TIME and SINGULARITY*

Jacques Demaret
Institut d'Astrophysique, Université de Liège,
B-4200 Cointe-Ougrée, Belgium

Mark J. Gotay
Department of Mathematics and Statistics, University
of Calgary, Calgary, Alberta, Canada T2N 1N4

Abstract: We show that the occurrence of quantum gravitational collapse and, more generally, the validity of Wheeler's "rule of unanimity" are inextricably linked to the classical choice of time. The crucial distinction is between "fast" and "slow" times, that is, between times which give rise to complete or incomplete classical evolution respectively. We conjecture that unitary slow-time quantum dynamics is always non-singular, while unitary fast-time quantum dynamics inevitably leads to collapse. These findings are illustrated by an analysis of the dust-filled Friedmann-Lemaître-Robertson-Walker universes.

* Dedicated to the memory of a colleague and friend, Arsène Boury (J.D.)

A. Berger (ed.), The Big Bang and Georges Lemaitre, 123–131.

It was a most ancient ... tradition amongst the Pagans ... that the cosmogonia ... took its first beginning from a chaos.

Cudworth (1678)

What is the genesis of the Universe? This fundamental question has never ceased to arouse man's curiosity and fascinate theologians and scientists alike. That there are two distinct issues here, the *beginning* of the Universe as opposed to its *creation*, was already pointed out by St. Thomas Aquinas 700 years ago. The creation of the Cosmos is an a priori philosophical concept, and its elucidation belongs to the realm of metaphysics. In contrast, the beginning of the Universe is an empirical concept and thus amenable to scientific analysis.[1]

With the prodigious development of relativistic cosmology since 1915, the traditional Western belief in the permanence of the heavens has gradually yielded to the notion that the Universe had an absolute beginning. Indeed, the first successful relativistic model of our expanding Universe, due to Friedmann, possessed an infinite density, infinite curvature cataclysm a finite proper time in the past. Insofar as one can reject the possibility of a closed cyclic universe with an infinitude of past cycles -- the "phoenix" universe of Monseigneur Lemaître -- such an *initial singularity* must represent the beginning of the Universe.[2]

Although the first detailed analysis of this phenomenon of catastrophic spacetime collapse was given by Monseigneur Lemaître, it was not realized until the late 1960s to what extent singularities form an essential element of modern cosmology [4]. In view of the celebrated theorems of Hawking, Penrose and Geroch, it is now clear that singularities must occur in all "physically reasonable" spacetimes. Moreover, the existence and isotropy of the cosmic microwave background strongly imply the presence of a singularity in the past of our Universe.

Still, the case for the initial singularity is not ironclad, since the singularity theorems are classical constructs and as such do not take into account quantum phenomena which are expected to be important during the exotic early stages of the Universe. As one extrapolates further into the past, it is therefore conceivable that quantum effects could modify -- or perhaps prevent altogether -- gravitationally induced spacetime collapse. On the other hand, Wheeler [5] has recently proposed a "rule of unanimity" which, if valid, would shatter this hope: "Given that all solutions of the equations of motion run into a singularity (or are free of singularity) except a set of measure zero. Then all solutions of the corresponding quantum-mechanical problem are singular (or free of singularity)."

In the absence of a complete consistent quantum theory of the gravitational field and its interactions, research on quantum singularity avoidance has proceeded along two lines: a search for semi-classical mechanisms for suppressing the formation of singularities and model calculations within the framework of quantum cosmology. Despite considerable effort, however, a satisfactory resolution of the quantum collapse problem remains a chimera. On the semiclassical level, in which the matter is quantized but the gravitational field is treated classically, attempts to eliminate the classical singularities by inducing violations of the positive energy conditions in the singularity theorems remain inconclusive [6]. Quantum cosmological studies, which include the quantum effects of both matter and gravity in the analysis (albeit at the expense of "freezing out" all but a finite number of degrees of freedom), are similarly beset with a host of technical and conceptual difficulties [7]. The foremost among these stems from the freedom in the classical choice of time: different such choices often lead to wildly divergent quantum behaviors [7-15].

The work which we now briefly summarize [8] is devoted to a study of the relationship between quantum gravitational collapse and the choice of time. We find that whether *quantum* collapse occurs is effectively predetermined, on the *classical* level, by this very choice. The crucial distinction is between *fast* and *slow* times, that is, between times which give rise to complete or incomplete classical evolution respectively.[3] More precisely, we conjecture that *unitary slow-time quantum dynamics is always non-singular*, while *unitary fast-time quantum dynamics inevitably leads to collapse*. These results indicate that the quantum collapse question is really quite intricate and also help to reconcile the heretofore bewildering array of "answers" to this question.

