

ON A FULL QUANTIZATION OF THE TORUS

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Abstract

I exhibit a prequantization of the torus which is actually a "full" quantization in the sense that a certain minimal complete set of classical observables is irreducibly represented. Thus in this instance there is no Groenewold-Van Hove obstruction to quantization.

1. INTRODUCTION

The prequantization procedure produces a faithful representation of the entire Poisson algebra of a quantizable symplectic manifold.¹ In general, these prequantization representations are flawed physically; for instance, the prequantization of \mathbf{R}^{2n} with its standard symplectic structure is not unitarily equivalent to the Schrödinger representation. One usually remedies this by imposing an irreducibility requirement. But there is seemingly a price to be paid for irreducibility: one can no longer quantize *all* classical observables, but rather only proper subalgebras thereof.

This "obstruction" to quantization has been known since the 1940s. In a series of papers, Groenewold and later Van Hove showed that it is impossible to quantize the entire Poisson algebra of polynomials on \mathbf{R}^{2n} in such a way that the Heisenberg $\mathfrak{h}(2n)$ subalgebra of inhomogeneous linear polynomials is irreducibly represented.²⁻⁴ Some of the maximal subalgebras of polynomials that *can* be consistently quantized subject to this irreducibility requirement are the inhomogeneous quadratic polynomials, and polynomials which are at most affine in the momenta or the configurations. See Refs. 5-7 for further discussions of this example. Recently, a similar phenomenon was observed for S^2 . In this case it was shown that the maximal subalgebra of the Poisson algebra of spherical harmonics that can be consistently quantized while irreducibly representing the $\mathfrak{u}(2)$ subalgebra of spherical harmonics of degree at most one is just this $\mathfrak{u}(2)$ subalgebra itself.⁸

Based on these results as well as general quantization theory, it would seem reasonable to conjecture that a "no-go" theorem must always hold, to wit:

It is impossible to quantize the entire Poisson algebra of any symplectic manifold subject to an irreducibility requirement.

To make this conjecture precise, I introduce some terminology. Let (M, ω) be a connected symplectic manifold, and let \mathcal{O} be a Poisson subalgebra of $C^\infty(M)$.

Definition 1. A prequantization of \mathcal{O} is a linear map \mathcal{Q} from \mathcal{O} to an algebra of (essentially) self-adjoint operators^a on a Hilbert space \mathcal{H} such that

$$(i) \quad \mathcal{Q}(\{f, g\}) = 2\pi i [\mathcal{Q}(f), \mathcal{Q}(g)],$$

where $\{, \}$ denotes the Poisson bracket and $[,]$ the commutator.^b If \mathcal{O} contains the constant function 1, then also

$$(ii) \quad \mathcal{Q}(1) = I.$$

As the nomenclature suggests, it appears necessary in practice to supplement these conditions, often by requiring that a certain subset \mathcal{F} of observables be represented irreducibly. In favorable circumstances one can take \mathcal{F} to consist of the components of a momentum map associated to a transitive Lie symmetry group. In the nonhomogeneous case, the corresponding notion is a "complete set of observables." This is a subset \mathcal{F} of $C^\infty(M)$ with the property that the Hamiltonian vector fields of elements of \mathcal{F} span the tangent spaces to M at every point.^c This means in particular that $\{f, g\} = 0$ for all $f \in \mathcal{F}$ implies that $g = \text{const}$. I say that \mathcal{F} is *minimal* if, modulo the constants, the linear span of \mathcal{F} contains no proper complete subspace.

Similarly, a set of operators \mathcal{A} on \mathcal{H} is *irreducible* provided the only (bounded) operator which commutes with each $A \in \mathcal{A}$ is a multiple of the identity. Since irreducibility is the quantum analogue of completeness, I make

Definition 2. Let $\mathcal{F} \subset \mathcal{O}$ be a complete set of observables. A prequantization \mathcal{Q} of \mathcal{O} is a quantization of the pair $(\mathcal{O}, \mathcal{F})$ provided

(iii) the corresponding operators $\{\mathcal{Q}(f) \mid f \in \mathcal{F}\}$ form an irreducible set.

