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A classical theorem of Van Hove in conjunction with a formalism developed by Weinstein is used to prove that a quantization functor does not exist. In the proof a category of exact transverse Lagrangian submanifolds is introduced which provides a functorial link between Schrödinger quantization and the prequantization/polarization theory of Kostant and Souriau.

1. INTRODUCTION

The quantization of a classical system has proved to be a delicate as well as difficult problem. In the past decade, however, considerable progress has been made by returning to an examination of the mathematical foundations of classical physics and noting that they can be simply and elegantly phrased in terms of symplectic geometry (Abraham and Marsden, 1978). The resulting "symplectic" quantization theory, *geometric quantization*, is an outgrowth of work by Souriau (1970) on the symplectic formulation of classical mechanics on one hand, and by Kostant (1970, 1978) and his collaborators (Auslander and Kostant, 1971) on group representation theory on the other.

Recently, Weinstein (1977) has developed a functorial formulation of quantization theory which clarifies the origins of the Kostant–Souriau procedure. According to this view, the *quantization problem* can be stated simply as follows: *Does there exist a "reasonable" functor Δ from the classical category*

$$\mathcal{C} = (\textit{symplectic manifolds}, \textit{symplectomorphisms})$$

to the quantum category

$$\mathfrak{Q} = (\textit{complex Hilbert spaces}, \textit{unitary transformations})?$$

As indicated, the functor Δ cannot be entirely arbitrary; it must satisfy certain restrictions dictated by both physical and mathematical considerations. Of primary importance among these limitations is that the correspondence $\Delta: \mathcal{C} \to \mathfrak{Q}$ be in some sense consistent with the standard Schrödinger quantization procedure.

It is widely recognized that such a functor does not exist. However, it is not necessarily obvious that this must be the case: the inability of the Schrödinger theory to provide an adequate quantization procedure in no *a priori* way precludes the existence of a quantization functor. Such a functor would presumably generalize the Schrödinger procedure while at the same time retaining the latter's successes and discarding its failures.

However, Van Hove (1951a, 1951b) has proved a theorem which effectively shows that the successes and failures of the Schrödinger theory are inseparably intertwined. Nonetheless, this theorem, like the quantization method it discusses, is essentially a local construct. Consequently, it is not immediately clear to what extent Van Hove's results clarify *global* questions. To obtain a suitable functorial restatement of Van Hove's theorem, it is therefore first necessary to develop an abstract formulation of the Schrödinger quantization procedure.

Weinstein (1977) has studied these problems, and, in this paper, I pursue his program with the following goals:

1. To specifically set forth the global conditions that the functor Δ must satisfy in order to qualify as a physically acceptable quantization procedure, and
2. To use Van Hove's theorem in conjunction with Weinstein's techniques to prove that such a functor cannot exist.

Having thus shown the impossibility of finding a quantization functor per se, it is natural to search for some generalization of this concept that will yield an admissible quantization procedure. Unfortunately, Van Hove's theorem, as he presents it, does not provide any insight as to how one might improve or generalize the Schrödinger theory. In standard geometric quantization, on the other hand, one first prequantizes the classical system, at which point the connection with the Schrödinger procedure is somewhat tenuous; the constraints imposed by Schrödinger quantization appear only when one proceeds further (polarization).

The major advantage of Weinstein's functorial formalism—when supplemented by the results of this paper—is that it readily provides a suitable generalization of the Schrödinger theory. In fact, it will be shown that these techniques yield a *direct* link between the Schrödinger theory and the more refined global methods of Kostant and Souriau. Consequently, the formalism developed in this paper could be used as a basis for a mathematically intuitive and physically reasonable "derivation" of geometric quantization which appears to have at least as much merit as the more traditional approaches (Simms and Woodhouse, 1976; Śniatycki, 1980; Abraham and Marsden, 1978[1]).

Section 2 briefly reviews the application of symplectic geometry to mechanics. Section 3 discusses the quantization problem in general terms, while Section 4 is concerned with obtaining a global symplectic formulation of the Schrödinger quantization procedure. Functorial geometric quantization is the subject of Section 5, and in Section 6 Van Hove's theorem is used to prove the functorial "nonexistence theorem." Finally, the last two sections examine the connection between functorial geometric quantization and the Kostant–Souriau theory, culminating in a heuristic demonstration of the "necessity" of both prequantization and polarization.

2. SYMPLECTIC GEOMETRY AND CLASSICAL PHYSICS

The natural mathematical model of a conservative classical system is a *symplectic manifold*, that is, a (connected) manifold P together with a distinguished closed nondegenerate 2-form ω. The manifold P represents the phase space of a physical system, while the *symplectic form* ω generalizes the Poisson bracket.

The standard (and almost physically universal) example of a symplectic manifold is the cotangent bundle $\mathfrak{T}^*M$ of a manifold M. The space $\mathfrak{T}^*M$ in fact carries a canonical 1-form θ defined by the universal property

$$\mathfrak{T}^*\alpha(\theta)=\alpha \tag{2.1}$$

where α is any 1-form on M. The symplectic structure on $\mathfrak{T}^*M$ is then simply $\omega=d\theta$. However, not every symplectic manifold is a cotangent bundle nor is ω always exact [e.g., (S^2,ω) where ω is any volume on S^2]. Such "exotic" symplectic manifolds occasionally appear in physics, cf. Souriau (1970).

The symplectic analog of the classical configuration space is a *Lagrangian submanifold*, that is, a maximally isotropic submanifold M of (P,ω). If P is a cotangent bundle, $P=\mathfrak{T}^*M$, then the zero section M is a Lagrangian submanifold. Furthermore, each fiber $\pi_M{}^{-1}(m)$ for $m\in M$ is one also, where π_M: $\mathfrak{T}^*M\to M$ is the projection.

A diffeomorphism f: $(P,\omega)\to(R,\Omega)$ of symplectic manifolds such that $\mathfrak{T}^*f(\Omega)=\omega$ is a *symplectomorphism*. The automorphisms of a symplectic manifold (P,ω) form an infinite-dimensional group denoted $Sym(P,\omega)$;

[1]See the section entitled "Quantization" in Chapter 5 of Abraham and Marsden (1978).

these are the analogs of classical canonical transformations. On the infinitesimal level, the important objects are the *locally Hamiltonian vector fields*, viz., the Lie algebra $\chi(P,\omega)$ of vector fields $X \in \mathfrak{X}(P)$ satisfying $\pounds_X \omega = 0$. Using the bundle isomorphism $\flat$: $\mathfrak{T}P \to \mathfrak{T}^*P$ given by $\flat(X) := i(X)\omega$, it follows that X is locally Hamiltonian iff the 1-form $\flat(X)$ is closed. A locally Hamiltonian vector field generates a local one-parameter group of local symplectomorphisms.

The classical observables are realized as the set $C^\infty(P)$ of smooth real-valued functions on phase space. A close correspondence between observables and locally Hamiltonian vector fields is established via the map ξ: $C^\infty(P) \to \chi(P,\omega)$ defined by $\xi_\phi = \flat^{-1}(d\phi)$. It is possible to endow $C^\infty(P)$ with a Lie algebra structure by defining the *Poisson bracket* $\{\phi, \rho\}$ of two functions ϕ, ρ to be $\{\phi, \rho\} := -\omega(\xi_\phi, \xi_\rho)$. The map ξ is then a Lie algebra homomorphism, and one obtains the exact sequence

$$0 \to \mathbb{R} \to C^\infty(P) \xrightarrow{\xi} \chi(P,\omega) \tag{2.2}$$

of Lie algebras.

Certain Lie subalgebras of observables play distinguished roles in quantization theory, notably the "Heisenberg" and "Schrödinger" subalgebras (Hermann, 1970). A linear subspace F of $C^\infty(P)$ is said to be a *Heisenberg subalgebra* of $C^\infty(P)$ if it has a basis $(f_i, g_i, 1)$, $i = 1, \ldots, \frac{1}{2}\dim P$, of elements satisfying

$$\{f_i, f_j\} = 0 = \{g_i, g_j\}$$

$$\{f_i, g_j\} = \delta_{ij}$$

A *Schrödinger subalgebra* of $C^\infty(P)$ is a pair (S,F), where S is a Lie subalgebra of $C^\infty(P)$ and F is a Heisenberg subalgebra contained in S. Several examples of Schrödinger subalgebras are given by Van Hove (1951a, 1951b).