We substantiate these contentions with an analysis of the classically collapsing Friedmann-Lemaître-Robertson-Walker (FLRW) universes in two time gauges, one fast and the other slow (cf. [8]).[4] These homogeneous and isotropic cosmologies are described by the metrics

$$ds^2 = -N(t)^2 dt^2 + e^{2\mu(t)} d\Sigma^2 ,$$

where $d\Sigma^2$ is the line element for a 3-manifold of constant curvature $k = +1$, 0 or -1. The matter content is taken to be dust with density ρ and 4-velocity $u = -d\varphi$, φ being the only nonvanishing Seliger-Whitham velocity potential. The super-hamiltonian constraint, characteristic of the general relativistic Hamiltonian formalism, is

$$p_\varphi - \frac{1}{24} e^{-3\mu} p_\mu{}^2 - 6ke^{\mu} = 0 , \qquad (1)$$

where

$$p_\mu = -\frac{12}{N} e^{3\mu}\dot{\mu} \quad \text{and} \quad p_\varphi = \rho u^0 N e^{3\mu}$$

are the momenta canonically conjugate to μ and φ, respectively. Since these models are in parametrized form, they admit an Arnowitt-Deser-Misner reduction. This consists of two steps: choosing a time t and then solving the constraint (1) in the form $p_t - H = 0$, thereby determining the effective Hamiltonian H.

We first choose the time from among the matter variables: $t = -\varphi$. This is essentially cosmic time, and hence slow. After performing the canonical transformation

$$x = \frac{4}{3}\sqrt{6}e^{3\mu/2}, \quad p_x = \frac{\sqrt{6}}{12} e^{-3\mu/2} p_\mu ,$$

reduction yields the phase space $(0,\infty) \times \mathbb{R}$ and the Hamiltonian

$$H(x,p_x) = p_x{}^2 + Kx^{2/3} , \tag{2}$$

where $K = \frac{3}{2}\sqrt[3]{6k}$. The dynamics is thus equivalent to that of a particle on the half-line $(0,\infty)$ moving in a potential $V(x) = Kx^{2/3}$.

Upon quantizing we find that the quantum Hilbert space is $L^2(0,\infty)$ and that the operator corresponding to the Hamiltonian (2) has an infinite number of self-adjoint extensions

$$\hat{H}_\alpha = -\hbar^2 \frac{d^2}{dx^2} + Kx^{2/3}$$

determined by the boundary conditions

$$\psi'(0) = \alpha\psi(0) , \tag{3}$$

where the parameter $\alpha \in (-\infty,\infty]$. Since the operators $\hat{H}_\alpha$ are rather complicated, we illustrate the qualitative features of the dynamics via the motion of a $k = 0$ wave packet.

Fix $\beta = b + iB$, $b > 0$, and consider the initial state

$$\psi(x,0) = \left(\frac{8b}{\pi}\right)^{\frac{1}{4}} e^{-\beta x^2} . \tag{4}$$

Taking $\alpha = 0$, this evolves according to

$$\psi(x,t) = \left(\frac{8b}{\pi}\right)^{\frac{1}{4}} [1 + 4i\hbar\beta t]^{-\frac{1}{2}} e^{-\beta x^2/[1+4i\hbar\beta t]} . \tag{5}$$

To check for collapse, we study the expectation value

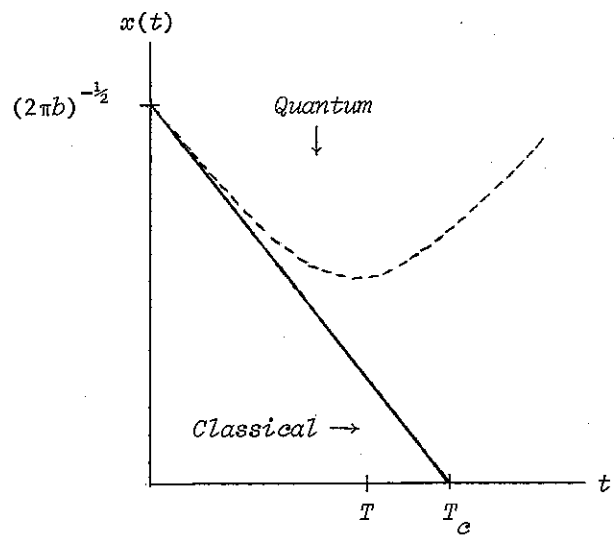
$$\langle\psi(t)|\hat{x}|\psi(t)\rangle = (2\pi b)^{-\frac{1}{2}}[1 - 8B\hbar t + 16(b^2+B^2)\hbar^2 t^2]^{\frac{1}{2}}$$

of the quantum "radius" operator, recalling that classically $x \propto e^{3\mu/2}$ measures the expansion of our model.

