

Quantum Cosmology and Geometric Quantization¹

JAMES A. ISENBERG

*Department of Applied Mathematics, University of Waterloo,
Waterloo, Ontario, Canada N2L 3G1*

and

MARK J. GOTAY

*Department of Mathematics and Statistics, University of Calgary,
Calgary, Alberta, Canada T2N-1N4*

Abstract

The Kostant–Souriau method of geometric quantization is applied to homogeneous and isotropic cosmological models with positive intrinsic curvature and a massless Klein–Gordon scalar field. These models are studied because classically they collapse to a singularity. It is rigorously shown that the quantized models collapse as well (so that there is no “quantum bounce”). This work demonstrates the practical usefulness of geometric quantization for the study of physical systems.

Can quantum effects prevent the “big crunch”? Even before it was recognized that the gravitationally induced collapse of the universe is a characteristic feature of classical general relativistic cosmology,² this question spurred major efforts to make space-time gravitational physics compatible with the quantum principle.³ While interesting results regarding “quantum bounce” have been obtained semiclassically [3], ultimately one would like to base one’s answer to this question on a complete and consistent quantum treatment of the gravitational field and its interactions. Such a theory is not yet available. However, we have recently made some progress toward this goal, using the geometric Kostant–Souriau quantization procedure [4]. In this essay, we briefly describe the application of geometric quantization to a simple model cosmology, which we

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²See Chap. 8 of [1].

³Useful review articles of the early as well as recent work can be found in [2].

rigorously show *does* exhibit quantum (as well as classical) collapse, and we discuss why geometric quantization promises to be a powerful tool in quantum gravity.

Geometric quantization is a well-defined procedure for obtaining a Hilbert space of states and a set of quantum observables for a given physical system all in terms of the underlying symplectic geometry of classical physics. For the purposes of this essay, it is not necessary to dwell upon the technical aspects of the Kostant–Souriau procedure.⁴ Just to give the flavor of it, however, we note that geometric quantization involves three major elements: (1) “prequantization,” which yields a preliminary Hilbert space and a complete but reducible representation of the classical observables; (2) “polarization,” which (locally) defines a complete commuting set of observables and thereby reduces the prequantization representation; and (3) the introduction of a “metaplectic structure,” which provides the measure in terms of which the quantum Hilbert space inner product is defined.

Geometric quantization is essentially a rigorous global generalization of the canonical quantization technique. As such, it is not a conceptually new approach to quantum mechanics: It does not alter the way in which quantum dynamics is analyzed (via the Schrödinger equation) or the way in which measurements are theoretically made.

One can best appreciate geometric quantization by comparing it to the standard canonical quantization procedure.⁵ The latter, we recall, is successful in treating systems which satisfy two conditions: (A) the classical phase space P is the cotangent bundle of a Euclidean configuration space C ; and (B) the observables to be quantized are essentially no more complicated than $p^2 + V(q)$. If either of these two conditions is violated, canonical quantization encounters severe difficulties. Specifically, it provides no way to define the quantum Hilbert space if $P \neq T^*C$ or if C is not Euclidean; and if complicated observables are quantized, then factor-ordering problems (and sometimes outright inconsistencies⁶) arise.

Since geometric quantization is designed to take into account the topological and geometrical structure of the classical phase space, the latter no longer need be Euclidean. Actually, geometric quantization can be applied to any physical system as long as its phase space is a symplectic manifold (a configuration space need not exist).⁷ Furthermore, there are no factor-ordering ambigu-

⁴ A complete discussion of the geometric quantization procedure (including details necessarily omitted here) may be found in [5]. For the details of its application to the cosmological models discussed in this essay, see the articles cited in [4].

⁵ For a thorough discussion of canonical quantization (including its limitations) see [6].

⁶ See Section 5.4 of Abraham and Marsden [6].

⁷ Modulo technical considerations of a global nature which will not be pursued here (cf. [5]).

ties in geometric quantization. Last, but not least, geometric quantization has proved to be an effective computational tool. Indeed, previous applications [4] have found it capable of quantizing systems which are otherwise intractable.

