KOSTANT-SOURIAU QUANTIZATION OF ROBERTSON-WALKER COSMOLOGIES WITH A SCALAR FIELD

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INTRODUCTION

The application of the Kostant-Souriau method of quantization[1] to homogeneous cosmologies enables us to study two important problems of theoretical physics: (1) the compatibility of gravity and quantum theory, and (2) the usefulness of the K-S procedure for quantizing classical systems. The K-S technique is an important tool for studying quantum gravity since (as we show) it quantizes some gravitational systems which cannot be easily handled by traditional methods. The homogeneous cosmologies, on the other hand, are interesting systems for testing the K-S method as they are simple, yet nontrivial and in some sense physically realistic.

The homogeneous cosmologies[2] have long been used in studies of the quantization of gravity, since their phasespaces are finite-dimensional. By using them instead of general spacetime models, we can set aside the difficulties inherent to systems with an infinite number of degrees of freedom, and focus instead on problems associated with the choice of gauge, the constraints, and the nonlinearities of Einstein's theory. These problems are severe within the traditional quantization schemes: (a) If we make no choice of gauge, we have a vanishing Hamiltonian. It can only generate quantum evolution via some Klein-Gordon perspective with its attendant complications; (b) If we do choose a gauge then the Hamiltonian tends to be time-dependent, non-commuting for different times, and square root in form. The Schrodinger equation is then nearly impossible to solve; (c) Different choices of gauge give us various (usually inequivalent) quantizations; (d) Factor-ordering ambiguities also lead to different quantum systems.

While the K-S quantization procedure avoids some (and perhaps all) of these problems, it introduces a few new ones. In particular, for a given classical system there may exist more than one of each of the necessary prequantization line bundles, metaplectic frame bundles, and quantum Hilbert spaces. We thereby obtain inequivalent quantizations. Further ambiguities are introduced due to the freedom in the choice of polarization. However, unlike their counterparts in canonical quantization schemes, the inequivalent quantizations of the K-S method can be classified, at least partially, by specific Čech cohomology classes. This classification, however, does not apply to the alternate choices of polarization. Nor does it in any way indicate which one of the possible quantizations is "physically correct."

CLASSICAL RWϕ MODELS

The "RWϕ" model cosmologies -- Robertson-Walker spacetimes containing a Klein-Gordon scalar field coupled to gravity -- are of special interest as they are the simplest cosmologies which are dynamically nontrivial. The classical system corresponding to the RWϕ universe, described by the metric

Published in: Group Theoretical Mathods in Physics, W. Beigelbock,
A. Bohm & E. Takasugi, eds., Springer LNP 94, 293-295 (1979).

$$ds^2 = - N^2(t)dt \otimes dt + R^2(t)g_{ij}\,\sigma^i \otimes \sigma^j,$$

consists of a 6-dimensional phasespace T^*R^3, a symplectic form

$$\omega = h^{-1}(d\pi_N \wedge dN + d\pi_R \wedge dR + d\pi_\phi \wedge d\phi), \qquad (1)$$

a Hamiltonian

$$H = - N\kappa = - N\{\frac{1}{24R}\ \pi_R^2 - \frac{1}{2R^3}\ \pi_\phi^2 + 6KR - \frac{1}{2} m^2\phi^2R^3\}, \qquad (2)$$

and the set of constraints

$$\pi_N = 0 \quad , \quad \kappa = 0. \qquad (3a,b)$$

Here ϕ is the scalar field with mass m, and h is Planck's constant.

The vanishing Hamiltonian indicates that the system is in parameterized form, and therefore admits reduction via "choice of time."[2] This is effected by specifying t as a function of the canonical variables, solving the constraint (3b) in the form $\kappa = \pi_t + H = 0$, and then eliminating the redundant variables. One obtains a two-dimensional phasespace M with Hamiltonian $H = - \pi_t$. There are many possible choices of time, including $t = R$, $t = \pi_R$ and $t = \phi$. We are studying the geometric quantization of all of these, as well as that of the unreduced (parameterized) system itself. Here, we restrict our attention to one specific reduction.

GEOMETRIC QUANTIZATION OF INTRINSIC-TIME MODEL

If we choose an "intrinsic-time" gauge $t = R$, and carry through the standard reduction, we obtain a 2-dimensional phasespace M [coordinates (ϕ,π_π)]. The symplectic structure on M is $\omega = h^{-1}d\pi_\phi \wedge d\phi$, and the unconstrained Hamiltonian is

$$H = [24t(\frac{1}{2t^3}\ \pi_\phi^2 + \frac{1}{2} m^2t^3\phi^2 - 6Kt)]^{1/2} .$$

Choosing $K = -1$ or 0 to ensure that H is well-defined, and treating t simply as a parameter, we see that in the massless case this system resembles a free particle, while in the massive case it mimics a harmonic oscillator.