If $B > 0$, this wave packet represents a universe which is initially contracting. But as t approaches the "turn-around time"

$$T = B/4(b^2+B^2)\hbar \ ,$$

quantum effects are decisive: the universe decelerates, "bounces," and expands thereafter! In contrast, the classical model corresponding to the initial state (4) contracts uniformly and collapses after a time $T_c = 1/4B\hbar > T$. This behavior is displayed below.



Classical/quantum correspondence for the wave packet (5)

In fact, this instance of quantum singularity avoidance is not exceptional: since $\hat{x}$ is a positive operator and as each $\hat{H}_\alpha$ is self-adjoint, $<\psi(t)|\hat{x}|\psi(t)>$ can *never* vanish in finite time for *any* evolving state $\psi(t)$. Consequently, *no nontrivial state can evolve into a singularity* so that, within this dynamical framework, *quantum gravitational collapse is strictly forbidden.*

An unexpected corollary is that *this phenomenon of quantum singularity avoidance is independent of the choice of boundary condition* (3). This is contrary to widespread belief, which holds that an evolving state $\psi(x,t)$ is non-singular if and only if $\psi(0,t) = 0$ for all t [9]. In particular, note that the wave packet (5) is certainly non-singular, even though $\psi(0,t) \neq 0$ always.

Another important consequence of our analysis is the breakdown of Wheeler's rule of unanimity. To regain it there are only two options: modify either the classical or the quantum formalism. But which one, and how? The key observation is that, ultimately, the cause of the disparity between the classical and the quantum predictions is that *the quantum evolution persists eternally, whereas the classical evolution does not.* Since self-adjointness *guarantees* that the quantum dynamics is defined for all time,[5] it is apparent that we should "complete" the classical dynamics.[6]

As this "paradox" of incomplete classical versus complete quantum evolution arises whenever one makes a *slow* choice of time, one might expect results more in agreement with the unanimity principle when one quantizes in a *fast*-time gauge. Then both the corresponding classical and quantum dynamics are complete, although the physical implications of this completion are rather surprising. Classically, of course, the system is still singular. Quantum mechanically, however, completeness in fast time has a quite different meaning than it does in slow time. Eternal slow-time quantum evolution implies that collapse is impossible. But quantum completeness in fast time, being physically equivalent to incompleteness in slow time, can only signal the presence of a singularity. In other words, it is plausible that fast-time quantum dynamics incorporates collapse in much the same way that slow-time dynamics prohibits it.

We verify this assertion for the $k = -1,0$ dust-filled FLRW universes in the intrinsic-time gauge $t = \mu$. Since $t = -\infty$ corresponds to the initial singularity, $t = \mu$ is a fast clock. The reduced phase space is $\mathbb{R} \times (0,\infty)$ and, from (1), the effective Hamiltonian is

$$H(\varphi,p_\varphi,t) = 2\sqrt{6}e^{3t/2}[p_\varphi - 6ke^{t}]^{\frac{1}{2}} . \qquad (6)$$

Quantizing in the momentum representation, the time-dependent quantum Hamiltonian $\hat{H}(t)$ is represented by multiplication by $H(t)$ on the Hilbert space $L^2(0,\infty)$.

It is straightforward to check that the resulting quantum dynamics is unitary, so that these models must evolve to the $t = -\infty$ limit. Furthermore, since classically $H(t) \to 0$ as $t \to -\infty$ and as $\hat{H}(t)$ is a positive operator, the expectation value $\langle\psi(t)|\hat{H}(t)|\psi(t)\rangle$ is a good indicator of quantum collapse. Then (6) and the dominated convergence theorem yield

$$\lim_{t \to -\infty} \langle\psi(t)|\hat{H}(t)|\psi(t)\rangle = 0 ,$$

so that these models asymptotically collapse. But asymptotic collapse in this fast intrinsic time means that *these quantum models become singular in finite proper time* so that, within this dynamical setting, *quantum gravitational collapse is inevitable*.

As this analysis demonstrates, the validity of Wheeler's rule of unanimity depends critically upon the choice of time. This classically innocuous choice is the decisive factor governing the occurrence of quantum gravitational collapse. Although our conclusions are motivated in the context of the FLRW models, a moment's reflection shows that they will apply, *mutatis mutandis*, to any spatially homogeneous cosmology.

