MOMENTUM MAPS & CLASSICAL FIELDS

2. Covariant Field Theory

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I develop some basic CFT from a covariant viewpoint, including:

- Geometry of the jet bundle and the Euler–Lagrange equations (analogous to that of the tangent bundle & the Lagrange equations in mechanics)

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- Geometry of the jet bundle and the Euler–Lagrange equations (analogous to that of the tangent bundle & the Lagrange equations in mechanics)
- Multisymplectic geometry (analogous to the geometry of the cotangent bundle)
- Conservation laws and Noether's theorem using covariant momentum maps (generalizing the concept of momentum map familiar from mechanics)

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Both are useful and have their own advantages.

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- Sections $\phi : X \to Y$ are the physical fields
- Compare $Y = \mathbb{R} \times Q \to \mathbb{R}$ in (time-dependent) mechanics
- Coordinates $(x^\mu, y^A) = (x^0, x^1, \ldots, x^n, y^1, \ldots, y^N)$ on Y.

- Conventions

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The Jet Bundle JY

For first order theories:

- $J_x Y = \{[\phi] \mid \phi_1 \equiv \phi_2 \text{ at } x \text{ iff } \phi_1(x) = \phi_2(x) \text{ and } T_x\phi_1 = T_x\phi_2\}$

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- $JY \to Y$ is an affine bundle, with fiber over $y \in Y_x$ being

$$J_yY = \{\gamma \in L(T_xX, T_yY) \,|\, T\pi_{X,Y} \circ \gamma = \mathrm{Id}_{T_xX}\}$$

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- Compare mechanics: $JY \approx \mathbb{R} \times TQ$

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Use affine maps as JY is an affine bundle.
- Fiber coordinates on $JY^\star \to Y$ are $(p, p_A{}^\mu)$, corresponding to the affine map
$$v^A{}_\mu \mapsto (p + p_A{}^\mu v^A{}_\mu)\, d^{n+1}x$$
where
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- $JY^\star$ is a vector bundle.

- In mechanics, $JY^\star \approx T^*\mathbb{R} \times T^*Q$

Alternate description:

Proposition: $JY^\star \approx Z$, where

$$Z_y = \{z \in \Lambda_y^{n+1} Y \mid i_v i_w z = 0 \text{ for all } v, w \in V_y Y\}.$$

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- $z \in Z$ takes the form

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where $d^n x_\mu = \partial_\mu \lrcorner\, d^{n+1}x$.

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- Intrinsically, the isomorphism $\vartheta : Z \to JY^\star$ is

$$\langle \vartheta(z), \gamma \rangle = \gamma^* z \in \Lambda_x^{n+1} X$$

  where $z \in Z_y$, $\gamma \in J_y Y$ and $x = \pi_{XY}(y)$.

Canonical forms:

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- Θ is the multi-Liouville form, $\Omega = -d\Theta$ is the multisymplectic form. (Z, Ω) is the covariant or multi- phase space.

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- Poisson brackets?

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- Coordinates on Z: $(x^\mu, \phi^A, h_{\sigma\rho}, p, p_A{}^\mu, \rho^{\sigma\rho\mu})$

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- $\mathcal{L} : JY \to \Lambda^{n+1}X$
- In coordinates $\mathcal{L} = L(x^\mu, y^A, v^A{}_\mu)\, d^{n+1}x$.
- No regularity assumption on $\mathcal{L}$; it would fail in almost all examples

The Legendre transformation:

- $\mathbb{F}\mathcal{L} : JY \rightarrow JY^\star$ defined by

$$\langle \mathbb{F}\mathcal{L}(\gamma), \gamma' \rangle = \mathcal{L}(\gamma) + \frac{d}{d\varepsilon}\mathcal{L}(\gamma + \varepsilon(\gamma' - \gamma))\,|_{\,\varepsilon=0}.$$

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$$p_A{}^\mu = \frac{\partial L}{\partial v^A{}_\mu} \quad \text{and} \quad p = L - \frac{\partial L}{\partial v^A{}_\mu} v^A{}_\mu$$