Generalizing the notion of a Heisenberg subalgebra, define a *polarization* of (P,ω) to be a foliation of P by Lagrangian submanifolds. A polarization is the classical counterpart of a complete set of commuting quantum observables. Any cotangent bundle $\mathfrak{T}^*M$ has a naturally defined polarization (the *vertical* polarization), the leaves of which are the fibers of the projection π_M: $\mathfrak{T}^*M \to M$.

Most of the phase spaces that will be encountered in this article are cotangent bundles. These symplectic manifolds have a rich geometric structure which will be continually utilized in the sequel. Therefore, it is worthwhile at this point to discuss the cotangent bundle case in some detail.

We begin by globalizing the notions of canonical coordinates and momenta. An element of $C^{\infty}(\mathfrak{T}^*M)$ is said to be a *configuration observable* if it is of the form $\psi\circ\pi_M$ for some $\psi\in C^{\infty}(M)$. If $X\in\mathfrak{X}(M)$, then the *momentum* P_X *associated to the vector field* X is defined by

$$P_X(\alpha_m)=\langle X(m)|\alpha_m\rangle$$

for each $\alpha_m\in\mathfrak{T}_m^*M$. One has the following Poisson bracket relations[1]:

$$\{P_X,P_Y\}=P_{[X,Y]}$$

$$\{\psi\circ\pi_M,\phi\circ\pi_M\}=0$$

$$\{P_X,\psi\circ\pi_M\}=X(\psi)\circ\pi_M$$

Therefore, the set $S(\mathfrak{T}^*M)$ of all observables of the form $P_X+\psi\circ\pi_M$ for $X\in\mathfrak{X}(M)$ and $\psi\in C^{\infty}(M)$ forms a Lie subalgebra of $C^{\infty}(\mathfrak{T}^*M)$.

This subalgebra has an important group theoretical interpretation. To see this, consider the subgroup $F(\mathfrak{T}^*M,\omega)$ of $Sym(\mathfrak{T}^*M,\omega)$ consisting of symplectomorphisms which preserve the fibers of $\mathfrak{T}^*M$. These mappings are combinations of (a) lifts of diffeomorphisms of the base M, and (b) translations along the fibers by closed 1-forms on M. More precisely, $F(\mathfrak{T}^*M,\omega)$ is the semidirect product of the additive group $B^1(M)$ of closed 1-forms on M with the diffeomorphism group $Diff(M)$, where the latter acts on $B^1(M)$ by pullback. An important simplification is achieved if one further restricts attention to the subgroup $E(\mathfrak{T}^*M,\omega)$ of $F(\mathfrak{T}^*M,\omega)$ obtained by requiring that the translations along the fibers be given by *exact* 1-forms[2]; thus

$$E(\mathfrak{T}^*M,\omega)=\hat{C}^{\infty}(M)\circledS Diff(M)$$

where $\hat{C}^{\infty}(M)=C^{\infty}(M)/\mathbb{R}$.

The Lie algebra of $E(\mathfrak{T}^*M,\omega)$ is isomorphic to $\hat{C}^{\infty}(M)\times\mathfrak{X}(M)$. To obtain a realization of this Lie algebra as vector fields on $\mathfrak{T}^*M$, consider the one-parameter group of symplectomorphisms generated by $(\psi,X)\in\hat{C}^{\infty}(M)\times\mathfrak{X}(M)$. If $\{f_t\}$ is the flow of X on M, then the locally Hamiltonian vector field ξ_{P_X} generates the induced flow $\{\mathfrak{T}^*f_t\}$ on $\mathfrak{T}^*M$. On the other hand, ψ generates the one-parameter group of translations $\alpha\mapsto\alpha+d\psi$ along the fibers of $\mathfrak{T}^*M$. The locally Hamiltonian vector field generating

[2]The group $E(\mathfrak{T}^*M,\omega)$ is discussed in Chapter 4 of Abraham and Marsden (1978); see specifically exercises 4.1G and 4.2C. The reason one restricts consideration to the subgroup $E(\mathfrak{T}^*M,\omega)$ is that its action on $\mathfrak{T}^*M$ admits a momentum mapping, whereas the action of $F(\mathfrak{T}^*M,\omega)$ does not.

this flow is simply $\flat^{-1}(d(\psi\circ\pi_M))$. Thus an element (ψ, X) has the representation

$$\xi_{P_X}+\flat^{-1}(d(\psi\circ\pi_M))$$

as a vector field on $\mathfrak{T}^*M$. The preimage under ξ of this set of vector fields is the subalgebra $S(\mathfrak{T}^*M)$ of $C^\infty(\mathfrak{T}^*M)$ encountered earlier.

Finally, it is necessary to distinguish certain types of Lagrangian submanifolds of $\mathfrak{T}^*M$. Let α be a closed 1-form on M. Viewing α as an imbedding of M into $\mathfrak{T}^*M$, it follows from (2.1) that

$$\mathfrak{T}^*\alpha(d\theta)=0$$

Thus, the graph $L\subset\mathfrak{T}^*M$ of α is a Lagrangian submanifold of M. The submanifold L is clearly *transverse* to the fibers of π_M in the sense that L projects diffeomorphically onto the zero section. Conversely (Abraham and Marsden, 1978), if L is a transverse Lagrangian submanifold of $\mathfrak{T}^*M$, then there exists a closed 1-form α on M such that L is the graph of α. Since α is closed, locally $\alpha=d\psi$ for some $\psi\in C^\infty(M)$; ψ is a (local) *generating function* of L. If α is exact, then L is said to be an *exact* transverse Lagrangian submanifold.

Geometrically, exact transverse Lagrangian submanifolds are important since they "behave properly" under the action of $E(\mathfrak{T}^*M,\omega)$ in the following sense: if $f\in E(\mathfrak{T}^*M,\omega)$ and L is an exact transverse Lagrangian submanifold, then $f(L)$ is also.

3. THE QUANTIZATION PROBLEM

In physics, the important quantities are the "infinitesimal" objects, viz., the classical observables and the quantum operators. Reflecting its physical origins, therefore, the *quantization problem* as it is usually stated (Guest, 1974) consists of finding a rule which, subject to certain limitations which will be outlined shortly, simultaneously assigns

($\mathfrak{Q}$1) to each symplectic phase space a complex Hilbert space, and
($\mathfrak{Q}$2) to each classical observable a self-adjoint operator on the quantum Hilbert space.

This algebraic version of the quantization problem gains in elegance and clarity when rephrased on a group-theoretical level. To do this, it is necessary to abstract requirement ($\mathfrak{Q}$2) above. By exploiting the duality ξ of observables and locally Hamiltonian vector fields, and recalling that the latter generate symplectomorphisms, a suitable global restatement is found to be the following:

($\mathcal{Q}$2′) to each symplectomorphism of phase space there is associated a unitary transformation of the quantum Hilbert space.

The correspondence described by ($\mathcal{Q}$1) and ($\mathcal{Q}$2′) can be best stated in functorial language. Defining the classical and quantum categories $\mathcal{C}$ and $\mathcal{Q}$ as in the Introduction, Weinstein (1977) formulates the quantization problem as follows: *Does there exist a "reasonable" functor* Δ: $\mathcal{C} \to \mathcal{Q}$? The main constraint that Δ must satisfy in order to qualify as an acceptable quantization procedure is "consistency with Schrödinger quantization." Beyond this, there are a number of other requirements that any "reasonable" quantization procedure should respect. These, however, are not directly relevant here and are discussed in Weinstein (1977).

The next two sections will be devoted to investigating the role of Schrödinger quantization in functorial geometric quantization.

4. SCHRÖDINGER QUANTIZATION

We for the moment restrict consideration to physical systems whose phase spaces are Euclidean spaces. Following Abraham and Marsden (1978), define a *full quantization* of the symplectic manifold $(\mathbb{R}^{2n}, \omega)$ to be a representation δ of classical observables $\phi \in C^{\infty}(\mathbb{R}^{2n})$ as self-adjoint operators $\delta(\phi)$ on some Hilbert space H such that

($\mathcal{F}\mathcal{Q}$1): $\delta(\phi + \psi) = \delta(\phi) + \delta(\psi)$
($\mathcal{F}\mathcal{Q}$2): $\delta(\lambda\phi) = \lambda\delta(\phi), \qquad \lambda \in \mathbb{R}$
($\mathcal{F}\mathcal{Q}$3): $\delta(\{\phi, \psi\}) = (1/i)[\delta(\phi), \delta(\psi)]$
($\mathcal{F}\mathcal{Q}$4): $\delta(1) = id$
($\mathcal{F}\mathcal{Q}$5): $\delta(x^i)$ and $\delta(\xi_i)$ are represented irreducibly on H.

