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$\mathbf{R}^{2n}$ is a Universal Symplectic Manifold for Reduction

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Abstract. We show that $\mathbf{R}^{2n}$ with its standard symplectic structure is 'universal' in that, subject to a mild topological restriction, essentially all symplectic manifolds can be obtained from it by reduction.

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1. Introduction

Let (M, ω) be a symplectic manifold and let N be a submanifold. Suppose that the pullback ω_N of ω to N has constant rank and that its characteristic distribution $\ker \omega_N$ is fibrating. Set $M_N = N/(\ker \omega_N)$; it inherits the structure of a symplectic manifold from N. We say that M_N is the *reduction of M by N.*

Many interesting symplectic manifolds arise via reduction [1]. Here we show that essentially *all* symplectic manifolds can be obtained in this way. To this end, we say that a closed k-form Ω on a manifold Q has *finite integral rank* if its de Rham class $[\Omega] \in H^k(Q, \mathbf{R})$ lies in $H^k(Q, \mathbf{Z}) \otimes \mathbf{R}$. Then we prove:

THEOREM 1. *Every symplectic manifold (Q, Ω) with Ω of finite integral rank can be realized as a reduction of some $\mathbf{R}^{2n}$ with its standard symplectic structure.*

Thus, $\mathbf{R}^{2n}$ is *universal* for symplectic geometry insofar as reduction is concerned.

An important instance of reduction is when N is a level set of the momentum map for a Hamiltonian action of a Lie group on M (*Marsden–Weinstein reduction* [2]). In this context, we have the following result:

THEOREM 2. *Every noncanonical cotangent bundle $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$, with Ω a closed 2-form on Q of finite integral rank, can be obtained by a Marsden–Weinstein reduction of some $\mathbf{R}^{2n}$ with its standard symplectic structure relative to the Hamiltonian action of a connected Abelian Lie group.*

We emphasize that the 'finite integral rank' condition in Theorems 1 and 2 is nearly always satisfied. In particular, if Q is of finite type then, by the universal

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coefficient theorem $H^k(Q, \mathbf{R}) = H^k(Q, \mathbf{Z}) \otimes \mathbf{R}$ so that *every* closed k-form has finite integral rank. Since all well-behaved manifolds are of finite type, it is clear that this is a very minor restriction.

2. Proof of Theorem 1

If Q is a manifold, then θ_Q will denote the canonical 1-form on T^*Q and $\tau_Q : T^*Q \to Q$ the cotangent projection. We call $(T^*Q, \mathrm{d}\theta_Q)$ a *canonical* cotangent bundle. If Ω is any closed 2-form on Q, then $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ is also symplectic; we refer to this as a *noncanonical* cotangent bundle. When $Q = \mathbf{R}^n$, we write $(T^*Q, \mathrm{d}\theta_Q) = (\mathbf{R}^{2n}, \mathrm{d}\theta_n)$ and denote the cotangent projection $\mathbf{R}^{2n} \to \mathbf{R}^n$ by τ_n.

The proof will proceed in three stages, in which we successively realize complicated symplectic manifolds as reductions of simpler ones. In the first stage, we show that any symplectic manifold arises as a reduction of a noncanonical cotangent bundle. In the second stage, we realize certain noncanonical cotangent bundles as reductions of canonical ones. In the last stage, we demonstrate that every canonical cotangent bundle can be obtained by reducing some $\mathbf{R}^{2n}$ with its standard symplectic structure. To finish the proof, we combine these three reductions into a single one.

(1) If (Q, Ω) is symplectic then, since the zero section Z_Q is a *symplectic* submanifold of $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$, we have that (Q, Ω) is canonically symplectomorphic to the reduction of $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ by Z_Q.

(2) In this step we utilize a 'toral' generalization of the prequantization procedure of Kostant and Souriau [3]; for this we need the finite integral rank assumption.

Let Ω be a closed (but not necessarily nondegenerate) 2-form on Q. Suppose that Ω is of finite integral rank so that there exist integral closed 2-forms $c_1, \ldots, c_k$ and real constants $\mu_1, \ldots, \mu_k$ such that $\Omega = \mu_1 c_1 + \cdots + \mu_k c_k$.

Now, according to the prequantization construction, the integrality of each c_i guarantees the existence of a principal $\mathbf{T}^1$-bundle $p_i : P_i \to Q$ with connection α_i whose curvature $\mathrm{d}\alpha_i$ satisfies $\mathrm{d}\alpha_i = p_i^* c_i$. Perform this construction for every $i = 1, \ldots, k$, and let $p : P \to Q$ be the fiber product of the corresponding P_i. It is a principal $\mathbf{T}^k$-bundle with connection $\alpha = (\alpha_1, \ldots, \alpha_k)$ and curvature $\mathrm{d}\alpha = (p^*c_1, \ldots, p^*c_k)$.