If \mathcal{F} is minimal and one can take $\mathcal{O} = C^\infty(M)$, the quantization is full.^d

Thus there do not exist full quantizations of either $(\mathbf{R}^{2n}, h(2n))$ or $(S^2, u(2))$. According to the conjecture above, it should be impossible to fully quantize *any* symplectic manifold. But here I show that this is *false*: there exists a full quantization of the torus T^2 . The existence of this full quantization, which can be obtained as a particular Kostant-Souriau prequantization, is surprising, especially given the Groenewold-Van Hove obstruction to a full quantization of its covering \mathbf{R}^2 . The dichotomy stems from the fact that, unlike on \mathbf{R}^2 , any prequantum bundle on T^2 is nontrivial. This twisting forces the wave functions to be *quasi-periodic*, and these boundary conditions effectively override the obstruction in the case of interest. That the prequantization I consider may indeed provide a full quantization is signalled by the observation that it has, in a sense, the Schrödinger representation of the Heisenberg algebra built in. It turns out that this representation of the Heisenberg algebra is well known in solid state physics, where it goes under the name "*kq*-representation".¹⁰

^a For the most part technical difficulties with unbounded operators will be ignored, as they are not essential for what follows.

^b I use units in which Planck's constant $h = 1$.

^c This definition of a complete set of observables is slightly stronger than that given in Ref. 1.

^d Without some sort of minimality condition on \mathcal{F} , it is often possible to find full quantizations; indeed, it may happen that a prequantization representation is itself irreducible.^{4,9}

2. PREQUANTIZATION

It is convenient to realize the torus T^2 as $\mathbf{R}^2/\mathbf{Z}^2$ with symplectic form

$$\omega = N dx \wedge dy,$$

where N is a nonzero integer. Then one can identify $C^\infty(T^2)$ with the space of doubly periodic functions f on the plane:

$$f(x+m, y+n) = f(x, y), \quad m, n \in \mathbf{Z}.$$

For the prequantization of the torus, I follow Ref. 1, §2.3. (See also Ref. 11 for a gauge-theoretic treatment.) Let L_N be a prequantum bundle over T^2 with Chern class N .^e To explicitly construct it,^f introduce the sets

$$U_- = \{(x, y) \in (-\delta, 1-\delta) \times [0, 1]\} \quad \text{and} \quad U_+ = \{(x, y) \in (\delta, 1+\delta) \times [0, 1]\}$$

for $\delta > 0$ small. Identifying $y = 0$ with $y = 1$ gives a pair of cylinders, and then identifying x with $x + 1$ for $x \in (-\delta, \delta)$ pastes the cylinders together into a torus. On U_\pm define the connection potentials

$$\theta_\pm = Nx dy.$$

On the overlap $U_+ \cap U_-$,

$$\theta_- - \theta_+ = \begin{cases} 0, & x \in (\delta, 1-\delta) \\ -N dy, & x \in (-\delta, \delta) \end{cases}$$

whence the transition functions are

$$c(x, y) = \begin{cases} 1, & x \in (\delta, 1-\delta) \\ e^{-2\pi i N y}, & x \in (-\delta, \delta). \end{cases}$$

The space $\Gamma(L_N)$ of smooth sections of L_N may thus be identified with the space of smooth 'quasi-periodic' complex-valued functions ϕ on the plane:

$$\phi(x+m, y+n) = e^{2\pi i N m y} \phi(x, y), \quad m, n \in \mathbf{Z}. \quad (2.1)$$

The corresponding prequantum Hilbert space \mathcal{H}_N consists of those $\phi(x, y)$ satisfying (2.1) which are square integrable over $[0, 1]^2$.

The prequantization map \mathcal{Q}_N is

$$\mathcal{Q}_N(f) = \frac{1}{2\pi i N} \left(\frac{\partial f}{\partial x} \left(\frac{\partial}{\partial y} - 2\pi i N x \right) - \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \right) + f. \quad (2.2)$$

^e Up to equivalence, there is exactly one line bundle per value of N , but each L_N carries a two-parameter family of inequivalent connections. Below I fix a particular connection.

