PRESYMPLECTIC HAMILTON AND LAGRANGE SYSTEMS, GAUGE TRANSFORMATIONS AND THE DIRAC THEORY OF CONSTRAINTS

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1. INTRODUCTION Differential equations of the form

$$i_X\omega = \alpha \tag{1}$$

where α is a closed 1 form (Hamiltonian) and ω is a closed 2 form (Lagrange bracket) frequently arise in physics. If ω is *symplectic* such equations have unique solutions.[1] However ω is often *pre*symplectic (i.e. $\beta : X \to i_X\omega$ is not an isomorphism). This occurs, e.g., (i) for many infinite dimensional systems, (ii) *a priori* - as in Künzle's[2] treatment of a spinning particle in curved space-time, (iii) for systems derived from degenerate Lagrangians (i.e., the fiber derivative[1] $FL : TQ \to T^*Q$ is not a diffeomorphism) as in gravity and electromagnetism. If ω is presymplectic neither existence nor uniqueness of solutions to the system (1) is guaranteed.

We present a geometric constraint algorithm which provides the necessary and sufficient conditions for the existence of solutions to the generalized Hamilton equations (1) on a (finite or infinite dimensional) *pre*symplectic manifold (M,ω). We discuss the application of our algorithm to Lagrangian and Hamiltonian systems and indicate its relationship to the Dirac theory of constraints. We also discuss the non-uniqueness of the solutions of (1) and present an algorithm for determining the associated "space of true degrees of freedom".

2. THE CONSTRAINT ALGORITHM[3] Our constraint algorithm finds the *unique maximal* submanifold $N \xrightarrow{i_N} M$ along which (1) possesses solutions *tangent* to N. This final

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constraint submanifold is the limit $N \equiv \cap M_\ell$ of a string of sequentially constructed constraint submanifolds[4]

$$M_{\ell+1} \equiv \{m \in M_\ell \mid \alpha_m \in \beta(T_m M_\ell)\} \tag{2}$$

(with $M_1 \equiv M$) which follow from applying consistency conditions[5] to (1).

<u>Theorem</u> The presymplectic Hamiltonian system (1) has solutions if N is non-empty, in which case we have consistent equations of motion along the final constraint submanifold N

$$0 = (i_X\omega - \alpha)|_N \in T_N^*M \tag{3}$$

with solutions $X \in TN$.

If β is closed and TM is reflexive Banach (which we henceforth assume) an equivalent but much more convenient characterization of the constraint submanifolds is

$$M_{\ell+1} \equiv \{m \in M_\ell \mid \langle TM_\ell^\perp|\alpha\rangle_m = 0\} \tag{4}$$

where $TW^\perp \equiv \{Z \in TM \mid \omega(Y,Z) = 0 \ \forall\ Y \in TW\}$ is the presymplectic orthogonal complement.

3. <u>DIRAC CONSTRAINT THEORY</u> When applied to systems described by degenerate Lagrangians our algorithm significantly generalizes as well as globalizes the local Dirac-Bergmann (DB) theory of constraints.[6,7] In this case $M \equiv FL(TQ) \xrightarrow{j} T^*Q$ is Dirac's primary constraint submanifold, $\omega \equiv j^*\Omega$ with Ω being the canonical symplectic structure on T^*Q, and $\alpha = dH$ where the Hamiltonian H satisfies

$$H \circ FL = E_L \tag{5}$$

with $E_L(v) \equiv \langle v|FL(v)\rangle - L(v)$.

The DB constraint theory works on all of phasespace (T^*Q) and proceeds to find (analytically) an extension of H to T^*Q and secondary constraints which will guarantee preservation of all constraints. Our (geometric) algorithm works directly on the primary constraint submanifold M. The constraint functions $\langle TM_\ell^\perp|\alpha\rangle$ in (4) correspond to Dirac's secondary constraints.

In recent years there has been some nice work done in giving a <u>symplectic</u> geometric description of the <u>results</u> of the DB constraint theory. (See the works of Śniatycki[8] Tulczjew,[9] Lichnerowicz[10] Marsden and Weinstein[11] and ref. 3.) Our work (following the presymplectic approach of Hinds[12]) geometrizes the constraint algorithm itself and thereby complements and extends this description.

In constrast to the Dirac method, our algorithm does not *a priori* require M to be a submanifold of some given symplectic manifold (e.g. T*Q). This fact is responsible for the wide range of applicability of the algorithm.

4. LAGRANGIAN SYSTEMS Our constraint algorithm can be applied directly to the inherently presymplectic Lagrange equations

$$i_X \Omega_L = dE_L \tag{6}$$

on velocity phase space TQ. The presymplectic form Ω_L is $FL^*\omega$ or equivalently $-dJ^*dL$ where J^* is the adjoint of Klein's[13] *almost tangent structure*

$$J : TTQ \longrightarrow TTQ$$

which satisfies

$$\ker J = \operatorname{Im} J = V(TQ) \equiv \ker T\tau_Q \ .$$

The ability of our algorithm to deal directly with Lagrangian systems on TQ is of particular importance because the fiber derivative FL may be too pathological for a Hamiltonian formalism to exist in T*Q. (For example (5) may not define a single valued function H.) There are two special cases of interest.