The cotangent lift of the $\mathbf{T}^k$-action on P to $(T^*P, \mathrm{d}\theta_P)$ is a symplectic action with canonically defined momentum map $J : T^*P \to \mathbf{R}^k$ given by $J(\beta) = (\beta(\xi_1), \ldots, \beta(\xi_k))$ for $\beta \in T^*P$, where ξ_i is the fundamental vector field for the T^k-action on P defined by $\alpha_i(\xi_j) = \delta_{ij}$. By the cotangent bundle reduction theorem [1, 4], it follows that $(T^*P, \mathrm{d}\theta_P)$ reduced by the coisotropic submanifold $J^{-1}(\mu_1, \ldots, \mu_k)$ is symplectomorphic to $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*(\mu_1 c_1 + \cdots + \mu_k c_k))$. Thus we have proved Proposition 3.

PROPOSITION 3. *Provided Ω is of finite integral rank, every noncanonical cotangent bundle $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ arises as a Marsden–Weinstein reduction of some canonical cotangent bundle $(T^*P, \mathrm{d}\theta_P)$ relative to a Hamiltonian torus action.*

REMARK. In general, for each i neither P_i nor α_i (and hence neither P nor α) is unique. Nonetheless, for a fixed choice of (P, α), the proof of the cotangent bundle reduction theorem shows that the identification of the reduced space with $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ is canonical.

(3) This can be accomplished using the notion of 'holonomic constraints' familiar from mechanics [1]. Take any embedding $P \to \mathbf{R}^n$ for some appropriate n, and observe that $(T^*P, \mathrm{d}\theta_P)$ is symplectomorphic to the reduction of $(\mathbf{R}^{2n}, \mathrm{d}\theta_n)$ by the coisotropic submanifold $\tau_n^{-1}(P)$.

To prove Theorem 1, it suffices to consolidate these steps. The following result shows how this can be done.

PROPOSITION 4. *Suppose the symplectic manifold (M_{i+1}, ω_{i+1}) is the reduction of the symplectic manifold (M_i, ω_i) by the submanifold $N_i \subset M_i$; let $\pi_i : N_i \to M_{i+1}$ be the canonical projection. Then M_2 is the reduction of M_0 by $\pi_0^{-1}(N_1)$.*

The proof is routine.

3. Proof of Theorem 2

From Proposition 3 we know that $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ can be realized as a Marsden–Weinstein reduction of $(T^*P, \mathrm{d}\theta_P)$ relative to a $\mathbf{T}^k$-action, where P is a principal $\mathbf{T}^k$-bundle over Q. To obtain this noncanonical cotangent bundle directly as a Marsden–Weinstein reduction of some $\mathbf{R}^{2n}$, we must first provide a group theoretic version of the third stage in the above and then appropriately combine the resulting Marsden–Weinstein reduction with that of the second stage.

We prepare several lemmas. Let N be a submanifold of a manifold M; $\mathscr{I}(N)$ will denote the ideal in $C^\infty(M)$ of all functions vanishing on N. We say that $\mathscr{I}(N)$ is generated by $f_1, \ldots, f_l \in \mathscr{I}(N)$ over $C^\infty(M)$ if for all $g \in \mathscr{I}(N)$ there exist $h_1, \ldots, h_l \in C^\infty(M)$ such that $g = \Sigma_{i=1}^l h_i f_i$.

LEMMA 5. *$\mathscr{I}(N)$ is generated by $f_1, \ldots, f_l$ over $C^\infty(M)$ iff the following two conditions hold*:

(i) $N = \{m \in M \mid f_i(m) = 0$ for all $i\}$.
(ii) *For all $m \in N$, $rk(\mathrm{d}f_1(m), \ldots, \mathrm{d}f_l(m)) = \operatorname{codim} N$.*

Proof. Elementary. □

LEMMA 6. *Let N be a closed submanifold of M. Then $\mathscr{I}(N)$ is finitely generated over $C^\infty(M)$.*

Proof. By the tubular neighborhood theorem, there exists an open neighborhood V of N in M which can be identified with an open neighborhood of the zero section of the normal bundle $\nu: \nu(N) \to N$ of N in M. By Lusternik–Schnirelmann theory [5] there exists a *finite* open cover $\{U_1, \ldots, U_r \mid r \leqslant \dim N + 1\}$ of N by contractible (in N) sets. For each $\alpha = 1, \ldots, r$, set $M_\alpha = \nu^{-1}(U_\alpha) \cap V$. As U_α is contractible in

N, $\nu^{-1}(U_\alpha)$ is trivial; it follows that each ideal $\mathscr{I}(N \cap M_\alpha)$ is finitely generated over $C^\infty(M_\alpha)$ by the coordinate functions $\{x_{\alpha i} \mid i = 1, \ldots, \operatorname{codim} N\}$ along the fibers of ν.