^f The associated principal circle bundle over T^2 may be realized as $\Gamma_N \backslash H_N(2)$, where $H_N(2)$ is the (polarized) Heisenberg group consisting of all matrices of the form

$$\begin{pmatrix} 1 & a & -c/N \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in \mathbf{R}$, and Γ_N is the subgroup of $H_N(2)$ consisting of those matrices for which a, b and c are integers, cf. Ref. 12, §2.

These operators are essentially self-adjoint on $\Gamma(L_N)$.

Now define, for each $N \neq 0$, the minimal complete sets⁹

$$\mathcal{F}_N = \{1, e^{\pm 2\pi i N x}, e^{\pm 2\pi i N y}\}.$$

Then (2.2) gives

$$\mathcal{Q}_N(e^{\pm 2\pi i N x}) = e^{\pm 2\pi i N x} \left(1 \mp 2\pi i N x \pm \frac{\partial}{\partial y}\right) \quad (2.3)$$

$$\mathcal{Q}_N(e^{\pm 2\pi i N y}) = e^{\pm 2\pi i N y} \left(1 \mp \frac{\partial}{\partial x}\right). \quad (2.4)$$

Note that $\mathcal{Q}_N(e^{-2\pi i N x}) = \mathcal{Q}_N(e^{2\pi i N x})^*$ and $\mathcal{Q}_N(e^{-2\pi i N y}) = \mathcal{Q}_N(e^{2\pi i N y})^*$.

3. THE GO THEOREM

The key observation is that the operators

$$\hat{X} = \frac{1}{2\pi i} \left(\frac{\partial}{\partial y} - 2\pi i N x \right), \quad \hat{Y} = -\frac{1}{2\pi i} \frac{\partial}{\partial x}, \quad \hat{Z} = -\frac{N}{2\pi i}, \quad (3.1)$$

the first two of which appear in (2.2), provide a representation of the Heisenberg algebra

$$\mathfrak{h}(2) = \{X, Y, Z \in \mathbf{R}^3 \mid [X, Y] = Z, [X, Z] = [Y, Z] = 0\}$$

on \mathcal{H}_N . For $|N| = 1$ this is equivalent to the Schrödinger representation, as I now show.

For the moment fix $N = 1$. ($N = -1$ will work as well.) The analysis is simplified by applying the Weil-Brezin-Zak transform W [cf. Refs. 6, §1.10, and 10] to the data in the previous section. Let $\mathcal{S}(\mathbf{R})$ be the Schwartz space. Define a linear map $W : \mathcal{S}(\mathbf{R}) \rightarrow \Gamma(L_1)$ by

$$(W\psi)(x, y) = \sum_{k \in \mathbf{Z}} \psi(x + k) e^{-2\pi i k y}$$

with inverse

$$(W^{-1}\phi)(x) = \int_0^1 \phi(x, y) dy.$$

That W extends to a unitary map of $L^2(\mathbf{R})$ onto \mathcal{H}_1 is readily checked, cf. Thm. 1.109 in Ref. 6. Under this transform, \hat{X} and \hat{Y} go over to the operators $-x$ and $-\frac{1}{2\pi i} \frac{d}{dx}$ on $L^2(\mathbf{R})$, respectively. Thus the $N = 1$ prequantization carries the Schrödinger representation of the Heisenberg algebra.

This leads one to suspect that \mathcal{Q}_1 might yield a full quantization of the torus. Indeed this is the case:

Theorem 1 (Go Theorem). *The prequantization \mathcal{Q}_1 gives a full quantization of $(C^\infty(T^2), \mathcal{F}_1)$.*

⁹Although in principle I consider only real-valued observables, here it is convenient to use complex notation.