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- Cool fact: $\mathcal{L}(j\phi) = (j\phi)^*\Theta_{\mathcal{L}}$

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- So the Cartan form is

$$\Theta_{\mathcal{L}} = \sqrt{-h}\left(-h^{\mu\nu} g_{AB} v^B{}_\nu d\phi^A \wedge d^1x_\mu + \frac{1}{2}\sqrt{-h}\,h^{\mu\nu} g_{AB} v^A{}_\mu v^B{}_\nu d^2x\right).$$

The E–L equations $\delta L/\delta\phi^A = 0$ and $\delta L/\delta h_{\alpha\beta} = 0$ are

$$(h^{\mu\nu} g_{AB}(\phi)\phi^B{}_{,\nu})_{;\mu} = 0 \tag{1}$$

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  - and it determines the conformal factor: $\Lambda^2 = \frac{1}{2} h^{\mu\nu} g_{AB}(\phi)\phi^A{}_{,\mu}\phi^B{}_{,\nu}$

Covariant Momentum Maps & Noether's Theorem

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Proposition

Canonical lifts are special covariant canonical transformations:

$$\eta_Z^*\Theta = \Theta$$

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J intertwines the group action with the multisymplectic structure via the above equation.

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- An alternate formula: $J(\xi)(z) = \pi^*_{YZ}(\xi_Y \lrcorner z)$
- In coordinates: if we write $\xi_Y = \xi^\mu \frac{\partial}{\partial x^\mu} + \xi^A \frac{\partial}{\partial y^A}$,

$$J(\xi) = (p_A{}^\mu \xi^A + p\xi^\mu) d^n x_\mu - p_A{}^\nu \xi^\nu \, dy^A \wedge d^{n-1}x_{\mu\nu}$$

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- $\mathrm{Diff}(X)$ acts on $C^\infty(X, \mathbb{R}^+)$ by $\eta \cdot \Lambda = \Lambda \circ \eta^{-1}$
- $(\eta, \Lambda) \in \mathcal{G}$ sends $(\phi, h) \in Y_x$ to

$$(\eta, \Lambda) \cdot (\phi, h) = \Big(\phi, \Lambda^2(\eta(x))(\eta^{-1})^* h\Big)$$

in $Y_{\eta(x)}$

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- $\mathrm{Diff}(X)$ acts on $C^\infty(X, \mathbb{R}^+)$ by $\eta \cdot \Lambda = \Lambda \circ \eta^{-1}$
- $(\eta, \Lambda) \in \mathcal{G}$ sends $(\phi, h) \in Y_x$ to

$$(\eta, \Lambda) \cdot (\phi, h) = \left(\phi, \Lambda^2(\eta(x))(\eta^{-1})^* h\right)$$

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- The multimomentum map is

$$J(\xi, \lambda) = \left[\rho^{\sigma\rho\mu} \left(2\lambda h_{\sigma\rho} - h_{\sigma\nu} \xi^\nu{}_{,\rho} - h_{\rho\nu} \xi^\nu{}_{,\sigma} \right) + p\, \xi^\mu \right] d^1 x_\mu$$

$$- \left(p_A{}^\mu \xi^\nu d\phi^A + \rho^{\sigma\rho\mu} \xi^\nu dh_{\sigma\rho} \right) \epsilon_{\mu\nu}$$

where $d^2 x_{\mu\nu} = \epsilon_{\mu\nu}$.

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- Infinitesimally, this is $\delta_\xi L = 0$, where

$$\delta_\xi L = \frac{\partial L}{\partial x^\mu}\xi^\mu + \frac{\partial L}{\partial y^A}\xi^A + \frac{\partial L}{\partial v^A{}_\mu}\left(\xi^A{}_{,\mu} - v^A{}_\nu \xi^\nu{}_{,\mu} + v^B{}_\mu \frac{\partial \xi^A}{\partial y^B}\right) + L\,\xi^\mu{}_{,\mu}$$

  is the variation of L.

