

MOMENTUM MAPS & CLASSICAL FIELDS

1. Introduction

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- Study the Lagrangian and Hamiltonian structures of **classical field theories** (CFTs) with constraints



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- Study the Lagrangian and Hamiltonian structures of **classical field theories** (CFTs) with constraints
- Explore connections between **initial value constraints** & **gauge transformations**
- Tie together & understand many different and apparently unrelated facets of CFTs
- Focus on roles of gauge symmetry and **momentum maps**



Methods & Tools:

- calculus of variations



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- initial value analysis, Dirac constraint theory



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- energy-momentum maps

Generic Properties of Classical Field Theories

Gleaned from extensive study of standard examples:

- electromagnetism, Yang–Mills
- gravity
- strings
- relativistic fluids
- topological field theories . . .

Pioneers: Choquet-Bruhat, Lichnerowicz, Dirac–Bergmann, and Arnowit–Deser–Misner (ADM).



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- This is because the theory has **gauge freedom**.
- The corresponding **gauge group** is known at the outset.
- **Kinematic fields** have no significance.
- **Dynamic fields** ψ , conjugate momenta ρ have physical meaning.



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- So initial data $(\psi(0), \rho(0))$ cannot be freely specified
- Elliptic system
- Assume all constraints are **first class** in the sense of Dirac



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— So the presence of constraints \longleftrightarrow gauge freedom



4. The Hamiltonian (with respect to a slicing) has the form

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depending *linearly* on the *atlas fields* α_j



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Atlas fields:

- closely related to kinematic fields
- arbitrarily specifiable
- “drive” the entire gauge ambiguity of the CFT



5. The evolution equations for the dynamic fields (ψ, ρ) take the *adjoint form*

$$\frac{d}{d\lambda} \begin{pmatrix} \psi \\ \rho \end{pmatrix} = \mathbb{J} \cdot \sum_i \left[D\Phi^i(\psi(\lambda), \rho(\lambda)) \right]^* \alpha_i. \quad (2)$$



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- hyperbolic system (typically)



Adjoint form displays, in the clearest and most concise way, the interrelations between the

- dynamics
- initial value constraints, and
- gauge ambiguity of a theory



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- The constraints are preserved by the evolution equations



7. The space of solutions of the field equations is not necessarily smooth. It may have quadratic singularities occurring at symmetric solutions.



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- symplectic reduction
- linearization stability
- quantization



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Noether's theorem and the Dirac analysis of constraints do much to predict and explain features 1–6.

- I wish to go further and provide (realistic) sufficient conditions which guarantee that they **must** occur in a CFT.
- I provide such criteria for 1–6 and lay the groundwork for 7.

A key objective is thus to **derive** the adjoint formalism for CFTs.



Example: (Abelian) Chern–Simons Theory

A topological field theory on 3-dimensional “spacetime” $X = \mathbb{R} \times \Sigma$.



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- Lagrangian density is $\mathcal{L} = dA \wedge A$.
- Gauge group is $\text{Diff}(X)$: $\eta \cdot A = \eta_* A$
- E–L equations: $dA = 0$.



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- Slicing of $\Lambda^1(X) \rightarrow X$ generated by:

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- Hamiltonian: $H = -2 \int_{\Sigma} (dA)_{12} (\zeta^\mu A_\mu) d\Sigma$



— Atlas field: $\zeta^\mu A_\mu$



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- Atlas field: $\zeta^\mu A_\mu$
- Evolution equations in adjoint form reduce to:

$$\frac{d}{d\lambda} \begin{pmatrix} A_i \\ \rho^i \end{pmatrix} = \begin{pmatrix} D_i(\zeta^\mu A_\mu) \\ \epsilon^{0ij} D_j(\zeta^\mu A_\mu) \end{pmatrix}$$

This is equivalent to $(dA)_{0i} = 0$.

- N.B. We also have **primary constraints** $\rho^0 = 0$ and $\rho^i = \epsilon^{0ij} A_j$.



