STRESS-ENERGY-MOMENTUM TENSORS & THE BELINFANTE-ROSENFELD FORMULA

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Joint work with

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Metatheorem:

MOMENTUM MAPS ARE EVERYTHING!

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- $\mathcal{T}^{\mu}{}_{\nu}$ is a component of J
- automatically capture the "Belinfante–Rosenfeld formula"

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Attempts to repair [Anderson,. . .] ad hoc.

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- A century-old problem!

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- Then the Hilbert formula follows as a theorem, not a definition
- $\mathfrak{T}$ will satisfy a generalized version of the B–R formula in *all* cases

Setup

Covariant field theory à la GiMmsy:

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- $Z = J^\star Y$ is the **multi-phase space** ('covariant' cotangent bundle) with w/ fiber coordinates $\phi^A, p_A{}^\mu, p$

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- Finally, the 'covariant Legendre transform' $\mathbb{F}\mathcal{L} : JY \to Z$ is

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- $J^{\mathcal{L}} = \mathbb{F}\mathcal{L}^* J$ (formula later) is the multimomentum map in the Lagrangian representation

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- More precisely, assume there is a group embedding $\mathrm{Diff}_c(X) \to \mathcal{G}$. Infinitesimally, $\mathfrak{X}_c(X) \to \mathrm{Lie}(\mathcal{G}) \subset \mathrm{aut}(Y)$ is

$$\xi \mapsto \xi_Y = \xi^\mu \partial_\mu + \xi^A(x^\mu, \phi^A, [\xi^\nu])\partial_A$$

where ξ^A is a linear differential function of the ξ^μ:

$$\xi^A = B^A_\mu \xi^\mu + C^{A\nu}_\mu \xi^\mu{}_{,\nu} + D^{A\nu\tau}_\mu \xi^\mu{}_{,\nu\tau} + \cdots$$

- $B \sim$ continua with internal structure
- $C \sim$ tensor fields
- $D \sim$ linear connections
- ...

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- So

$$\xi_Y = \xi^\mu \partial_\mu - \left(A_\beta \xi^\beta{}_{,\alpha} + \lambda_{,\alpha\beta} \xi^\beta + \lambda_{,\beta} \xi^\beta{}_{,\alpha} \right) \frac{\partial}{\partial A_\alpha}$$

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Theorem: Suppose $\mathcal{L}$ is $\mathcal{G}$-covariant. Then for any hypersurface $i_\Sigma : \Sigma \to X$,

$$\boxed{\int_\Sigma {i_\Sigma}^*(j\phi)^*\langle J^{\mathcal{L}}, \xi\rangle = \int_\Sigma \mathfrak{T}^\mu{}_\nu(\phi)\xi^\nu d^n x_\mu}$$

uniquely defines the SEM tensor (density) $\mathfrak{T}(\phi)$.

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yields the…

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But *why* does it work?

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$$\frac{1}{8\pi}\mathfrak{G}^\mu{}_\nu \xi^\nu = \mathfrak{T}^\mu{}_\nu \xi^\nu$$

The $\mathfrak{G}^\mu{}_\nu$ are thus the Noether conserved quantities for vacuum gravity corresponding to $\mathrm{Diff}_c(X)$.

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- Higher derivative correction terms $\sim D, E, \ldots$ are 'hidden' in the symmetric part of $\partial_\alpha(C_\nu^{A(\mu} L_A{}^{\alpha)})$

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  for all ‘internal’ gauge transformations $\eta \in \mathcal{G}_{\mathrm{Id}}$

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- No metric—necessarily—in any of this!

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With the nonstandard imbedding $\mathrm{Diff}_c(X) \to \mathcal{G}$ described earlier

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& + F^{\alpha\mu}{}_{;\alpha}(A_{\nu} + \lambda_{,\nu}) && \longleftarrow C \\
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gives the correct result.

The Hilbert Formula

Theorem If the system is coupled to a metric g on X, then

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Proof:

Covariance of $\mathcal{L}$ under $\eta \in \mathrm{Diff}_c(X)$ means

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Use above expression for ξ^A and group; the coefficients of

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must individually vanish. After rewriting, get [at most second derivatives in ξ^A for simplicity]:

$$3^{\text{rd}}: \qquad D_\alpha^{A(\rho\tau} L_A{}^{\mu)} = 0$$

$$2^{\text{nd}}: \qquad \partial_\mu \left(D_\alpha^{A\rho\tau} L_A{}^\mu \right) + C_\alpha^{A(\rho} L_A{}^{\tau)} = 0$$

$$1^{\text{st}}: \qquad L\delta^\rho{}_\alpha - L_A{}^\rho \phi^A{}_{,\alpha} + L_A{}^\rho B_\alpha^A + \partial_\mu \left(C_\alpha^{A\rho} L_A{}^\mu \right) - 2\frac{\delta L}{\delta g_{\rho\tau}} g_{\alpha\tau} = 0$$

$$0^{\text{th}}: \qquad L_{,\alpha} + \partial_\rho \left(B_\alpha^A L_A{}^\rho \right) = 0$$

and compare with what went before.

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- 3^{rd} & 2^{nd} are used to prove the main theorem stated previously.

Examples:

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  In the usual 1×1 formulation ('material representation'), with $X = \mathbb{R}$, the scalar $\mathfrak{T} = 0$ as the system is parametrized.

  But in the 4×4 formulation ('inverse material representation'), with X being Minkowski spacetime, and $z : \mathbb{R} \to X$ the particle placement field, $\mathfrak{T}$ is the *kinetic tensor*

$$\Theta^{ab}(x) = \int_{\mathbb{R}} \frac{\dot{z}^a(\lambda)\dot{z}^b(\lambda)}{\|\dot{z}(\lambda)\|} \, \delta^4(x - z(\lambda)) \, d\lambda.$$

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  - Note that $\mathfrak{T}$ vanishes on shell (viz., when $F = 0$) — TFTs have no local physics
  - But there is *global* physics:

$$\int_\Sigma {i_\Sigma}^*(jA)^*\langle J^{\mathcal{L}}, \xi\rangle = -\int_{\partial\Sigma} (\xi \lrcorner A)A$$

for $\xi \in \mathfrak{X}(X)$, not just $\mathfrak{X}_c(X)$! The RHS is 'topological energy-momentum at infinity'

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  - Gravitational energy can't be localized
  - No pseudo-tensors
  - In Schwarzschild, the boundary term gives the Komar mass