Is geometric quantization needed for quantizing the gravitational field? The difficulties in gravity theory are primarily due to the presence of the super-Hamiltonian \mathcal{H} and supermomentum \mathcal{H}_m constraints which reflect the diffeomorphism gauge freedom (“coordinate choice” freedom) of the theory. If one solves the constraints and fixes the coordinates before quantizing (“ADM” approach), then generally there is no configuration space and the ADM Hamiltonian is a rather complicated function to quantize. If instead one leaves the coordinates free and treats the constraints as quantum observables (“Dirac approach”), then the gauge freedom must be eliminated from the phase space (leaving a structure—“super-phase-space”—which is not a manifold) and one still has complicated functions—namely \mathcal{H} and \mathcal{H}_m —to quantize.⁸

It is only when one looks at the simplest of models—those with a considerable amount of symmetry built in *before* quantizing (“minisuperspaces”)—that one has a well-defined Euclidean configuration space (and, even in these simple models, devastating factor-ordering ambiguities appear⁹). Thus, unless symmetries are imposed which make the constraints vacuous, the classical description of gravity satisfies neither condition (A) nor (B). So one should not be surprised to encounter difficulties in applying canonical quantization to gravity; a more complete and consistent theory (e.g., geometric quantization) is needed.

Whether or not geometric quantization can be profitably applied to quantum gravity in all generality is not yet clear. We have, however, begun to apply this method successfully to simple systems. The one we focus on here is the positive curvature Robertson-Walker cosmology with Einstein gravitation and a massless Klein-Gordon scalar field. The reason for studying this “RW ϕ ” model is threefold: First, such models are well understood classically, they are roughly compatible with observational cosmology, and they exhibit classical collapse. Second, this RW ϕ model has been canonically quantized (although with inconclusive results) [7]. Finally, the mathematics—albeit fearsome—is doable.

Since the starting point of the Kostant-Souriau analysis is a classical description of the system in terms of symplectic geometry, let us give that now for our RW ϕ model. The RW ϕ space-times are homogeneous and isotropic and so are described completely by the “radius” $R(t)$, the scalar field $\phi(t)$ and the (auxiliary) lapse function $N(t)$. The phase space is $T^*\mathbb{R}_+^2$ with coordinates $\{R, \phi, \pi_R, \pi_\phi; R > 0\}$, the symplectic form is

$$\tilde{\omega} = d\pi_R \wedge dR + d\pi_\phi \wedge d\phi$$

⁸Both approaches, and their problems, are discussed in all the review articles listed in Reference 2.

⁹M. MacCallum [2].

and the Hamiltonian is

$$\tilde{H} = -N\mathcal{H} = -N \left\{ \frac{1}{24R} \pi_R^2 - \frac{1}{2R^3} \pi_\phi^2 + 6R \right\}$$

Since \mathcal{H} is the super-Hamiltonian, it is constrained to vanish. There are no other constraints.¹⁰

We must now choose between the ADM and Dirac approaches. We take the ADM approach, since it is better understood formally (especially in light of certain problems with the Dirac scheme recently noted by Komar and Bergmann [8]) and since the calculations are simpler. We reduce our RW ϕ system by choosing the gauge $t = \phi$ (thereby fixing $N = -R^3/\pi_\phi$) and solving $\mathcal{H} = 0$ for π_ϕ . We select this particular reduction since (i) ϕ -time covers the entire classical evolution of the model, and (ii) with ϕ -time, we may quantize the radius R and monitor the asymptotic temporal behavior of its expectation value as a test for collapse. The unconstrained phase space resulting from this reduction is the half-plane \mathbb{R}_+^2 ; the symplectic form is now

$$\omega = d\pi_R \wedge dR = (r/12) dr \wedge d\theta$$

and the ADM Hamiltonian takes the form

$$H = R \left(\frac{1}{12} \pi_R^2 + 12R^2 \right)^{1/2} = \frac{r^2}{24\sqrt{3}} \sin\theta$$

(Here, we have introduced polar coordinates

$$r = (\pi_R^2 + 144R^2)^{1/2}, \quad \theta = \frac{1}{2i} \ln \left(\frac{\pi_R + 12iR}{\pi_R - 12iR} \right)$$

on \mathbb{R}_+^2 .)