Here (x^i, ξ_i) are Cartesian coordinates on $\mathbb{R}^{2n}$.

According to the Stone–von Neumann theorem,[1] condition ($\mathcal{F}\mathcal{Q}$5) implies that $H = L^2(\mathbb{R}^n)$ up to equivalence and that

$$\delta(x^i) = x^i \tag{4.1}$$

and

$$\delta(\xi_i) = \frac{1}{i} \frac{\partial}{\partial x^i} \tag{4.2}$$

Groenwold (1946) showed that a full quantization of $\mathbb{R}^{2n}$ in the above sense is impossible if one insists on quantizing *every* classical observable.[1]

However, it is possible to obtain a quantization satisfying ($\mathcal{FQ}1$)–($\mathcal{FQ}5$) provided one restricts to a subalgebra S of $C^\infty(\mathbb{R}^{2n})$ (Van Hove, 1951b).[3] Noting that $\{1, x^i, \xi_i\}$ spans a Heisenberg subalgebra of $C^\infty(\mathbb{R}^{2n})$, conditions ($\mathcal{FQ}4$) and ($\mathcal{FQ}5$) imply that the subalgebra S is a Schrödinger subalgebra. These considerations motivate the following definition (Hermann, 1970; Streater, 1966):

A *Schrödinger quantization* of the symplectic phase space $(\mathbb{R}^{2n}, \omega)$ is a representation of a Schrödinger subalgebra (S, F) of $C^\infty(\mathbb{R}^{2n})$ by self-adjoint operators on $L^2(\mathbb{R}^n)$ which is irreducible when restricted to the Heisenberg subalgebra F.

Schrödinger quantization is not a "quantization procedure" in the strict sense of ($\mathcal{Q}1$) and ($\mathcal{Q}2$) as it does not attempt to quantize *all* classical phase spaces, but rather only those that are Euclidean spaces. Furthermore, it is not capable of consistently quantizing *all* the observables of a classical system, instead limiting consideration to Schrödinger subalgebras.

Our first objective is to find a global reformulation of the Schrödinger theory that will bring the underlying symplectic geometry to the fore. Since $\mathbb{R}^{2n} = \mathcal{T}^*\mathbb{R}^n$, the natural way to accomplish this is to generalize the Schrödinger quantization procedure to cotangent bundles.

Unfortunately, there is no symplectic analog of a Heisenberg subalgebra—in fact, there exist cotangent bundles (e.g., $\mathcal{T}^*S^2$) that do not admit the existence of Heisenberg subalgebras. Geometrically, the reason for this is as follows: Consider a Heisenberg subalgebra F of $C^\infty(\mathcal{T}^*\mathbb{R}^n)$. The Lie subalgebra $\xi(F)$ of $\chi(\mathcal{T}^*\mathbb{R}^n, \omega)$ generates a transitive action of $\mathbb{R}^{2n}$ on $\mathcal{T}^*\mathbb{R}^n$, so that the global counterpart to a Heisenberg subalgebra is a translation group. In general, however, it is not possible to define a transitive action of $\mathbb{R}^{2n}$ on a cotangent bundle.

On the other hand, there is a natural extension of the notion of a Schrödinger subalgebra to cotangent bundles: the subalgebra $S(\mathcal{T}^*M)$ discussed in Section 2. Indeed, if $M = \mathbb{R}^n$, then $S(\mathcal{T}^*\mathbb{R}^n)$ is actually a Schrödinger subalgebra of $C^\infty(\mathcal{T}^*\mathbb{R}^n)$ since it contains the Heisenberg subalgebra spanned by $\{1, x^i, \xi_i\}$. In the general case, note that *locally* $S(\mathcal{T}^*M)$ is a Schrödinger subalgebra as it contains the (locally defined)

[3]On the other hand, if one discards the requirement ($\mathcal{FQ}5$) then it is possible to obtain a quantization of *all* of $C^\infty(\mathbb{R}^{2n})$ satisfying ($\mathcal{FQ}1$)–($\mathcal{FQ}4$). The so-called *Dirac problem*, i.e., the construction of such a quantization, was solved independently by Van Hove (1951b), Souriau (1970), and Kostant (1970). Thus the results of Groenwold present one with two options: drop ($\mathcal{FQ}5$) and quantize all of $C^\infty(\mathbb{R}^{2n})$, or retain ($\mathcal{FQ}5$) and be content with quantizing subalgebras of observables. I choose the latter alternative, since it is my opinion that ($\mathcal{FQ}5$) is the most physically significant of all the quantization axioms (e.g., it is this axiom that incorporates the uncertainty principle). Regardless of how one proceeds, however, one is eventually led to prequantization (Section 7). The Dirac problem is treated in Abraham and Marsden (1978), and the two alternatives are compared in Streater (1966).

Heisenberg subalgebra generated by $\{1, q^i \circ \pi_M, P_{\partial/\partial q^i}\}$, where the q^i are local coordinates on M.

Turning now to quantization, we must find an appropriate generalization of $L^2(\mathbb{R}^n)$ when $\mathbb{R}^n$ is replaced by an arbitrary configuration space M. Such a manifold M does not carry a canonically defined measure; nonetheless, it is possible to intrinsically associate to M a Hilbert space $\mathcal{H}(M)$, the space of L^2 *half-densities* on M (Weinstein, 1977).[1] Define an *α-density* at $m \in M$ to be a complex-valued function ν_m on the fiber $\mathcal{B}_m(M)$ of the linear frame bundle of M which satisfies the equivariance condition

$$\nu_m(b \cdot g) = |det\, g|^{\alpha} \nu_m(b)$$

where $b \in \mathcal{B}_m(M)$, $g \in GL(n, \mathbb{C})$ and $n = dim\, M$. The collection of all α-densities ν_m at all points $m \in M$ forms a complex line bundle $|M|^{\alpha}$ over M; an *α-density* on M is a section ν of this bundle. Let $\mathcal{D}_0^{\infty}(|M|^{1/2})$ denote the pre-Hilbert space of all smooth compactly supported half-densities on M with the inner product

$$(\mu, \nu) = \int_M \bar{\mu} \cdot \nu$$

The completion of $\mathcal{D}_0^{\infty}(|M|^{1/2})$ with respect to (,) is the Hilbert space $\mathcal{H}(M)$.

It is now possible to formulate an algebraic version of the Schrödinger procedure applied to cotangent bundles: a *Schrödinger quantization* of $(\mathcal{T}^*M, \omega)$ is an irreducible representation of $S(\mathcal{T}^*M)$ by self-adjoint operators on the Hilbert space $\mathcal{H}(M)$ of L^2 half-densities. Globalizing this statement in the fashion of ($\mathcal{Q}$2)→($\mathcal{Q}$2′), one requires that the group $E(\mathcal{T}^*M, \omega)$ be represented irreducibly by unitary operators on $\mathcal{H}(M)$.

5. FUNCTORIAL GEOMETRIC QUANTIZATION

Having clarified the symplectic aspects of the Schrödinger quantization theory, we now turn to the task of elucidating in functorial terms the requirement that the functor Δ: $\mathcal{C} \to \mathcal{Q}$ be consistent with Schrödinger quantization.

For this purpose, it is convenient to introduce the category

$$\mathcal{M} = (C^{\infty} \text{ manifolds, } C^{\infty} \text{ diffeomorphisms})$$

We have the *cotangent functor* $\mathcal{T}^*$: $\mathcal{M} \to \mathcal{C}$ defined by

$$\mathcal{T}^*(M, f) = (\mathcal{T}^*M, \mathcal{T}^*f)$$

and the *half-density functor* $\mathcal{H}$: $\mathfrak{M} \to \mathfrak{Q}$ given by

$$\mathcal{H}(M,f) = (\mathcal{H}(M), f^*)$$

where f^* acts unitarily on half-densities by pullback.[4] Both $\mathfrak{T}^*$ and $\mathcal{H}$ are contravariant functors.