Proof: Since \mathcal{Q}_1 is a prequantization, conditions (i) and (ii) are automatically satisfied. Thus it is only necessary to verify (iii), i.e.,

$$\{\mathcal{Q}_1(e^{\pm 2\pi i x}), \mathcal{Q}_1(e^{\pm 2\pi i y})\}$$

forms an irreducible set. Applying W to (2.3) and (2.4), one has, as operators on $\mathcal{S}(\mathbf{R})$,

$$\begin{aligned} (\mathcal{Q}_1(e^{\pm 2\pi i x})\psi)(x) &= e^{\pm 2\pi i x}(1 \mp 2\pi i x)\psi(x) \\ (\mathcal{Q}_1(e^{\pm 2\pi i y})\psi)(x) &= \left(1 \mp \frac{d}{dx}\right)\psi(x \pm 1). \end{aligned}$$

Suppose that T is a bounded linear operator on $L^2(\mathbf{R})$ which commutes with both $A := \mathcal{Q}_1(e^{2\pi i x})$, $B := \mathcal{Q}_1(e^{2\pi i y})$ and their adjoints. Then T must commute with $A^*A = 1 + 4\pi^2 x^2$ and $B^*B = 1 - d^2/dx^2$. So $[T, x^2] = 0$ and $[T, d^2/dx^2] = 0$, whence T must commute with $[d^2/dx^2, x^2] = 4x d/dx + 2$. Thus T commutes with the generators of the symplectic algebra $\mathfrak{sp}(2, \mathbf{R})$ in the metaplectic representation μ . Since $\mathcal{S}(\mathbf{R})$ contains a dense subspace of analytic vectors for μ (viz. the Hermite functions, cf. Ref. 6, §4.3), it follows that $[T, \mu(\mathcal{M})] = 0$ for all \mathcal{M} in some neighborhood of the identity in the metaplectic group $\text{Mp}(2, \mathbf{R})$ and hence, as this group is connected, for all \mathcal{M} in $\text{Mp}(2, \mathbf{R})$.

Although the metaplectic representation is reducible, the subrepresentations μ_e and μ_o on each invariant summand of $L^2(\mathbf{R}) = L_e^2(\mathbf{R}) \oplus L_o^2(\mathbf{R})$ of even and odd functions are irreducible [cf. Ref. 6, §4.4]. Writing $T = P_e T + P_o T$, where P_e and P_o are the even and odd projectors, one has

$$[P_e T, \mu(\mathcal{M})] = P_e [T, \mu(\mathcal{M})] + [P_e, \mu(\mathcal{M})]T = 0 \quad (3.2)$$

for any $\mathcal{M} \in \text{Mp}(2, \mathbf{R})$. It then follows from the irreducibility of the subrepresentation μ_e that $P_e T = k_e P_e + R P_o$ for some constant k_e and some operator $R : L_o^2(\mathbf{R}) \rightarrow L_e^2(\mathbf{R})$. Substituting this expression into (3.2) yields $[R P_o, \mu(\mathcal{M})] = 0$, and Schur's Lemma then implies that R is either an isomorphism or is zero. But R cannot be an isomorphism as the representations μ_e and μ_o are inequivalent [cf. Ref. 6, Thm. 4.56]. Thus $P_e T = k_e P_e$. Similarly $P_o T = k_o P_o$, whence $T = k_e P_e + k_o P_o$.

But now a short calculation shows that T commutes with

$$A - A^* = 2i(\sin 2\pi x - 2\pi x \cos 2\pi x)$$

only if $k_e = k_o$. Thus T is a multiple of the identity, and so $\{A, A^*, B, B^*\}$ is an irreducible set, as was to be shown. \square

This proof is not valid for $|N| > 1$. The problem is that the map $W : L^2(\mathbf{R}) \rightarrow \mathcal{H}_N$, which in order to maintain (2.1) now takes the form

$$(W\psi)(x, y) = \sum_{k \in \mathbf{Z}} \psi(x + k) e^{-2\pi i N k y},$$

is no longer onto; indeed, from this formula it is apparent that the image of W consists of those functions in \mathcal{H}_N which have period $1/N$ in y . Denote the subspace of all such functions by \mathcal{P}_N . Since the operators \hat{X} and \hat{Y} preserve periodicity, $\Gamma(L_N) \cap \mathcal{P}_N$ is invariant and it follows that the representation (3.1) of $\mathfrak{h}(2)$ on \mathcal{H}_N is no longer irreducible. While these facts do not *a priori* preclude the existence of a full quantization for $|N| > 1$, they do yield the following.