Definition L is *almost regular*[14] if FL is a submersion onto its image and the fibers of FL are connected.

Definition L is *quasi-regular* if the leaf space $TQ/(\ker FL_*)$ of the foliation of TQ generated by the involutive distribution $\ker FL_*$ is a manifold such that the canonical projection is a submersion. (Note almost regular implies quasi-regular.)

Theorem[15] If L is almost regular there exists a corresponding Hamiltonian formulation of the dynamics on a submanifold of T*Q and the Lagrangian and Hamiltonian formalisms are *equivalent* (i.e. each constraint submanifold of the Lagrangian system maps onto a corresponding constraint submanifold for the Hamiltonian system).

Theorem If L is quasi-regular there is a corresponding equivalent generalized Hamiltonian system; however it is not given on a submanifold of T*Q but rather on the leaf space $TQ/(\ker FL_*)$.

4A. Second Order Equation Problem Lagrangian systems differ from Hamiltonian systems in that from both physical[1] and variational[16] viewpoints the Lagrange equations (6) must be supplemented[17] by the 2nd order equation condition

$$T\tau_Q X = \tau_{TQ} X \quad . \tag{7}$$

This condition was first investigated for homogeneous degenerate Lagrangians by Künzle[18] who considered only degeneracies which arise from the homogeneity. He developed an algorithm which in principle solves the problem, but he gave no specific results in the general case. We have taken a different approach and considered the general degenerate Lagrangian, finding that (7) is compatible with our constraint algorithm.

Theorem[19] If L is quasi-regular there exists a 1 to 1 correspondence between solutions to (6,7) and solutions to the corresponding generalized Hamiltonian system on $TQ/(\ker FL_*)$.

5. NON-UNIQUENESS OF SOLUTIONS Presymplectic Hamiltonian systems in general do not have unique solutions. To any vector field $X \in TN$ satisfying (3) along the final constraint submanifold $N \xrightarrow{N} M_1$ one can add any element Y in

$$K_1 \equiv TN \cap TM_1^{\perp} \quad . \tag{8}$$

(These vector fields are generated by Dirac's primary 1st class constraints.) The presence of these "gauge" vector fields indicates that N contains redundant information. For certain purposes (e.g. quantization) one would like to eliminate this redundant information by constructing a *reduced phase space* (RPS) i.e. a "space of true degrees of freedom". One would like the RPS to be a symplectic manifold with a unique evolution determined by a Hamiltonian system.

5A. Standard RPS A reduced phase space can be constructed as follows: pull back (3) to N obtaining

$$i_X \omega_N = \alpha_N \tag{9}$$

where $\alpha_N \equiv i_N^* \alpha$ and $\omega_N \equiv i_N^* \omega$. The solutions to (9) are unique up to elements in

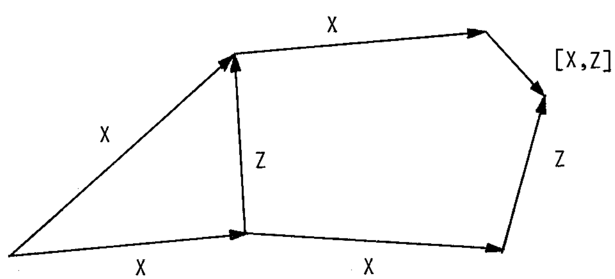
$$K_N \equiv TN \cap TN^{\perp}, \tag{10}$$

the characteristic vector fields of ω_N. These vector fields form an involutive distribution which determines a foliation of N. On the leaf space of this foliation $\overline{N} \equiv N/(K_N)$ there is a symplectic (hence unique solution) Hamiltonian system

$$i_{\overline{X}}\omega_{\overline{N}} = \alpha_{\overline{N}} \tag{11}$$

induced by (9) - i.e. our constraint algorithm guarantees that α is *admissible* for N in the sense of Lichnerowicz.[10] We call $\overline{N}$ the *standard* RPS. Regarding $\overline{N}$ as the "space of true degrees of freedom" of the system is equivalent to regarding all of K_N as gauge vector fields which corresponds to Dirac's extended Hamiltonian using all 1st class constraints (primary and secondary) as gauge generators. Dirac himself was uncertain as to whether this was correct - "I think it may be that all the first-class secondary constraints should be included among the transformations which don't change the physical state, but I haven't been able to prove it."[20]

5B. Gauge Vector Field Algorithm (GVA) RPS[21] In order to see how the manifest K_1 gauge invariance of (3) could imply K_N gauge invariance we have developed a gauge vector field algorithm (GVA) for extracting the "hidden" gauge transformations in (3). The associated GVA RPS is obtained by "moding out" the "gauge directions" in N: i.e. one wants to identify points connected by "gauge vector fields" (GVF's) since such points can be reached from the same initial data. The set of GVF's can be considerably larger than K_1 for if Z is a GVF then so is [X,Z] where X is any solution to (3). This follows from the diagram



A similar construction shows that [Z,Y] is a GVF whenever Z and Y are GVF's. Starting from $G_1 \equiv K_1$ (the manifestly obvious GVF's) and iterating one obtains the *implicit* GVF's

$$G_{\ell+1} \equiv G_\ell + [G_\ell, G_\ell] + [X, G_\ell] \tag{12}$$

whose limit is an involutive distribution G - the Lie module generated by $(\mathcal{L}_X)^\ell K_1$. The associated GVA RPS is the leaf space $\tilde{N} \equiv N/(G)$ of the foliation generated by G. Is the GVA RPS the same as the standard RPS?