The auxiliary category $\mathfrak{M}$ serves as an intermediary through which cotangent bundles can be associated with Hilbert spaces of half-densities. We thus obtain the diagram

$$\mathcal{C} \overset{\mathfrak{T}^*}{\leftarrow} \mathfrak{M} \overset{\mathcal{H}}{\rightarrow} \mathfrak{Q} \qquad (\mathfrak{W})$$

This defines a *functorial relation* in $\mathcal{C} \times \mathfrak{Q}$, that is, a subcategory of $\mathcal{C} \times \mathfrak{Q}$ which is not necessarily the graph of a functor.

This functorial relation allows us to quantize those symplectomorphisms of $\mathfrak{T}^*M$ that are lifts of diffeomorphisms of the base. Indeed, if M is a manifold and $f \in Diff(M)$, then $(\mathfrak{W})$ quantizes $\mathfrak{T}^*M$ and $\mathfrak{T}^*f$ via the diagrams

$$\mathfrak{T}^*M \overset{\mathfrak{T}^*}{\leftarrow} M \overset{\mathcal{H}}{\rightarrow} \mathcal{H}(M) \tag{5.1}$$

and

$$\mathfrak{T}^*f \overset{\mathfrak{T}^*}{\leftarrow} f \overset{\mathcal{H}}{\rightarrow} f^* \tag{5.2}$$

In particular, if X is a complete vector field on M, then $(\mathfrak{W})$ quantizes the momentum P_X associated to X as follows. Let $\{f_t\}$ be the flow of X on M. Then $\{\mathfrak{T}^*f_t\}$ is a symplectic one-parameter group on $\mathfrak{T}^*M$ with generating vector field ξ_{P_X} (cf. Section 2), and the corresponding one-parameter unitary group on $\mathcal{H}(M)$ is $\{f_t^*\}$. The self-adjoint differential operator $\delta(P_X)$ generating this unitary flow has been calculated by Śniatycki (1980) and Abraham and Marsden (1978)[1]; we give a local expression for the action of $\delta(P_X)$ on $\mathfrak{D}_0^\infty(|M|^{1/2})$. Let $U \subset M$ be open, and let ζ: $U \to \mathfrak{B}(M)$ be a local linear frame field, i.e., $\zeta(m) = [\zeta^1(m), \ldots, \zeta^n(m)]$,

[4] A diffeomorphism f: $M \to N$ induces a map $f^\#$: $\mathfrak{B}(M) \to \mathfrak{B}(N)$ of the respective linear frame bundles defined by

$$f^\#([\varphi^1(m), \ldots, \varphi^n(m)]) = [\mathfrak{T}f(\varphi^1(m)), \ldots, \mathfrak{T}f(\varphi^n(m))]$$

If $\nu \in \mathcal{H}(N)$, define the pullback $f^*\nu \in \mathcal{H}(M)$ by

$$(f^*\nu)(b) = \nu(f^\# b)$$

for all $b \in \mathfrak{B}(M)$.

where $n = \dim M$. Without loss of generality, we may choose $\nu \in \mathfrak{D}_0^\infty(|M|^{1/2})$ so that $\nu \circ \zeta = 1$. Let A_X denote the matrix with i,j entry equal to the jth component of the vector field $[X, \zeta^i]$. Then, if $\phi \in C_{\mathbf{C}}^\infty(M)$,

$$\delta(P_X)[\phi\nu]|U = \frac{1}{i}\left[X(\phi) + \frac{1}{2}(\operatorname{tr} A_X)\phi\right]\nu \tag{5.3}$$

If g is a Riemannian metric on M, (5.3) becomes

$$\delta(P_X)[\phi\nu] = \frac{1}{i}\left[X(\phi) + \frac{1}{2}(\operatorname{div}_g X)\phi\right]\nu$$

where div_g denotes the covariant divergence with respect to g.

Weinstein's functorial relation ($\mathfrak{W}$) thus correctly globalizes certain aspects of the Schrödinger theory. It is, however, incomplete as we have not yet quantized all of $E(\mathfrak{T}^*M, \omega)$, but rather only its subgroup *Diff*(M). It remains to quantize the subgroup $\hat{C}^\infty(M)$, or in infinitesimal terms, the configuration observables. But the Hamiltonian vector fields of such functions generate symplectomorphisms of $\mathfrak{T}^*M$ which leave the fibers of π_M invariant, and such symplectomorphisms cannot be realized as the lifts of diffeomorphisms of the base M. Thus ($\mathfrak{W}$) is inadequate; we need a "bigger" functorial relation.

To quantize the translations along the fibers, we recall from Section 2 that if $\Psi \in E(\mathfrak{T}^*M, \omega)$ is the translation by the exact 1-form $d\psi$, then $\Psi(M)$ is an exact transverse Lagrangian submanifold of $\mathfrak{T}^*M$ with generating function ψ. Moreover, if $L \subset \mathfrak{T}^*M$ is an exact transverse Lagrangian submanifold with generating function ψ_L, then $\Psi(L)$ is an exact transverse Lagrangian submanifold as well, with generating function $\psi_L + \psi$.

It is apparent, then, that the new functorial relation should involve a category consisting of exact transverse Lagrangian submanifolds of cotangent bundles. Symplectic translations along the fibers will then be included among the morphisms of this category.

We now define the *Lagrangian submanifold category* $\mathfrak{LSM}$. Its objects are pairs $(L, \mathfrak{T}^*M)$, where L is an exact transverse Lagrangian submanifold of $\mathfrak{T}^*M$. The morphisms of $\mathfrak{LSM}$ are pairs of mappings (f,g): $(L, \mathfrak{T}^*M) \to (J, \mathfrak{T}^*N)$ such that f: $L \to J$ is a diffeomorphism, g: $\mathfrak{T}^*M \to \mathfrak{T}^*N$ is a fiber-preserving symplectomorphism, and $g|L = f$.

We relate this category to all the other categories and the functorial relation ($\mathfrak{W}$) by means of various functors. In particular, it is possible to realize $\mathfrak{M}$ as a subcategory of $\mathfrak{LSM}$ via the covariant functor $\mathfrak{L}$: $\mathfrak{M} \to \mathfrak{LSM}$, where

$$\begin{aligned} \mathfrak{L}(M) &= (M, \mathfrak{T}^*M) \\ \mathfrak{L}(f) &= (f, \mathfrak{T}_*f) \end{aligned} \tag{5.4}$$

Here, we have set $\mathcal{T}_* f := \mathcal{T}^*(f^{-1})$. Conversely, we have the functor π: $\mathcal{L}\mathcal{S}\mathcal{M} \to \mathcal{M}$ such that

$$\pi(L, \mathcal{T}^*M) = M \tag{5.5}$$

Given a morphism (f,g): $(L, \mathcal{T}^*M) \to (J, \mathcal{T}^*N)$ of $\mathcal{L}\mathcal{S}\mathcal{M}$, define a map $M \to N$ via

$$M \overset{\Psi_L}{\to} L \overset{f}{\to} J \overset{-\Psi_J}{\to} N$$

where Ψ_L denotes translation by $d\psi_L$, ψ_L being a generating function of L etc. Set

$$\pi(f,g) = -\Psi_J \circ f \circ \Psi_L \tag{5.6}$$

The functor π so defined is clearly covariant. Finally, by "forgetting" the Lagrangian submanifolds, one obtains a contravariant functor $\mathcal{L}\mathcal{T}^*$: $\mathcal{L}\mathcal{S}\mathcal{M} \to \mathcal{C}$; specifically,

$$\begin{aligned} &\mathcal{L}\mathcal{T}^*(L, \mathcal{T}^*M) = \mathcal{T}^*M \\ &\mathcal{L}\mathcal{T}^*(f,g) = g^{-1} \end{aligned} \tag{5.7}$$

The quantization in the new functorial relation is provided by $\mathcal{L}\mathcal{H}$: $\mathcal{L}\mathcal{S}\mathcal{M} \to \mathcal{Q}$, where

$$\mathcal{L}\mathcal{H}(L, \mathcal{T}^*M) = \mathcal{H}(M) \tag{5.8}$$

and, if (f,g): $(L, \mathcal{T}^*M) \to (J, \mathcal{T}^*N)$ is a morphism of $\mathcal{L}\mathcal{S}\mathcal{M}$ and $\nu \in \mathcal{H}(N)$,

$$\mathcal{L}\mathcal{H}(f,g)[\nu] = \exp[-i(\psi_J \circ \pi(f,g) - \psi_L)]\pi(f,g)^*\nu \tag{5.9}$$