In the massless case, the phasespace can be taken to be R^2 and the simplest choice of the polarization P (which diagonalizes H) is the horizontal polarization. It follows that the prequantization line bundle L, the metaplectic frame bundle, and the bundle of 1/2-forms $N^{1/2}(P)$ must be trivial. The corresponding quantum pre-Hilbert space consists of compactly supported (modulo P) sections of $L \otimes N^{1/2}(P)$ of the form:

$$\psi = f(\pi_\phi)exp[(-i/\hbar)\pi_\phi\phi].\nu,$$

where f is arbitrary, and ν is the appropriate 1/2-form.

Since the polarization diagonalizes H, the Hamiltonian operator on our Hilbert space acts simply by multiplication $\tilde{\delta}_H\psi = H.\psi$. This operator commutes for different times, and we can solve the Schordinger equation by expanding in an evolving complete set of energy eigenfunctions $\{\psi_{E(t)}\}$. These eigenstates satisfy

$\psi_{E(t)} = exp[\frac{i}{\hbar} \int_{t_o}^{t} E(s)ds]\psi_{E(t_o)}$, where $\tilde{\delta}_H(t_o)\psi_{E(t_o)} = E(t_o)\psi_{E(t_o)}$. From (4), we find that

$$\psi_{E(t_o)} = f(\pi_\phi) exp[(-i/\hbar)\pi_\phi \phi]\delta(24t_o\ \{(\pi_\phi^{\ 2}\ /(2t_o^3) - 6Kt_o\} - E^2(t_o)).\nu$$

and that the spectrum of $H^2(t_o)$ is $(-144Kt_o^2\ ,\ \infty)$.

Blyth and Isham have quantized this system in the coordinate representation using canonical techniques.[3] We have demonstrated that the difference between their approach and ours lies in the choice of polarization only. Using the BKS transform we have shown that the K-S results are in exact agreement with theirs.

Blyth and Isham (or anybody else, to our knowledge) have not been able to quantize in the massive case because the canonically quantized Hamiltonian operator does not commute for different times. A major advantage of geometric quantization is that it allows one to quantize using a Hamiltonian polarization (that is, one which contains the Hamiltonian vectorfield of H), in which the K-S version of this operator *does* commute for all times.

Taking $M = R^2-\{0\}$ in the massive case (so that the Hamiltonian polarization is well-defined), the prequantization line bundle is still unique (and trivial), but now there are *two* metalinear frame bundles for every polarization. Following Simms,[1] we find that the quantum Hilbert space (in either case) must be identified with the cohomology group $H^1(M,S_P)$ [since $H^0(M,S_P) = 0$], where S_P is the sheaf of germs of covariantly constant sections of $L \otimes N_{1/2}(P)$. Explicitly, the state functions corresponding to the trivial metalinear frame bundle take the form $\psi = [\sum_{n=0}^{\infty} f(n) e^{in\theta}].\nu$, where n ranges through the positive integers, f is an arbitrary (ℓ^2-bounded) function of n, ν is the appropriate 1/2-form, and θ is related to the original coordinates on phasespace by

$$\theta = \frac{1}{2i} ln[(\pi_\phi - it^6 m^2 \phi)/(\pi_\phi + it^6 m^2 \phi)].$$

For the other metalinear structure, which is a bundle over $R^2-\{0\}$ with one "twist," the wave functions take a similar form; but now $n = 1/2\ ,\ 3/2, \cdots$. For both quantizations, the Hamiltonian is "diagonal". Thus the Hamiltonian operator commutes for different times. and the Schrodinger equation can be solved. Unlike the (evolving) energy eigenstates of the massless case, however, those in the massive case are energy quantized. For the trivial metalinear frame bundle

$$E_n(t) = \frac{2}{t}\sqrt{6}\ [\frac{nt^6m^2}{4\pi} - 6t^4K]^{1/2}.$$

The eigenvalues for the nontrivial structure are obtained by replacing n by $n + 1/2$.

The physical implications of these quantizations are presently under study. We are also comparing these quantizations with those obtained via different choices of polarization and time gauge.

REFERENCES

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