the bundle $\mathcal{A}$ of smooth connections on a fixed principal G-bundle P over X, where G is a Lie group whose Lie algebra $\mathfrak{g}$ carries an adjoint-invariant inner product. The phase space is the "L^2"-cotangent bundle $T^*\mathcal{A}$ (described below) with its canonical symplectic structure.

Let $ad\ P = P \times_{Ad} \mathfrak{g}$ be the adjoint bundle, and let Λ^k be the bundle of k-forms on X. Then $T_\alpha\mathcal{A} \approx \Gamma(\Lambda^1 \otimes ad\ P)$, so that locally $\mathbf{a} \in T_\alpha\mathcal{A}$ is just a $\mathfrak{g}$-valued 1-form on X. Similarly $T_\alpha{}^*\mathcal{A} \approx \Gamma(\Lambda^2 \otimes ad^*P)$, where $ad^*P = P \times_{Ad*} \mathfrak{g}^*$ is the coadjoint bundle. The pairing between $\mathbf{a} \in T_\alpha\mathcal{A}$ and its conjugate "electric" field $\mathbf{e} \in T_\alpha{}^*\mathcal{A}$ is given by

$$\mathbf{e}(\mathbf{a}) = \int_X \mathbf{e} \wedge \mathbf{a}.$$

In a formula such as this, a contraction on Lie algebra indicies is implicitly assumed.

We observe that each of the spaces $\Gamma(\Lambda^k \otimes ad\ P)$ carries a natural L^2-inner product defined by the expression

$$\int_X \star\mathbf{a} \wedge \mathbf{b}.$$

Here $\star\colon \Gamma(\Lambda^k \otimes ad\ P) \to \Gamma(\Lambda^{3-k} \otimes ad^*P)$ is the Hodge star operator with respect to the Riemannian metric on X, with the added twist of interchanging $\mathfrak{g}$- and $\mathfrak{g}^*$-indices using the adjoint-invariant inner product on $\mathfrak{g}$.

The Lie group $\mathcal{G}$ of automorphisms of P which cover the identity on X can be viewed as smooth sections φ of $P \times_{Ad} G$. The Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ can then be identified with $\Gamma(ad\ P)$, and its L^2-dual $\mathfrak{g}^*$ with $\Gamma(\Lambda^3 \otimes ad^*P)$, where again the pairing is given by contraction and integration. Now $\mathcal{G}$ acts on $\mathcal{A}$ by gauge transformations:

$$\Phi(\varphi, a) = \varphi^{-1} a \varphi + \varphi^{-1} d\varphi$$

where G is represented as a matrix group on g. The lifted action on $T^*\mathcal{A}$ has momentum map $J\colon T^*\mathcal{A} \to \Gamma(\Lambda^3 \otimes ad^*P)$ given by

$$J(a, e) = D_a e.$$

Here $D_a\colon \Gamma(\Lambda^2 \otimes ad^*P) \to \Gamma(\Lambda^3 \otimes ad^*P)$ is defined by

$$D_a e \cdot \xi = de \cdot \xi - e \wedge [a, \xi]$$

for $\xi \in \Gamma(ad\ P)$. The condition $J = 0$ arising from the gauge invariance of the theory is an *initial value constraint.* That is, only those pairs $(a, e) \in T^*\mathcal{A}$ such that $D_a e = 0$ constitute (formally) admissible initial data for the Yang-Mills equations.

THEOREM 2: *The constraint set $J^{-1}(0)$ for vacuum Yang-Mills theory is coisotropic.*

Proof: Let $\mathcal{O}_a$ be the orbit of $a \in \mathcal{A}$ under $\mathcal{G}$. Using elliptic theory, it can be shown that $\mathcal{O}_a$ is in fact an embedded submanifold of $\mathcal{A}$.[11] Thus the first hypothesis of Theorem 1 is satisfied.

Next we verify hypothesis (ii). From the expression above for the action Φ it follows that

$$T_a\mathcal{O}_a = d_a(\Gamma(ad\ P)) = Im\ d_a,$$

where $d_a\colon \Gamma(ad\ P) \to \Gamma(\Lambda^1 \otimes ad\ P)$ takes the form

$$d_a\xi = d\xi + [a, \xi].$$

In other words, a linear gauge perturbation of a connection $a \in \mathcal{A}$ is locally the gauge covariant differential of a g-valued function on X. The annihilator $(T_a\mathcal{O}_a)^{\vdash}$ is

$$\left\{ e \in T_a{}^*\mathcal{A} \mid \int_X e \wedge d_a\xi = 0 \text{ for all } \xi \in \Gamma(ad\ P) \right\}.$$

An integration by parts gives

$$\int_X e \wedge d_a\xi = -\int_X D_a e \cdot \xi,$$

so that

$$(T_a\mathcal{O}_a)^{\vdash} = ker\ D_a \subseteq \Gamma(\Lambda^2 \otimes ad^*P).$$

Let $\delta_a\colon \Gamma(\Lambda^1 \otimes ad\ P) \to \Gamma(ad\ P)$ be the L^2-adjoint of d_a. Then we have the L^2-orthogonal splitting

$$\Gamma(\Lambda^1 \otimes ad\ P) = Im\ d_a \oplus ker\ \delta_a.$$

(This decomposition follows from the ellipticity of the gauge covariant Laplacian $\delta_a \circ d_a$; it is this part of the argument which requires compactness of X.) Thus any $a \in \Gamma(\Lambda^1 \otimes ad\ P)$ can be written $a = d_a\xi + b$ for some $b \in ker\ \delta_a$. Furthermore, a calculation shows that $\delta_a = -\star D_a \star$.

Now we claim that if $a \in (T_a\mathcal{O}_a)^{\vdash\vdash}$, then b in the decomposition of a above vanishes. Since $b \in ker\ \delta_a$ it follows that $\star b \in ker\ D_a = (T_a\mathcal{O}_a)^{\vdash}$. Consequently

$$0 = \int_X \star b \wedge a = \int_X \star b \wedge b,$$

so $b = 0$. Thus $(T_a\mathcal{O}_a)^{\vdash\vdash} = Im\ d_a = T_a\mathcal{O}_a$. Now apply the Theorem. ∎

Since the model spaces we are dealing with here are C^∞ tensor spaces, their L^2-duals are also C^∞ tensor spaces. Thus we have

$$[\Gamma(\Lambda^1 \otimes ad\ P)]^{**} = [\Gamma(\Lambda^2 \otimes ad^*P)]^*$$

$$= \Gamma(\Lambda^1 \otimes ad\ P),$$

so that the space $\mathcal{A}$ of C^∞ connections is modelled on a reflexive space. Then Theorem 2 and Corollary 1 yield

COROLLARY 2: *The constraint set* $J^{-1}(0)$ *for vacuum Yang-Mills theory is a first class variety.*

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