Proposition 2. For $|N| > 1$, \mathcal{Q}_N does not represent \mathcal{F}_N irreducibly.

Proof: It suffices to observe that the operators (2.3) and (2.4) commute with the orthogonal projector $\mathcal{H}_N \rightarrow \mathcal{P}_N$. \square

Thus \mathcal{Q}_N provides a full quantization of $(C^\infty(T^2), \mathcal{F}_N)$ iff $|N| = 1$.

4. DISCUSSION

The $|N| = 1$ quantization of the torus is curious in several respects. First, it does not mesh well with what one intuitively expects on the basis of geometric quantization theory.¹ For one thing, it is not necessary to introduce a polarization in order to obtain an acceptable quantization. (Since the prequantum wave functions are quasi-periodic – or, more crudely, since $\mathcal{H}_1 \approx L^2(\mathbf{R})$ – they are in effect “already polarized.”) And if one does polarize T^2 , then one is guaranteed to be able to consistently quantize only a substantially smaller set of observables – namely, those whose Hamiltonian vector fields preserve the polarization – but not necessarily any larger subalgebra. On the other hand, it is known that the torus admits a strict deformation quantization,¹³ so perhaps it is not entirely unexpected that it admits a full quantization as well.

A second point concerns the role of the Heisenberg group $H(2)$ in this example, especially as compared to \mathbf{R}^2 . In both cases, it acts transitively (and factors through a translation group.) On \mathbf{R}^2 , there is a momentum mapping for this action, and it is natural to insist that quantization provide an irreducible representation of $\mathfrak{h}(2)$. But on T^2 , there is no momentum map; the torus is “classically anomalous” in this regard.¹⁴ So it does not make sense to require that quantization produce a representation of $\mathfrak{h}(2)$ in this case. Nonetheless, there is the representation (3.1) of the Heisenberg algebra on \mathcal{H}_N which, for $|N| = 1$, is irreducible. The reason one can obtain a full quantization while irreducibly representing the Heisenberg algebra in this instance is because \hat{X} and \hat{Y} are not the quantizations of any observables; in other words, the representation (3.1) does not arise via quantization. Thus in this sense the quantization is also anomalous. On \mathbf{R}^2 this is not the case: demanding that \mathcal{Q} be $\mathfrak{h}(2)$ -equivariant is too stringent a requirement to allow the quantization of every observable in the Schrödinger representation.

As an aside, it is interesting to observe that if x and y were globally defined observables, then from (2.2) one would have

$$\mathcal{Q}(x) = \frac{1}{2\pi i N} \frac{\partial}{\partial y} \quad \text{and} \quad \mathcal{Q}(y) = -\frac{1}{2\pi i N} \left(\frac{\partial}{\partial x} - 2\pi i N y \right). \quad (4.1)$$

Of course the operators $\partial/\partial y$ and $\partial/\partial x - 2\pi i N y$ are not defined on $\Gamma(L_N)$,^h although they do make sense on $\Gamma(L_{-N})$ with the connection given by the potentials $\bar{\theta}_\pm = Ny dx$. In fact, one has^h

$$\frac{\partial}{\partial y} = \bar{\nabla}_{\frac{\partial}{\partial y}} \quad \text{and} \quad \frac{\partial}{\partial x} - 2\pi i N y = \bar{\nabla}_{\frac{\partial}{\partial x}}$$

where $\bar{\nabla}$ is the aforesaid connection on L_{-N} . This explains the remark of Asorey [Ref. 14, §4], to the effect that ‘the quantum generators of translation symmetries are

^h In Ref. 15 such operators are called “bad.”

given by multiples of $-i\bar{\nabla}_j$ for $j = 1, 2$.⁷ Furthermore, if one were on \mathbf{R}^2 rather than T^2 , the operators (4.1) would be well defined and would commute with \hat{X} and \hat{Y} , indicating that the representation (3.1) of $\mathfrak{h}(2)$ is reducible. So it is apparent how the nontriviality of the prequantum bundles removes this obstruction to the irreducibility of this representation of the Heisenberg algebra.