To demonstrate that $\mathcal{L}\mathcal{H}$ is a contravariant functor,[5] consider the sequence

$$(L, \mathcal{T}^*M) \overset{(f,g)}{\to} (J, \mathcal{T}^*N) \overset{(h,k)}{\to} (K, \mathcal{T}^*R)$$

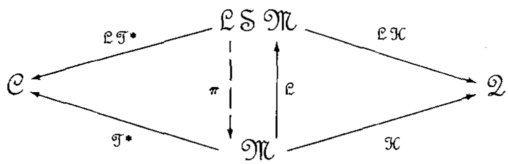
[5]Strictly speaking, $\mathcal{L}\mathcal{H}$ is actually multivalued, since the generating functions ψ_L and ψ_J appearing in (5.9) are not uniquely determined. This ambiguity is unimportant for the time being and discussion will be deferred until Section 7.

and suppose that $\nu \in \mathcal{H}(R)$. Then

$$\begin{aligned}
\mathcal{L}\mathcal{H}(f,g)\circ\mathcal{L}\mathcal{H}(h,k)[\nu] &= \mathcal{L}\mathcal{H}(f,g)\{\exp[-i(\psi_K\circ\pi(h,k)-\psi_J)]\pi(h,k)^*\nu\} \\
&= \exp[-i(\psi_J\circ\pi(f,g)-\psi_L)]\pi(f,g)^*\{\exp[-i(\psi_K\circ\pi(h,k)-\psi_J)]\pi(h,k)^*\nu\} \\
&= \exp[-i(\psi_K\circ\pi(h,k)\circ\pi(f,g)-\psi_L)][\pi(h,k)\circ\pi(f,g)]^*\nu \\
&= \exp[-i(\psi_K\circ\pi(h\circ f,k\circ g)-\psi_L)]\times\pi(h\circ f,k\circ g)^*\nu \\
&= \mathcal{L}\mathcal{H}(h\circ f,k\circ g)[\nu] \\
&= \mathcal{L}\mathcal{H}((h,k)\circ(f,g))[\nu]
\end{aligned}$$

as π is covariant.

To summarize, we have the diagram



(5.10)

Define the *Schrödinger functorial relation* to be

$$\mathcal{C} \overset{\mathcal{L}\mathcal{T}^*}{\longleftarrow} \mathcal{L}\mathcal{S}\mathcal{M} \overset{\mathcal{L}\mathcal{H}}{\longrightarrow} \mathcal{Q} \tag{$\mathcal{S}$}$$

We devote the remainder of this section to showing that ($\mathcal{S}$) is indeed the proper functorial generalization of the Schrödinger quantization procedure.

Begin by proving that ($\mathcal{S}$) contains ($\mathcal{W}$) as a sub-functorial relation, that is, the diagram (5.10) commutes provided one ignores the broken arrow π. From (5.1), (5.4), (5.7), and (5.8) one sees that the diagram (5.10) commutes on the objects of $\mathcal{M}$. Now, let $f\colon M\to N$ be a diffeomorphism; it induces via (5.4) the morphism $(f,\mathcal{T}_*f)\colon (M,\mathcal{T}^*M)\to(N,\mathcal{T}^*N)$ of $\mathcal{L}\mathcal{S}\mathcal{M}$. By (5.4)–(5.6) one has $\pi\circ\mathcal{L}=id_{\mathcal{M}}$, so that $\pi(f,\mathcal{T}_*f)=f$. Consequently, (5.7) and (5.9) yield the quantization

$$\mathcal{T}^*f \overset{\mathcal{L}\mathcal{T}^*}{\longleftarrow} (f,\mathcal{T}_*f) \overset{\mathcal{L}\mathcal{H}}{\longrightarrow} f^*$$

This is consistent with (5.2), and the diagram commutes on the morphisms of $\mathfrak{M}$. Thus ($\mathcal{S}$) is consistent with ($\mathfrak{W}$). As a corollary, the functorial relation ($\mathcal{S}$) quantizes the momentum P_X associated to a complete vector field X via (5.3), which in turn generalizes (4.2).

The above arguments show that ($\mathcal{S}$) is capable of quantizing the subgroup *Diff*(M) of $E(\mathcal{T}^*M,\omega)$. It remains to quantize the translations along the fibers. Thus, let Ψ denote translation by $d\psi$, and choose an arbitrary exact transverse Lagrangian submanifold L of $\mathcal{T}^*M$. The functorial relation ($\mathcal{S}$) quantizes the induced morphism $(-\Psi|L, -\Psi)$ of $\mathcal{L}\mathcal{S}\mathfrak{M}$ as follows: from (5.6), $\pi(-\Psi|L,-\Psi)=id_M$, so that by (5.7) and (5.9),

$$\Psi \overset{\mathcal{L}\mathcal{T}^*}{\leftarrow} (-\Psi|L,-\Psi) \overset{\mathcal{L}\mathcal{H}}{\rightarrow} e^{i\psi}id^* \tag{5.11}$$

Note that this result does not depend upon the choice of Lagrangian submanifold L.

Equation (5.11) enables us to trivially quantize the configuration observables: let $\psi\in C^\infty(M)$, and consider the one-parameter group

$$\Psi_t\colon \alpha\to\alpha+td\psi$$

of translations generated by ψ. Then

$$\Psi_t \overset{\mathcal{L}\mathcal{T}^*}{\leftarrow} (\Psi_{-t}|L,\Psi_{-t}) \overset{\mathcal{L}\mathcal{H}}{\rightarrow} e^{it\psi}id^*$$

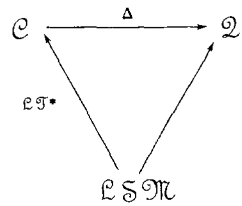
so that the self-adjoint operator $\delta(\psi\circ\pi_M)$ associated to $\psi\circ\pi_M$ via ($\mathcal{S}$) is the generator of the one-parameter unitary group $e^{it\psi}id^*$. Thus, if $\nu\in\mathfrak{D}_0^\infty(|M|^{1/2})$, we recover (4.1):

$$\delta(\psi\circ\pi_M)[\nu]=i\psi\nu \tag{5.12}$$

If we distinguish an element $\nu\in\mathcal{H}(M)$, then every half-density is of the form $\phi\nu$ for some $\phi\in C_{\mathbb{C}}^\infty(M)$. Since ν^2 defines a measure on M, it follows that there exists an isomorphism $\mathcal{H}(M)\to L^2(M,\nu^2)$ given by $\phi\nu\to\phi$. Equations (5.3) and (5.12) then realize the quantum operators $\delta(P_X)$ and $\delta(\psi\circ\pi_M)$ as differential operators on $L^2(M,\nu^2)$. Since the only differential operator that commutes with both $\delta(P_X)$ and $\delta(\psi\circ\pi_M)$ is a multiple of the identity operator, the representation of $E(\mathcal{T}^*M,\omega)$ on $\mathcal{H}(M)$ induced by ($\mathcal{S}$) is irreducible.

6. VAN HOVE'S THEOREM AND THE NONEXISTENCE THEOREM

We search for a functor Δ: $\mathcal{C} \to \mathcal{Q}$ consistent with Schrödinger quantization. In global terms, this means that the diagram



(6.1)

must commute. We will prove the following theorem:

Nonexistence Theorem. There does not exist a functor Δ compatible with (6.1).

This result is essentially a trivial consequence of the functorial formalism developed in the last section and a classical theorem of Van Hove.