Thm

Let $\mathcal{L}$ be $\mathcal{G}$-equivariant. Then:

- $\mathbb{F}\mathcal{L}$ is also equivariant, i.e., $\eta_Z \circ \mathbb{F}\mathcal{L} = \mathbb{F}\mathcal{L} \circ \eta_{JY}$

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- $\mathbb{F}\mathcal{L}$ is also equivariant, i.e., $\eta_Z \circ \mathbb{F}\mathcal{L} = \mathbb{F}\mathcal{L} \circ \eta_{JY}$
- The Cartan form $\Theta_{\mathcal{L}}$ is invariant, i.e., $\eta_{JY}^* \Theta_{\mathcal{L}} = \Theta_{\mathcal{L}}$
- The map $J^{\mathcal{L}}(\xi) := \mathbb{F}\mathcal{L}^* J(\xi) : JY \to \Lambda^n(JY)$ is a momentum map for the prolonged action of $\mathcal{G}$ on JY relative to $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}}$. That is to say,

  $$\xi_{JY} \,\lrcorner\, \Omega_{\mathcal{L}} = dJ^{\mathcal{L}}(\xi).$$

  Moreover,

  $$J^{\mathcal{L}}(\xi) = \xi_{J^1Y} \,\lrcorner\, \Theta_{\mathcal{L}}.$$

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If $\mathcal{L}$ is $\mathcal{G}$-covariant, then for each $\xi \in \mathfrak{g}$,

$$d\left[(j\phi)^* J^{\mathcal{L}}(\xi)\right] = 0$$

for any section ϕ of π_{XY} satisfying the Euler–Lagrange equations.

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Proof

If ϕ is a solution of the Euler–Lagrange equations, then

$$(j\,\phi)^*(W \,\lrcorner\, \Omega_{\mathcal{L}}) = 0$$

for any vector field W on JY. In particular, set $W = \xi_{JY}$ and simply apply $(j\phi)^*$ to

$$\xi_{JY} \,\lrcorner\, \Omega_{\mathcal{L}} = dJ^{\mathcal{L}}(\xi). \qquad \blacksquare$$

Local Expressions

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- the "Lagrangian multimomentum map" is

$$J^{\mathcal{L}}(\xi) =$$

$$\left(\frac{\partial L}{\partial v^A{}_\mu} \xi^A + \left[L - \frac{\partial L}{\partial v^A{}_\nu} v^A{}_\nu \right] \xi^\mu \right) d^n x_\mu - \frac{\partial L}{\partial v^A{}_\mu} \xi^\nu dy^A \wedge d^{n-1} x_{\mu\nu}$$

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- the Noether current is

$$(j^1\phi)^* J^{\mathcal{L}}(\xi) = \left[-\frac{\partial L}{\partial v^A{}_\mu} (j^1\phi)(\mathbb{L}_\xi \phi)^A + L(j^1\phi)\xi^\mu \right] d^n x_\mu$$

where the "Lie derivative of ϕ along ξ is

$$\mathbb{L}_\xi \phi = T\phi \circ \xi_X - \xi_Y \circ \phi; \quad \text{i.e.,} \quad (\mathbb{L}_\xi \phi)^A = \phi^A{}_{,\nu} \xi^\nu - \xi^A \circ \phi$$

A computation gives the useful expression for the Noether divergence:

$$d\left[(j\phi)^* J^{\mathcal{L}}(\xi)\right] = \left\{\frac{\delta L}{\delta \phi^A}(\mathbb{L}_\xi \phi)^A + \delta_\xi L\right\}(j^1\phi)\, d^{n+1}x$$

from which again Noether's theorem is immediate.

Bosonic String

The Noether current is:

$$j(\phi, h)^* J^{\mathcal{L}}(\xi, \lambda) = \sqrt{-h}\, g_{AB} \left(h^{\mu\nu} \phi^A{}_{,\rho} \phi^B{}_{,\nu} \xi^\rho - \frac{1}{2} h^{\sigma\rho} \phi^A{}_{,\sigma} \phi^B{}_{,\rho} \xi^\mu \right) d^1 x_\mu. \tag{3}$$

Note again that λ does not appear on the RHS.