Our claim that quantum collapse is strictly forbidden within the slow-time dynamical framework is supported by the work of DeWitt [9], Lund [10] and Lapchinskii and Rubakov [11] on the FLRW universes and Demaret's analyses [12] of several Bianchi models. On the other hand, our contention that the fast-time version of quantum cosmology does not significantly alter the classical behavior near the singularity is consistent with the findings of Misner and Ryan [13], Gotay and Isenberg [14] and Brill [15]. Thus, our conjectures are confirmed for a wide range of both cosmological models and (intrinsic-, extrinsic- and matter-) time gauges.

Of course, it remains to determine which of these classical/quantum formalisms is "correct". Philosophical considerations [2-4] aside, the answer must likely await the development of a complete quantum theory of gravity. Until then, one can only wonder, like philosophers of all ages, whether indeed "... the world has a beginning in time."

Notes

[1] This is so despite Scriven's claim [1] that the origin of the Universe "... is not within the power of science to determine, nor will it ever be." North refutes this assertion in Chap. 18 of [2].

[2] Whether it can be demonstrated, on a purely philosophical basis, that the Universe has either a finite or an infinite past remains open to question [3].

[3] We call a time variable t a *fast time* if the singularities always occur at either $t = -\infty$ or $t = +\infty$. If this is not the case, then t is said to be a *slow time*.

[4]We chose units so that $c = 1$ and $16\pi G = 1$.

[5]It is possible [8] to relax the requirement that the quantum Hamiltonian be self-adjoint by letting α in (3) be complex. The operators $\hat{H}_\alpha$ with Im $\alpha < 0$ will then generate *contraction semigroups* rather than unitary groups. In this case, the quantum models may *asymptotically collapse* in the sense that $<\psi(t)|\hat{x}|\psi(t)> \to 0$ as $|t| \to \infty$, although it still cannot be ensured that an initially contracting state will collapse in *finite* time.

[6]This is in keeping with Lund's suggestion [10] that one should always quantize on a geodesically complete minisuperspace. Here, however, the completion consists of modifying the choice of time rather than the minisuperspace itself.

References

[1] Scriven, L. (1954), Brit. J. Phil. Sci., *5*, p. 181.

[2] North, J.D. (1965), *The Measure of the Universe* (Clarendon Press, Oxford).

[3] Whitrow, G.J. (1978), Brit. J. Phil. Soc., *29*, p. 39; Popper, K. (1978), Ibid., *29*, p. 47; Bell, J. (1979), Ibid., *30*, p. 161; Craig, W.L. (1979), Ibid., *30*, p. 165; Weingard, R. (1979), Ibid., *30*, p. 170.

[4] Misner, C.W. (1969), Phys. Rev., *186*, p. 1328.

[5] Wheeler, J.A. (1977), Gen. Rel. Grav., *8*, p. 713.

[6] Barrow, J.D. and Matzner, R.A. (1980), Phys. Rev. D *21*, p. 336; Birrell, N.D. and Davies, P.C.W. (1982), *Quantum Fields in Curved Space*, Sec. 7.4 (Cambridge U. Press, Cambridge).

[7] MacCallum, M.A.H. (1975), in *Quantum Gravity*, eds. Isham, C.J., Penrose, R., and Sciama, D. (Clarendon Press, Oxford).

[8] Gotay, M.J. and Demaret, J. (1983), Phys. Rev., D*28*, p. 2402.

[9] DeWitt, B.S. (1967), Phys. Rev., *160*, p. 1113.

[10] Lund, F. (1973), Phys. Rev., D*8*, p. 3253.

[11] Lapchinskii, V.G. and Rubakov, V.A. (1977), Teor. i. Mat.

Fiz., *33*, p. 364.

[12] Demaret, J. (1979), Gen. Rel. Grav., *11*, p. 453; (1980), Phys. Lett., *95*B, p. 413; (1980), Bull. Soc. Math. Belgique, *32*, p. 99.

[13] Misner, C.W. (1969), Phys. Rev., *186*, p. 1319; Ryan, M. (1972), *Hamiltonian Cosmology*, Lecture Notes in Physics No. 13 (Springer, New York).

[14] Gotay, M.J. and Isenberg, J.A. (1980), Phys. Rev. D*22*, p. 235; (1980), In Lecture Notes in Physics No. 135, p. 418 (Springer, New York).

[15] Brill, D.R. (1975), In *Quantum Theory and the Structures of Space and Time*, eds. Castell, L., Drieschner, M., and von Weizsächer, C.F. (Carl Hansen Verlag, München).