The work of Groenwold (1946) and Abraham and Marsden (1978) alluded to in Section 4 denies the existence of a full quantization of $(\mathbb{R}^{2n}, \omega)$. However, as Abraham and Marsden (1978) have pointed out, from a global viewpoint this result is not necessarily disastrous as the Hamiltonian vector fields of the observables Groenwold employed as counterexamples are not complete.[6] Taking this into account, Van Hove sharpened Groenwold's results and proved the following (slightly restated) theorem:

Theorem (Van Hove, 1951b). Let $\tilde{C}^{\infty}(\mathbb{R}^{2n})$ denote the collection of observables on $\mathbb{R}^{2n}$ with complete Hamiltonian vector fields. Then there does not exist a quantization $(\mathbb{R}^{2n}, \omega) \to L^2(\mathbb{R}^n)$, associating to each $\phi \in \tilde{C}^{\infty}(\mathbb{R}^{2n})$ a self-adjoint operator $\delta(\phi)$ on $L^2(\mathbb{R}^n)$, which simultaneously satisfies the following two conditions:

($\mathcal{VH}$1) If $\lambda \in \mathbb{R}$ and $\phi, \rho \in \tilde{C}^{\infty}(\mathbb{R}^{2n})$ are such that

$$\lambda\phi + \rho \in \tilde{C}^{\infty}(\mathbb{R}^{2n})$$

and

$$\{\phi, \rho\} \in \tilde{C}^{\infty}(\mathbb{R}^{2n})$$

[6]From a global standpoint, it is the symplectomorphisms which are directly quantized, not the observables. Consequently, one would expect to be able to quantize an observable iff its Hamiltonian vector field is complete.

then

(i) $\delta(\lambda\phi+\rho)=\lambda\delta(\phi)+\delta(\rho)$

(ii) $\delta(\{\phi,\rho\})=\frac{1}{i}[\delta(\phi),\delta(\rho)]$

Furthermore, if $\{\rho_t\}$ is the symplectic flow generated by ξ_ρ, $\rho\in\tilde{C}^\infty(\mathbb{R}^{2n})$, then the correspondence $\{\rho_t\}\to e^{it\delta(\rho)}$ satisfies

(iii) $\chi_s=\rho_t\circ\phi_s\circ\rho_{-t}\to e^{is\delta(\chi)}=e^{it\delta(\rho)}e^{is\delta(\phi)}e^{-it\delta(\rho)}$

for $\chi,\rho,\phi\in\tilde{C}^\infty(\mathbb{R}^{2n})$.

($\mathcal{VH}2$) The correspondence δ is a Schrödinger quantization of $(\mathbb{R}^{2n},\omega)$ in the sense of Section 4.

We now use this result to prove the functorial nonexistence theorem. Suppose that there exists a functor Δ: $\mathcal{C}\to\mathcal{Q}$, and let P be an object in $\mathcal{C}$. Restricting consideration to one-parameter groups, we obtain a mapping δ from $\tilde{C}^\infty(P)$ to self-adjoint operators on $\Delta(P)$ by associating to ϕ the generator $\delta(\phi)$ of the one-parameter unitary group $\Delta(\{\phi_t\})$. Now, set $P=\mathbb{R}^{2n}$. Condition (iii) of ($\mathcal{VH}1$) must then hold by the definition of δ and the assumption that Δ is a functor. That (i) and (ii) of ($\mathcal{VH}1$) are satisfied as well follows similarly, taking into account the well-known properties of flows of vector fields and the fact that ξ is a Lie algebra homomorphism. On the other hand, it was shown in the last section that the functorial relation ($\mathcal{S}$) induces a Schrödinger quantization of $(\mathbb{R}^{2n},\omega)$. Thus, a functor Δ with the property that the diagram (6.1) commutes necessarily satisfies both ($\mathcal{VH}1$) and ($\mathcal{VH}2$), and this is impossible according to Van Hove's theorem. Consequently, no such functor can exist.

7. PREQUANTIZATION

Having shown that a functor $\mathcal{C}\to\mathcal{Q}$ consistent with Schrödinger quantization cannot exist, we now turn to the task of developing "the best possible" quantization procedure. At this point, the only quantization scheme available is

$$\mathcal{C}\overset{\mathcal{LT}^*}{\leftarrow}\mathcal{LS}\,\mathcal{M}\overset{\mathcal{LH}}{\rightarrow}\mathcal{Q} \qquad (\mathcal{S})$$

This functorial relation, while to some extent an improvement over the Schrödinger theory, is essentially a globalization thereof and hence suffers from many of the latter's defects.

In particular:

(i) ($\mathbb{S}$) can only quantize cotangent bundles. This lack of generality has serious physical repercussions since, as indicated in Section 2, there exist physical systems with phase spaces that are not cotangent bundles.

(ii) On the infinitesimal level, the only quantizable observables are elements of $\tilde{C}^{\infty}(\mathfrak{T}^*M)$ of the form $P_X+\psi\circ\pi_M$. It is often necessary to quantize a more general class of observables than this [e.g., ($\mathbb{S}$) would not, in general, allow one to quantize the Hamiltonian].

(iii) The functor $\mathfrak{L}\mathfrak{H}$—and consequently the quantization procedure ($\mathbb{S}$)—are multivalued. Since an exact transverse Lagrangian submanifold L does not uniquely determine a generating function, but rather only specifies ψ_L modulo an additive constant, (5.9) shows that $\mathfrak{L}\mathfrak{H}(f,g)$ is defined only up to a phase factor. Each symplectomorphism in $E(\mathfrak{T}^*M,\omega)$ therefore possesses a set of inequivalent quantizations—parametrized by the circle S^1—as a unitary operator on $\mathfrak{H}(M)$. It follows that ($\mathbb{S}$) actually yields a *projective* representation of $E(\mathfrak{T}^*M,\omega)$ on $\mathfrak{H}(M)$.

Following Weinstein (1977), we design a new functorial relation which does not possess these disadvantages. We deal with problem (iii) first. To obtain a single-valued quantization procedure, we search for a category $\mathfrak{P}$ whose objects and morphisms take into account the arbitrariness in the choice of generating functions. Quantization in the new functorial relation will then provide an *honest* representation of $E(\mathfrak{T}^*M,\omega)$. Thus, if Z is an object in $\mathfrak{P}$ associated to $\mathfrak{T}^*M$, its automorphism group—denoted $Quant(Z)$—must be a central extension of $E(\mathfrak{T}^*M,\omega)$ by S^1. In other words, the sequence

$$0\to S^1\to Quant(Z)\to E(\mathfrak{T}^*M,\omega) \tag{7.1}$$

must be exact. On the Lie algebra level, we correspondingly require a central extension of $\hat{C}^{\infty}(M)\times\mathfrak{X}(M)$ by $\mathbb{R}$. As a special case of (2.2), however, we have the exact sequence

$$0\to\mathbb{R}\to S(\mathfrak{T}^*M)\xrightarrow{\xi}\hat{C}^{\infty}(M)\times\mathfrak{X}(M) \tag{7.2}$$

This implies that the Lie algebra of $Quant(Z)$ is isomorphic to $S(\mathfrak{T}^*M)$.

With regard to point (ii) above, we attempt to quantize a more general class of observables by replacing $\hat{C}^{\infty}(M)\times\mathfrak{X}(M)$ with $\chi(\mathfrak{T}^*M,\omega)$ and $S(\mathfrak{T}^*M)$ with $C^{\infty}(\mathfrak{T}^*M)$ in (7.2). This in turn necessitates substituting $Sym(\mathfrak{T}^*M,\omega)$ in place of $E(\mathfrak{T}^*M,\omega)$ in (7.1). Finally, we take (i) into account by generalizing to the case of an arbitrary symplectic manifold (P,ω). These considerations lead us to replace the exact sequences (7.1) and (7.2) with

$$0 \to S^1 \to Quant(Z) \to Sym(P,\omega) \tag{7.3}$$

and

$$0 \to \mathbb{R} \to C^\infty(P) \xrightarrow{\xi} \chi(P,\omega) \tag{7.4}$$

respectively.

Thus, we will have remedied defects (*i*)–(*iii*) of the Schrödinger theory provided it is possible to find a category $\mathcal{P}$ with the property that the automorphism group $Quant(Z)$ of an object Z in $\mathcal{P}$ associated to the symplectic manifold (P,ω) renders the sequence (7.3) exact. This requirement suggests that Z should be a circle bundle over P which is in some way "tied down" to the symplectic structure of P. More precisely, Z should be a *regular contact manifold*, that is, a principal S^1 bundle together with a connection 1-form Θ satisfying

$$\ker d\Theta = \mathrm{Ver}(Z) \tag{7.5}$$

[here, $\mathrm{Ver}(Z)$ denotes the subbundle of $\mathcal{T}Z$ consisting of vectors tangent to the fibers of Z]. We thus take the regular contact manifolds as the objects of the *prequantization category* $\mathcal{P}$. The morphisms in this category—*quantomorphisms*—are connection-preserving principal bundle isomorphisms.

Prequantization was discovered by Van Hove (1951b) and subsequently rediscovered and generalized by both Souriau (1970) and Kostant (1970). Within the framework of the Kostant–Souriau theory, prequantization arose in an attempt to solve the so-called "Dirac problem" (cf. note 3). Here on the other hand, prequantization appears as the result of "repairing" the overly restrictive and multivalued Schrödinger functorial relation. Thus, from the standpoint of functorial geometric quantization, prequantization is the logical successor to the Schrödinger theory.