Traditional Approaches to CFT



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— Group-theoretical



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— Group-theoretical

- ▶ concerned with the gauge covariance of a CFT
- ▶ Lagrangian-oriented
- ▶ covariant
- ▶ based on Noether's theorem



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— Canonical



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— Canonical

- ▶ initial value analysis
- ▶ Hamiltonian-oriented
- ▶ **not** covariant
- ▶ based on space + time decomposition



Connections:

- These two aspects of a **mechanical** system are linked by the **momentum map**.



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Connections:

- These two aspects of a **mechanical** system are linked by the **momentum map**.
- One would like to have an analogous connection in **CFT** relating gauge symmetries to initial value constraints.



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Caveat!

The standard notion of a momentum map associated to a symplectic group action usually cannot be carried over to spacetime covariant field theory, because:

- spacetime diffeomorphisms move Cauchy surfaces, and
- the Hamiltonian formalism is only defined relative to a fixed Cauchy surface



Prime Example: Einstein's theory of vacuum gravity

- the gauge group is the spacetime diffeomorphism group



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- the gauge group is the spacetime diffeomorphism group
- the only remnants of this group on the *instantaneous* (i.e., space + time split) level are the *superhamiltonian* \mathfrak{H} and *supermomenta* \mathfrak{J}



Prime Example: Einstein's theory of vacuum gravity

- the gauge group is the spacetime diffeomorphism group
- the only remnants of this group on the **instantaneous** (i.e., space + time split) level are the **superhamiltonian** \mathfrak{H} and **supermomenta** \mathfrak{J}
 - ▶ interpreted as the generators of temporal and spatial deformations of a Cauchy surface
 - ▶ these deformations do not form a group
 - ▶ nor are \mathfrak{H} and \mathfrak{J} components of a momentum map.

This circumstance forces us to work on the **covariant** level.



The Way Out: Multisymplectic Field Theory

We must construct a covariant counterpart to the instantaneous Hamiltonian formalism.

In the **spacetime covariant** (or **multisymplectic**) framework we develop here—an extension and refinement of the formalism of Kijowski and Szczyrba—the gauge group **does** act.

So we can define a **covariant** (or **multi-**) **momentum map** on the corresponding **covariant** (or **multi-**) **phase space**.



The Energy-Momentum Map

Key fact: The covariant momentum map induces an **energy-momentum** map Φ on the instantaneous phase space.



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The Energy-Momentum Map

Key fact: The covariant momentum map induces an **energy-momentum** map Φ on the instantaneous phase space.

- Bridges the covariant & instantaneous formalisms
- Φ is the crucial object reflecting the gauge transformation covariance of a CFT in the instantaneous picture.
- In ADM gravity, $\Phi = -(\mathfrak{H}, \mathfrak{J})$, so that the superhamiltonian and supermomenta are the **components** of the energy-momentum map.



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— Hamiltonian (Item 4)



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- Hamiltonian (Item 4)
- gauge freedom (Item 3)
- initial value constraints (Item 2)
- stress-energy-momentum tensor
- ...



Indeed:

Energy-Momentum Theorem

The constraints (1) are given by the vanishing of the energy-momentum map associated to the gauge group of the theory.

Φ thus synthesizes the group-theoretical and canonical approaches to CFT.



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The theoretical underpinnings of the adjoint formalism consist of:

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- I. a covariant analysis of CFTs;
- II. a space + time decomposition of the covariant formalism followed by an initial value analysis;
- III. a study of the relations between gauge symmetries and initial value constraints



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The Plan

The theoretical underpinnings of the adjoint formalism consist of:

- I. a covariant analysis of CFTs;
- II. a space + time decomposition of the covariant formalism followed by an initial value analysis;
- III. a study of the relations between gauge symmetries and initial value constraints
- IV. the derivation of the adjoint formalism.

We will focus on I–III.



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Carry along an example:

- bosonic string
 - ▶ Mimics gravity to a significant extent
 - ▶ a first class first order theory
- Exercise: Work through abelian Chern–Simons!

Other sidelights might include:

- parametrization theory (à la Kuchař)
- covariantization theory (à la Yang–Mills)

