MOMENTUM MAPS & CLASSICAL FIELDS

3. Canonical Field Theory

MARK J. GOTAY

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- I'll recover instantaneous Lagrangian & Hamiltonian mechanics on appropriate spaces of fields
- discuss the standard initial value analysis
- define the energy-momentum mapping

Cauchy Surfaces & Spaces of Fields

- Let Σ be a closed n-manifold, and $\mathrm{Emb}(\Sigma, X)$ the space of all embeddings $\Sigma \to X$. For $\tau \in \mathrm{Emb}(\Sigma, X)$, view $\Sigma_\tau := \tau(\Sigma)$ as a Cauchy surface.

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- If $K \to X$ is a bundle, let K_τ be the restriction of K to $\Sigma_\tau \subset X$. Let $\mathcal{K}_\tau = \Gamma(K_\tau \to \Sigma_\tau)$.
- the instantaneous configuration space at "time" τ is $\mathcal{Y}_\tau$

The Tangent Space

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The tangent space to $\mathcal{K}_\tau$ at σ is

$$T_\sigma \mathcal{K}_\tau = \left\{ W : \Sigma_\tau \to VK \,\middle|\, W \text{ covers } \sigma \right\},$$

where VK denotes the vertical tangent bundle of K.

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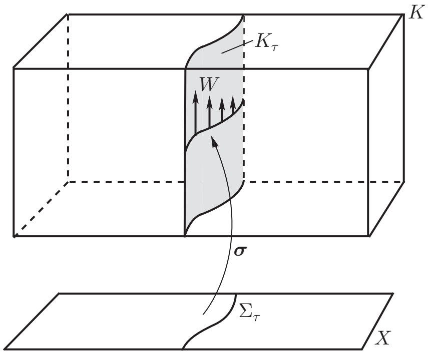

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- The smooth (or L^2) cotangent space to $\mathcal{K}_\tau$ at σ is

$$T^*_\sigma \mathcal{K}_\tau = \left\{ \pi : \Sigma_\tau \to L(VK, \Lambda^n \Sigma_\tau) \,\middle|\, \pi \text{ covers } \sigma \right\},$$

  where $L(VK, \Lambda^n \Sigma_\tau)$ is the vector bundle over K whose fiber at $k \in K_x$ is the set of linear maps from $V_k K$ to $\Lambda^n_x \Sigma_\tau$.

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- In adapted coordinates (Σ_τ is locally $x^0 = 0$), $\pi \in T^*_\sigma \mathcal{K}_\tau$ is

$$\pi = \pi_A \, dk^A \otimes d^n x_0$$

Canonical Forms

- The Liouville form on the τ-phase space $T^*\mathcal{Y}_\tau$ is given in the 'usual' manner:

$$\theta_\tau(\varphi, \pi)(V) = \int_{\Sigma_\tau} \pi(T\pi_{\mathcal{Y}_\tau, T^*\mathcal{Y}_\tau} \cdot V)$$

for $(\varphi, \pi) \in T^*\mathcal{Y}_\tau$.

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- In adapted coordinates,

$$\omega_\tau(\varphi,\pi) = \int_{\Sigma_\tau} (d\varphi^A \wedge d\pi_A) \otimes d^n x_0.$$

Multisymplectic $\to$ Instantaneous Phase Spaces

Multisymplectic → Instantaneous Phase Spaces

- Consider the vector bundle map

$$R_\tau : \mathcal{Z}_\tau \to T^*\mathcal{Y}_\tau$$

over $\mathcal{Y}_\tau$ given by

$$\langle R_\tau(\sigma), V \rangle = \int_{\Sigma_\tau} \varphi^*(V \lrcorner\, \sigma),$$

where $\varphi = \pi_{YZ} \circ \sigma$.

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where $\varphi = \pi_{YZ} \circ \sigma$. Locally,

$$R_\tau(\sigma) = \int_{\Sigma_\tau} {\sigma_A}^0 \, dy^A \otimes d^n x_0 \tag{1}$$

- R_τ is a submersion with kernel

$$\left\{ \sigma_A{}^i dy^A \wedge d^n x_i + (p \circ \sigma)\, d^{n+1} x \right\}.$$

so therefore

$$\ker TR_\tau(\sigma) = \left\{ \frac{\delta}{\delta p_A{}^i}, \frac{\delta}{\delta p} \right\} \qquad (2)$$

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- So we recover the instantaneous phase space from the covariant one. What about the canonical forms?

- On $\mathcal{Z}_\tau$ there is the canonical 1-form:

$$\Theta_\tau(\sigma)(V) = \int_{\Sigma_\tau} \sigma^*(i_V\Theta),$$

where $\sigma \in \mathcal{Z}_\tau$, $V \in T_\sigma\mathcal{Z}_\tau$, and Θ is the canonical $(n+1)$-form on Z. The canonical two-form Ω_τ on $\mathcal{Z}_\tau$ is $\Omega_\tau = -d\Theta_\tau$.