Let (Z,Θ) be a regular contact manifold, and denote by σ_Z the projection onto the orbit space Z/S^1. Condition (7.5) implies that $d\Theta$ projects onto a symplectic form ω_Z on Z/S^1 such that

$$\mathcal{T}^*\sigma_Z(\omega_Z) = d\Theta \tag{7.6}$$

Thus Z/S^1 is a symplectic manifold. Now, consider a quantomorphism g: $(Z,\Theta) \to (Y,\rho)$. Since g is fiber-preserving, it induces a diffeomorphism

$\sigma(g)$: $Y/S^1 \to Z/S^1$ of the respective orbit spaces. The relation (7.6) implies that $\sigma(g)$ is in fact a symplectomorphism, so that the correspondences $(Z,\Theta) \to (Z/S^1, \omega_Z)$ and $g \to \sigma(g)$ define a contravariant functor σ: $\mathcal{P} \to \mathcal{C}$.

Moreover, if γ_φ denotes the quantomorphism of (Z,Θ) associated with $\varphi \in S^1$, one obtains—as required—the exact sequence (7.3):

$$0 \to S^1 \xrightarrow{\gamma} Quant(Z,\Theta) \xrightarrow{\sigma} Sym(Z/S^1, \omega_Z)$$

The Lie algebra of $Quant(Z,\Theta)$ is the set of all connection-preserving vector fields on Z. The isomorphism of this Lie algebra with $C^\infty(P)$ implied by (7.4) is given by $\phi \to \xi_\phi{}^\#$, where $\#$ denotes the horizontal lift.

Quantization in the category $\mathcal{P}$ is defined as follows. Let $\mathfrak{D}_0^\infty(Z)$ denote the set of smooth compactly supported functions λ: $Z \to \mathbb{C}$ which are equivariant with respect to the S^1 action, i.e.,

$$\lambda(\gamma_\varphi(z)) = e^{-i\varphi}\lambda(z)$$

for all $z \in Z$. The *prequantization Hilbert space* $\mathcal{H}(Z,\Theta)$ is the completion of $\mathfrak{D}_0^\infty(Z)$ with respect to the inner product

$$\langle \lambda, \eta \rangle = \frac{1}{2\pi} \int_Z \bar{\lambda}\eta \; \Theta \wedge (d\Theta)^n$$

Since quantomorphisms act unitarily on the Hilbert spaces $\mathcal{H}(Z,\Theta)$ by pullback, we have a contravariant functor $\mathcal{H}$: $\mathcal{P} \to \mathcal{Q}$.

The *prequantization functorial relation* is

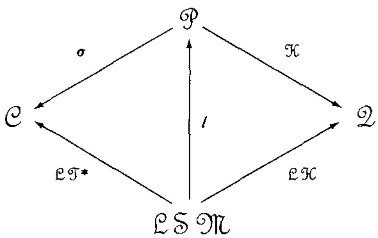
$$\mathcal{C} \xleftarrow{\sigma} \mathcal{P} \xrightarrow{\mathcal{H}} \mathcal{Q} \qquad (\mathcal{P})$$

At first glance, this functorial relation is quite remarkable since it is so general. If (P,ω) is a symplectic manifold, a *prequantization* of (P,ω) is an object (Z,Θ) in $\sigma^{-1}(P,\omega)$. A prequantization of (P,ω) exists iff ω is *integral*, i.e., $[\omega]$ lies in the image of $H^2(P,\mathbb{Z})$ in $H^2(P,\mathbb{R})$ (Kostant, 1970; Simms and Woodhouse, 1976). Thus, $(\mathcal{P})$ is capable of quantizing any integral symplectic manifold. On the infinitesimal level, $(\mathcal{P})$ quantizes *all* of $\tilde{C}^\infty(P)$ as essentially self-adjoint operators on $\mathfrak{D}_0^\infty(Z)$ via the correspondence $\phi \to \xi_\phi{}^\#$.[7]

Unfortunately, this generality is illusory since the quantization procedure $(\mathcal{P})$ fails to be consistent with the Schrödinger functorial relation $(\mathcal{S})$.

[7]Kostant (1970) shows that $\tilde{C}^\infty(P)$ is mapped onto the set of *complete* connection-preserving vector fields on Z.

According to general principles, ($\mathcal{P}$) qualifies as a physically acceptable quantization procedure only if the diagram



(7.7)

commutes, where l is the natural inclusion $\mathcal{L}\mathcal{S}\mathcal{M}\to\mathcal{P}$ given by

$$
\begin{aligned}
l(L,\mathcal{T}^*M) &= (\mathcal{T}^*M\times S^1,\theta+d\varphi)\\
l(f,g) &= g\times id_{S^1}
\end{aligned}
\tag{7.8}
$$

Here, φ denotes the coordinate on S^1, and θ is the canonical 1-form on $\mathcal{T}^*M$. Since $\mathcal{K}(\mathcal{T}^*M\times S^1,\theta+d\varphi)$ consists of square-integrable functions λ on $\mathcal{T}^*M\times S^1$ of the form $\lambda(\beta_m,\varphi)=e^{-i\varphi}\lambda(\beta_m,0)$, the prequantization Hilbert space may be identified with $L^2(\mathcal{T}^*M,\omega^n)$. From (7.8), then,

$$\mathcal{K}\circ l(L,\mathcal{T}^*M)=L^2(\mathcal{T}^*M,\omega^n)$$

whereas ($\mathcal{S}$) implies that

$$\mathcal{L}\mathcal{K}(L,\mathcal{T}^*M)=\mathcal{K}(M)$$

Furthermore, it may be shown (Simms and Woodhouse, 1976) that ($\mathcal{P}$) does not provide an irreducible representation of $E(\mathcal{T}^*M,\omega)$ or $S(\mathcal{T}^*M)$. Thus, diagram (7.7) does not commute.

8. POLARIZATION

The prequantization procedure must be modified in such a way as to reduce the prequantization representation from "L^2(phase space)" to "L^2(configuration space)." This necessitates polarizing the classical system. We now briefly describe the functorial aspects of the polarization procedure; motivation, proofs, and explicit constructions are omitted.[8]

A polarization $\mathcal{F}$ of (P,ω) is *admissible* provided the leaf space $P/\mathcal{F}$ has a manifold structure such that the canonical projection p: $P\to P/\mathcal{F}$ is

[8]Comprehensive treatments of this topic may be found in Weinstein (1977), Simms and Woodhouse (1976), and Śniatycki (1980). To simplify the presentation, I assume that all polarizations are real and that nontrivial Bohr–Sommerfeld conditions (leading to distributional wave functions) do not appear. Furthermore, I ignore complications due to both considerations of metalinear geometry and the half-form nature of the wave functions.

a submersion. A *geometric quantum system* is a triple $(Z,\Theta,\mathfrak{F})$, where (Z,Θ) is an object in $\mathcal{P}$ and $\mathfrak{F}$ is an admissible real polarization of $(P,\omega)=\sigma(Z,\Theta)$. The geometric quantum systems form a category $\mathcal{G}$; ignoring the polarizations gives a covariant functor τ: $\mathcal{G}\to\mathcal{P}$ and hence a contravariant functor $\sigma\circ\tau$: $\mathcal{G}\to\mathcal{C}$.

There is a preferred functor $\mathcal{V}$: $\mathcal{L}\mathcal{S}\mathcal{M}\to\mathcal{G}$ obtained by assigning to each object $(L,\mathfrak{T}^*M)$ in $\mathcal{L}\mathcal{S}\mathcal{M}$ its natural prequantization (7.8) and by equipping $\mathfrak{T}^*M$ with the vertical polarization $\mathfrak{F}_V$:

$$\mathcal{V}(L,\mathfrak{T}^*M)=\left(\mathfrak{T}^*M\times S^1,\,\theta+d\varphi,\mathfrak{F}_V\right) \tag{8.1}$$

Furthermore, if (f,g): $(L,\mathfrak{T}^*M)\to(J,\mathfrak{T}^*N)$ is a morphism in $\mathcal{L}\mathcal{S}\mathcal{M}$, then by definition g preserves $\mathfrak{F}_V$ so that the assignment

$$\mathcal{V}(f,g)=g\times exp\left[\,i(\psi_J\circ\pi(f,g)-\psi_L)\circ Pr\right] \tag{8.2}$$

defines a morphism of $\mathcal{G}$, where pr is the projection $\mathfrak{T}^*M\times S^1\to M$.