We now use a partition of unity to extend the locally defined generators of each $\mathscr{I}(N \cap M_\alpha)$ to all of M. If we set $M_0 = M \backslash N$, then $\{M_0, M_1, \ldots, M_r\}$ is a finite open cover of M and $\mathscr{I}(N \cap M_\alpha)$ is finitely generated over $C^\infty(M_\alpha)$ for each $\alpha = 0, 1, \ldots, r$. (For $\alpha = 0$, 1 is a generator because $N \cap M_0 = \emptyset$.) Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the cover $\{M_\alpha\}$. Then, using Lemma 5, it is straightforward to check that $\{\varphi_0, \varphi_\alpha x_{\alpha i} \mid \alpha = 1, \ldots, r, i = 1, \ldots, \operatorname{codim} N\}$ generates $\mathscr{I}(N)$ over $C^\infty(M)$. □

Now apply this result to the (closed) embedding $P \to \mathbf{R}^n$ of the third stage in the proof of Theorem 1, thereby obtaining a finite set of generators $f_1, \ldots, f_l$ for $\mathscr{I}(P)$ over $C^\infty(\mathbf{R}^n)$. Set $F_i = \tau_n^* f_i$ for each i and consider the Hamiltonian vector fields of the F_i on $\mathbf{R}^{2n}$. Since these vector fields generate translations along the fibers of τ_n, they commute and are complete. Hence, their flows define a symplectic $\mathbf{R}^l$-action on $\mathbf{R}^{2n}$ with momentum map $F\colon \mathbf{R}^{2n} \to \mathbf{R}^l$ given simply by $F = (F_1, \ldots, F_l)$. It follows that the Marsden–Weinstein reduction $F^{-1}(\mathbf{0})/\mathbf{R}^l$ of $\mathbf{R}^{2n}$ by $F^{-1}(\mathbf{0})$ is symplectomorphic to $(T^*P, \mathrm{d}\theta_P)$.

To combine this Marsden–Weinstein reduction with that of Proposition 3, it is necessary to extend the torus action on P to the ambient space. We appeal to a theorem of Mostow and Palais [6, 7] which states that P with its free $\mathbf{T}^k$-action can be equivariantly embedded as a closed invariant submanifold of an orthogonal representation space for $\mathbf{T}^k$. By choosing n appropriately, we may assume that this representation space is the same $\mathbf{R}^n$ that appears above. Moreover, we need the following lemma.

LEMMA 7. *$\mathscr{I}(P)$ is finitely generated over $C^\infty(\mathbf{R}^n)$ by $\mathbf{T}^k$-invariant functions.*

Proof. Let $K \subset \mathbf{R}^n$ be the set of all points where $\mathbf{T}^k$ acts freely; K is open and dense. Since $\mathbf{T}^k$ is compact, $K/\mathbf{T}^k$ is a manifold and $\pi\colon K \to K/\mathbf{T}^k$ is a submersion. Now P is a closed invariant submanifold of K so $\pi(P) = Q$ is a closed submanifold of $K/\mathbf{T}^k$. By Lemma 6, $\mathscr{I}(Q)$ is finitely generated over $C^\infty(K/\mathbf{T}^k)$, say by $g_1, \ldots, g_{l-1}$ for some l. Again by the invariance of P, the pullbacks $\pi^* g_1, \ldots, \pi^* g_{l-1}$ are invariant generators of $\mathscr{I}(P)$ over $C^\infty(K)$, but we are not guaranteed that they will extend to smooth functions on $\mathbf{R}^n$. To remedy this, let $\{\psi_1, \psi_2\}$ be a $\mathbf{T}^k$-invariant partition of unity subordinate to the open cover $\{K, \mathbf{R}^n \backslash P\}$ of $\mathbf{R}^n$. Then using Lemma 5 it follows that $\{\psi_1 \pi^* g_i, \psi_2 \mid i = 1, \ldots, l-1\}$ comprise an invariant set of generators for $\mathscr{I}(P)$ over $C^\infty(\mathbf{R}^n)$. □

We are now ready to prove Theorem 2. The cotangent lift of the torus action on $\mathbf{R}^n$ to $\mathbf{R}^{2n}$ is symplectic with a canonically defined momentum map which we still denote by J. On the other hand, we have a symplectic $\mathbf{R}^l$-action on $\mathbf{R}^{2n}$ with momentum map F. By Lemma 7, we may suppose that the f_i (and hence F) are $\mathbf{T}^k$-invariant, so that the $\mathbf{R}^l$-action commutes with the $\mathbf{T}^k$-action. Thus we obtain a symplectic action of $\mathbf{T}^k \times \mathbf{R}^l$ on $\mathbf{R}^{2n}$ with momentum map $J \times F$. It then follows from the group theoretic analogue of Proposition 4 that the Marsden–Weinstein reduction $(J \times F)^{-1}(\mu_1, \ldots, \mu_k, \mathbf{0})/\mathbf{T}^k \times \mathbf{R}^l$ is symplectomorphic to $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ (see also Theorem 2 in [2]).

REMARK. Unfortunately, we are unable to obtain a group theoretic interpretation of the first stage in the proof of Theorem 1, at least not without additional structure (e.g., (Q, Ω) is completely integrable). In any case, one would not expect to be able to combine such a Marsden–Weinstein reduction with that of Theorem 2 to obtain (Q, Ω) as a *single* Marsden–Weinstein reduction of some $\mathbf{R}^{2n}$. In fact, determining which symplectic manifolds arise via a Marsden–Weinstein reduction of some $\mathbf{R}^{2n}$ appears to be a difficult open problem.

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