Before proceeding with the quantization, we note that the model classically evolves according to

$$R(\phi) = \frac{R_{\max}}{\{\cosh [(\phi - \phi_0)/\sqrt{3}]\}^{1/2}} \quad (1)$$

so that a “big bang” occurs at $\phi = -\infty$, maximum expansion at $\phi = \phi_0$, and a “big crunch” at $\phi = +\infty$.

Now, depending upon the topology of the classical phase space, there may be alternative choices of the three geometric quantization structures discussed earlier. For the RW ϕ system, however, the prequantization and metaplectic structures are unique, and there is a natural choice of polarization. The Kostant-

¹⁰The symmetries force the supermomentum constraints to be trivially satisfied, which in turn makes it possible for these models to be canonically quantized.

Souriau analysis then leads to a quantum state space which is isomorphic to $L^2(0, \pi)$.

The polarization is chosen so that both H and R^2 are quantizable; the corresponding self-adjoint operators on $L^2(0, \pi)$ are

$$\mathfrak{Q}H = \frac{-i\hbar}{2\sqrt{3}} \left(2 \sin\theta \frac{d}{d\theta} + \cos\theta \right)$$

and

$$\mathfrak{Q}R^2 = \frac{-i\hbar}{6} \sin\theta \left(\sin\theta \frac{d}{d\theta} + \cos\theta \right)$$

respectively. The spectrum of $\mathfrak{Q}H$ is $(-\infty, +\infty)$. Solving the time-dependent Schrödinger equation, we obtain the evolution

$$\psi(t) = \int_{-\infty}^{+\infty} \left[g(E) e^{(iE/\hbar)(t-t_0)} \sin^{-1/2} \theta \left(\tan \frac{\theta}{2} \right)^{iE/4\sqrt{3}\hbar} \right] dE$$

where $g(E)$ is a Fourier amplitude. Note that we have imposed no boundary conditions.

We now examine the quantum dynamics of the $RW\phi$ system for evidence of a big crunch. We do so by studying the asymptotic ($t \rightarrow \infty$) behavior of the expectation value of $\mathfrak{Q}R^2(t)$ for a general square integrable wave packet. $\mathfrak{Q}R^2(t)$ is obtained by solving the Heisenberg equation for this operator, and then a careful study of the matrix elements of this operator shows that

$$\lim_{t \rightarrow \infty} \langle \mathfrak{Q}R^2(t) \rangle = 0 \tag{2}$$

In fact, we find that the rate of quantum collapse matches the classical rate (1) exactly.

Is this truly a big crunch? One might contend that we have merely shown that as ϕ classically grows very large, $\langle \mathfrak{Q}R^2(\phi) \rangle$ goes to zero. Why should the quantum state $\psi(\phi)$ evolve to $\phi = \infty$? We argue as follows: Recall that the quantum Hamiltonian $\mathfrak{Q}H$ is self-adjoint. It follows that probability is conserved, i.e., the norm of any state $\psi(\phi)$ is independent of ϕ . Therefore, there is no "leakage" of the states as $\phi \rightarrow \infty$, and so our quantum $RW\phi$ model necessarily evolves to the $\phi = \infty$ limit. Consequently, (2) implies that *all physically well-defined states of this $RW\phi$ model eventually collapse*.

As noted earlier, these positive-curvature $RW\phi$ models have been studied by Blyth and Isham using canonical quantization, but their results were not definitive. Similar systems were canonically quantized by DeWitt [2] and by Misner [9]. Their conclusions underscore the ambiguities which canonical quantization leads to, since Misner obtains a big crunch using one choice of

factor ordering while DeWitt avoids it using another choice. Our results, based upon geometric quantization, are unambiguous in this regard.

Our analysis supports the contention that, at least in some (highly symmetric) space-time models, quantum effects do *not* prevent the "big crunch." Of course, much work needs to be done before a consensus can even begin to emerge regarding quantum collapse. Furthermore, there are many theoretical issues that must be elucidated, such as the effects of different choices of time and polarization upon quantization. Finally, there are deep and largely unresolved questions regarding the physical interpretation of the quantum dynamics of these cosmological models in particular [4, 9] and quantum gravity in general.

Our $RW\phi$ model calculation indicates that geometric quantization should prove to be an important tool in quantum gravity since it can be applied to the nontrivial phase spaces and observables which appear in general relativistic field theories. We hope that this work will lead to an appreciation of the practical aspects of geometric quantization.

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