Let (Z,Θ) be a prequantization of (P,ω). The polarization $\mathfrak{F}$ induces a foliation $\hat{\mathfrak{F}}$ of (Z,Θ) which covers $\mathfrak{F}$. For each leaf D of $\mathfrak{F}$, consider those equivariant functions λ: $\sigma_Z{}^{-1}(D)\to\mathbb{C}$ which are constant on the leaves of $\hat{\mathfrak{F}}|\sigma_Z{}^{-1}(D)$. Assuming trivial Bohr–Sommerfeld conditions, the set of all such functions forms a one-dimensional complex vector space Q^D attached to the point $p(D)\in P/\mathfrak{F}$. The collection of all such vector spaces, one for each point of $P/\mathfrak{F}$, defines a complex line bundle $Q(Z,\Theta,\mathfrak{F})$ over $P/\mathfrak{F}$. Let $\mathfrak{D}_0^\infty(Z,\Theta,\mathfrak{F})$ denote the space of all smooth compactly supported sections of $Q(Z,\Theta,\mathfrak{F})\otimes|P/\mathfrak{F}|^{1/2}$. The product $\bar{\mu}\cdot\nu$ of two elements of $\mathfrak{D}_0^\infty(Z,\Theta,\mathfrak{F})$ can be thought as a section of $|P/\mathfrak{F}|$, so integration over $P/\mathfrak{F}$ gives $\mathfrak{D}_0^\infty(Z,\Theta,\mathfrak{F})$ a pre-Hilbert space structure; its completion is denoted $\Gamma(Z,\Theta,\mathfrak{F})$.

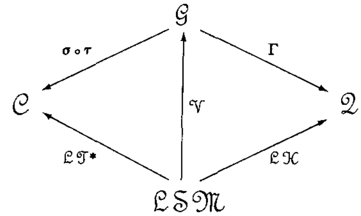
Let h: $(Z_1,\Theta_1,\mathfrak{F}_1)\to(Z_2,\Theta_2,\mathfrak{F}_2)$ be an isomorphism of geometric quantum systems, and set $(P_i,\omega_i)=\sigma(Z_i,\Theta_i)$. The symplectomorphism $(\sigma\circ\tau)(h^{-1})$: $(P_1,\omega_1)\to(P_2,\omega_2)$ preserves the polarizations in the sense that it maps the leaves of $\mathfrak{F}_1$ onto the leaves of $\mathfrak{F}_2$. Consequently, $(\sigma\circ\tau)(h^{-1})$ induces a diffeomorphism $h_{\mathfrak{F}}$: $P_1/\mathfrak{F}_1\to P_2/\mathfrak{F}_2$ which in turn gives rise to an isomorphism $h_{\mathfrak{F}}{}^*$: $|P_2/\mathfrak{F}_2|^{1/2}\to|P_1/\mathfrak{F}_1|^{1/2}$. On the other hand, let D_2 be a leaf of $\mathfrak{F}_2$ and $D_1=(\sigma\circ\tau)(h)[D_2]$ the corresponding leaf of $\mathfrak{F}_1$. The pullback $\mathfrak{T}^*(\tau(h))$ of the quantomorphism $\tau(h)$ induces a vector space isomorphism $Q^{D_2}\to Q^{D_1}$ and hence a bundle isomorphism h_Q: $Q(Z_2,\Theta_2,\mathfrak{F}_2)\to Q(Z_1,\Theta_1,\mathfrak{F}_1)$.

The considerations of the preceding two paragraphs enable us to define a contravariant functor Γ: $\mathcal{G}\to\mathfrak{Q}$ by associating to each geometric quantum system $(Z,\Theta,\mathfrak{F})$ the Hilbert space $\Gamma(Z,\Theta,\mathfrak{F})$ and to each morphism h of $\mathcal{G}$ the unitary transformation $h_Q\times h_{\mathfrak{F}}{}^*$.

The *Kostant–Souriau quantization procedure* is

$$\mathcal{C} \overset{\sigma\circ\tau}{\leftarrow} \mathcal{G} \overset{\Gamma}{\rightarrow} \mathcal{Q} \qquad (\mathcal{KS})$$

We now prove that the $(\mathcal{KS})$ procedure is in fact consistent with the functorial relation $(\mathcal{S})$. In other words, we show that the diagram



(8.3)

commutes.

The left-hand side of (8.3) clearly presents no problem. Consider the geometric quantum system $(\mathcal{T}^*M\times S^1, \theta+d\varphi, \mathcal{F}_V)$. The leaf space $\mathcal{T}^*M/\mathcal{F}_V$ is diffeomorphic to M, and the line bundle $Q(\mathcal{T}^*M\times S^1,\theta+d\varphi,\mathcal{F}_V)$ can be identified with $M\times\mathbb{C}$. Consequently,

$$Q(\mathcal{T}^*M\times S^1,\theta+d\varphi,\mathcal{F}_V)\otimes|\mathcal{T}^*M/\mathcal{F}_V|^{1/2}\approx|M|^{1/2}$$

and $\Gamma(\mathcal{T}^*M\times S^1,\theta+d\varphi,\mathcal{F}_V)\approx\mathcal{H}(M)$. Thus, from (5.8) and (8.1),

$$(\Gamma\circ\mathcal{V})(L,\mathcal{T}^*M)=\mathcal{L}\mathcal{H}(L,\mathcal{T}^*M)$$

On the other hand, consider the morphism $\mathcal{V}(f,g)$: $\mathcal{T}^*M\times S^1\to\mathcal{T}^*N\times S^1$ of $\mathcal{G}$ defined by (8.2). The induced mappings $(\sigma\circ\tau)[\mathcal{V}(f,g)]$: $\mathcal{T}^*N\to\mathcal{T}^*M$ and $\mathcal{V}(f,g)_{\mathcal{F}}^*$: $|N|^{1/2}\to|M|^{1/2}$ are simply g^{-1} and $\pi(f,g)^*$, respectively; it remains to calculate the induced morphism $\mathcal{V}(f,g)_Q$. Let λ_Q be a section of $Q(\mathcal{T}^*N\times S^1,\theta+d\varphi,\mathcal{F}_V)$. By construction, λ_Q may be identified with an equivariant function λ: $\mathcal{T}^*N\times S^1\to\mathbb{C}$ constant on the leaves of $\hat{\mathcal{F}}_V$. Thus, if $(\beta_m,\varphi)\in\mathcal{T}^*_mM\times S^1$, then by equivariance

$$\begin{aligned}
[\mathcal{V}(f,g)_Q(\lambda_Q)](m)&=[\mathcal{T}^*(\mathcal{V}(f,g))(\lambda)](\beta_m,\varphi)\\
&=\lambda(\mathcal{V}(f,g)(\beta_m,\varphi))\\
&=\lambda(g(\beta_m),\varphi+i[\psi_J\circ\pi(f,g)-\psi_L](m))\\
&=\exp[-i(\psi_J\circ\pi(f,g)-\psi_L)(m)]\lambda(g(\beta_m),\varphi)\\
&=\exp[-i(\psi_J\circ\pi(f,g)-\psi_L)(m)][\mathcal{T}^*(g\times id_{S^1})\lambda](\beta_m,\varphi)\\
&=\exp[-i(\psi_J\circ\pi(f,g)-\psi_L)(m)][\pi(f,g)^*\lambda_Q](m)
\end{aligned}$$

Consequently, if $\lambda_Q \otimes \nu \in \Gamma(\mathcal{T}^* N \times S^1, \theta + d\varphi, \mathcal{F}_V)$, then

$$[\mathcal{V}(f,g)_Q \times \mathcal{V}(f,g)_{\mathcal{T}^*}](\lambda_Q \otimes \nu)$$
$$= exp\Big[-i(\psi_J \circ \pi(f,g) - \psi_L)\Big]\pi(f,g)^*(\lambda_Q \otimes \nu)$$

which, with the identification $\Gamma(\mathcal{T}^* N \times S^1, \theta + d\varphi, \mathcal{F}_V) = \mathcal{H}(N)$, is clearly equivalent to (5.9). Thus diagram (8.3) commutes and the $(\mathcal{H}\mathbb{S})$ functorial relation is consistent with the Schrödinger quantization procedure.

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