It is perhaps also surprising that the irreducibility of the representation (3.1) of $\mathfrak{h}(2)$ on \mathcal{H}_1 apparently does not, in and by itself, imply that \mathcal{Q}_1 is a full quantization of $(C^\infty(T^2), \mathcal{F}_1)$. Indeed, that the prequantum Hilbert space carries the Schrödinger representation only seems to be a portent; the proof of the Go Theorem devolves instead upon the properties of the metaplectic representation. I do not know if this theorem can be proved directly using the irreducibility of $\mathfrak{h}(2)$. More generally, I wonder to what extent the fullness of the prequantization \mathcal{Q}_N , for a given minimal complete set \mathcal{F} , is correlated with the irreducibility of the $\mathfrak{h}(2)$ representation? For instance, is it possible to obtain a full quantization using \mathcal{Q}_N for $|N| > 1$? Or perhaps one could prove a no-go result in this context, thereby strengthening Proposition 2?

Thirdly, the requirement that quantization irreducibly represent a complete set \mathcal{F} typically leads to "von Neumann rules" for elements of \mathcal{F} .⁸ Roughly speaking, these rules govern the extent to which \mathcal{Q} preserves the multiplicative structure of the Poisson algebra. In particular, they determine how $\mathcal{Q}(f^2)$ is related to $\mathcal{Q}(f)^2$ for $f \in \mathcal{F}$. For \mathbf{R}^{2n} and S^2 this relation is relatively 'tight'. But for T^2 both $\mathcal{Q}_N(f^2)$ and $\mathcal{Q}_N(f)^2$ are completely determined for any observable f and any N by the simple fact that \mathcal{Q}_N is a prequantization; irreducibility is irrelevant. Moreover, one sees from (2.2) that $\mathcal{Q}_N(f^2)$ is a first order differential operator whereas $\mathcal{Q}_N(f)^2$ is of second order. This indicates that, in general, one cannot expect quantization to respect the classical multiplicative structure to any significant degree.

Finally, a salient feature of this example is that the Poisson subalgebra $\rho(\mathcal{F}_1)$ generated by the minimal complete set \mathcal{F}_1 – the trigonometric polynomials – is infinite dimensional, and is in fact dense in $C^\infty(T^2)$. This does not happen for either \mathbf{R}^{2n} or S^2 , in which cases the relevant subalgebras are finite dimensional. Since the irreducibility of \mathcal{F}_1 under \mathcal{Q}_1 is tantamount to that of $\rho(\mathcal{F}_1)$, it follows that the irreducibility requirement on T^2 is substantially weaker than the corresponding requirements on either \mathbf{R}^{2n} or S^2 . This, along with the *a posteriori* observation that \mathcal{Q}_1 irreducibly represents $C^\infty(T^2)$,ⁱ is likely the underlying reason why \mathcal{Q}_1 provides a full quantization of $(C^\infty(T^2), \mathcal{F}_1)$.^j On the other hand, there are prequantizations which do not represent $C^\infty(T^2)$ irreducibly, for instance, the prequantization of Avez.^{9,15} Consequently such a prequantization cannot yield a full quantization of $(C^\infty(T^2), \mathcal{F})$ for any subset \mathcal{F} . Notice, however, that Avez' prequantization is not a Kostant-Souriau prequantization.

An alternate approach to the quantization of the torus, which touches upon some of the issues discussed here, and which contains applications to the quantum Hall effect, is given in the recent paper by Aldaya et al.¹⁶

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ⁱThis follows immediately from Theorem 1.

^jI am indebted to G. Tuyenman for clarifying this point.

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