<u>Theorem</u> $G = K_N$ iff the *first class constraints* $\langle K_\ell | \alpha \rangle$ are "good" constraints (i.e. their differentials do not vanish along N) where $K_\ell \equiv TN \cap TM_\ell^\perp$.

In general G is a proper Lie submodule of K_N (i.e. some 1st class secondary constraints do not generate gauge transformations of (3)) and consequently $\tilde{N}$ is larger than $\bar{N}$. In general $\tilde{N}$ is <u>inherently</u> presymplectic. (In fact $\tilde{N}$ can be odd dimensional!) Whenever $G \neq K_N$ there <u>is</u> <u>no</u> <u>natural</u> symplectic Hamiltonian system on $\tilde{N}$ for the *unique* evolution $\tilde{X} \in T\tilde{N}$. Consequently regarding $\tilde{N}$ as the space of true degrees of freedom has some undesirable features (e.g. How would one quantize such a system?). On the other hand $\tilde{N}$ has the virtue of literally respecting the equations of motion (4).

Which RPS is the "space of true degrees of freedom"?

5C. <u>A Possible Resolution</u> If the Hamiltonian contains a built in gauge condition then G will respect it while K_N will not. So it may be possible to interpret $G \neq K_N$ as a consequence of built in gauge conditions. Two examples may clarify this point. Consider the Lagrangian densities

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\lambda} (\partial_\mu A^\mu)^2 \tag{13}$$

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu A_\nu) \partial^\mu A^\nu - A^\mu \partial_\mu \phi - \frac{\lambda}{2} \phi^2 \quad . \tag{14}$$

where λ is a Lagrange multiplier and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. From (13) one obtains Maxwell's equations and the Lorentz condition $\partial_\mu A^\mu = 0$. If we <u>interpret</u> this system as electromagnetism with the Lorentz gauge condition imposed then it is easy to understand why $G \neq K_N$, for G respects the Lorentz condition while K_N does not. The Lagrangian density (14) in fact yields the same effective equations as (13), though now one may be tempted to interpret it as what it appears to be - not E&M but an entirely different (massless, spin 1, divergence free) field. This is the GVA interpretation of both (13) and (14). Thus whether or not Dirac's conjecture is correct (and which RPS is correct) depends on one's physical interpretation of the system. In general our algorithm permits a whole class of interpretations bounded by one (G) which extracts the gauge invariance of a given Lagrangian (and yields an inherently presymplectic RPS) and the standard one (K_N) which extracts the maximum possible invariance of the given system (and has a symplectic Hamiltonian system for the unique evolution on its RPS). The former violates Dirac's conjecture while the latter satisfies it by interpreting $G \neq K_N$ to be entirely a consequence of built-in gauge conditions.

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   P. Chernoff and J. Marsden, Properties of Infinite Dimensional Hamiltonian Systems, Lecture notes in Math #425, Springer-Verlag 1974.
2. H.P. Künzle, J. Math. Phys. 13, 739 (1972).
3. M. Gotay, J. Nester, G. Hinds, "Presymplectic Manifolds and the Dirac-Bergmann Theory of Constraints", to appear J. Math. Phys. (1978).
4. We assume here and throughout that quotients and constraints yield sufficiently nice manifolds. For many interesting systems this is not true - one must then cut the system up into nice pieces. See refs. 8 and 12.
5. Actually this is a particular application of a general integrability algorithm that we have developed for systems of equations.
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13. J. Klein, Symposia Mathematica 14, 181 (1974).

14. This generalizes Śniatycki's definition.[8]

15. M. Gotay and J. Nester, "Presymplectic Lagrangian Systems I: The Constraint Algorithm and the Equivalence Theorem", submitted to Ann. Inst. Henri Poincaré.

16. J. Nester, "An Invariant Derivation of the Euler Lagrange Equations", in preparation.

17. If L is degenerate (7) is <u>not</u> a consequence of (6).

18. H.P. Künzle, Ann. Inst. Henri Poincaré <u>A11</u>, 393 (1969).

19. J. Nester and M. Gotay, "Presymplectic Lagrangian Systems II: The Second Order Equation Problem", in preparation for Ann. Inst. Henri Poincaré.

20. See ref. 5, pp. 23,24.

21. J. Nester and M. Gotay, "Presymplectic Hamilton Equations: Gauge Transformations, and the Reduced Phase Space", in preparation.