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Proposition

$$\Omega_\tau(\sigma)(V, W) = \int_{\Sigma_\tau} \sigma^*(\mathbf{i}_W\mathbf{i}_V\Omega)$$

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- So the instantaneous framework is obtained from the covariant formalism via symplectic reduction.
- From (1) the instantaneous momenta are just $\pi_A = {p_A}^0$.

Instantaneous Dynamics

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- We similarly slice bundles over X.

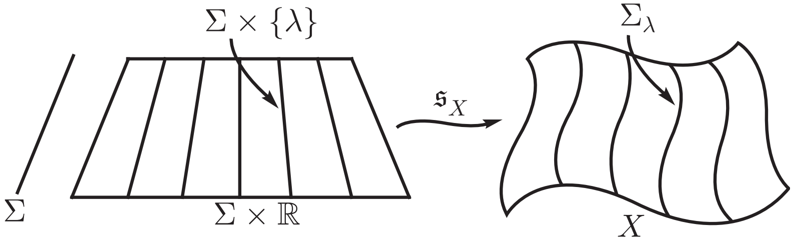

Figure: A slicing of spacetime

By general covariance, slicings ($\approx$ coordinates) have no physical significance. So the flow of ζ must consist of gauge transformations: $\zeta \in \mathfrak{g}$.

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Let $\zeta := \zeta_Y$ be the generator of the slicing on Y; ϕ a section of $Y \to X$. Set

$$\varphi := \phi \Big| \Sigma_\tau \quad \text{and} \quad \dot{\varphi} := \mathbb{L}_\zeta \phi \Big| \Sigma_\tau$$

(Recall: $\mathbb{L}_\zeta \phi := T\phi \cdot \zeta_X - \zeta \circ \phi$.)

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(Recall: $\mathbb{L}_\zeta \phi := T\phi \cdot \zeta_X - \zeta \circ \phi$.)

Proposition

The jet decomposition map $(JY)_\tau \to J(Y_\tau) \times VY_\tau$ given by

$$j\phi(x) \mapsto (j\varphi(x), \dot{\varphi}(x))$$

is an isomorphism.

Locally this splitting amounts to

$$(x^i, y^A, v^A{}_\mu) \mapsto (x^i, y^A, v^A{}_j, \dot{y}^A)$$

where

$$\dot{y}^A = \zeta^\mu v^A{}_\mu - \zeta^A$$

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Corollary

The jet decomposition map induces an isomorphism

$$(j\mathcal{Y})_\tau \approx T\mathcal{Y}_\tau$$

where $(j\mathcal{Y})_\tau$ is the collection of restrictions of holonomic sections of $JY \to X$ to Σ_τ.

The Instantaneous Lagrangian

The Instantaneous Lagrangian

- This is

$$L_{\tau,\zeta}(\varphi, \dot{\varphi}) = \int_{\Sigma_\tau} i_\tau^* \left(\zeta_X \,\lrcorner\, \mathcal{L}(j\phi) \right),$$

for $(\varphi, \dot{\varphi}) \in T\mathcal{Y}_\tau$ and where i_τ is the inclusion $\Sigma_\tau \to X$.

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- In adapted coordinates
$$L_{\tau,\zeta}(\varphi,\dot{\varphi}) = \int_{\Sigma_\tau} L(j\varphi,\dot{\varphi})\zeta^0 d^n x_0.$$

- Analysis now proceeds as always.

The Legendre Transformation

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- The diagram commutes:

$$
\begin{array}{ccc}
(j^1\mathcal{Y})_\tau & \xrightarrow{\mathbb{F}\mathcal{L}} & \mathcal{Z}_\tau \\
{\scriptstyle \beta_\zeta}\downarrow & & \downarrow{\scriptstyle R_\tau} \\
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- Define the covariant primary constraint set to be

$$N = \mathbb{F}\mathcal{L}(JY) \subset Z$$

and with a slight abuse of notation, set

$$\mathcal{N}_\tau = \mathbb{F}\mathcal{L}((j\mathcal{Y})_\tau) \subset \mathcal{Z}_\tau.$$

- Then

$$R_\tau(\mathcal{N}_\tau) = \mathcal{P}_{\tau,\zeta}.$$

where $\mathcal{P}_{\tau,\zeta}$ is the instantaneous τ-primary constraint set.

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- So we equally well arrive at the cusp of the Hamiltonian formalism covariantly or instantaneously.

Hamiltonian Dynamics

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As always, the instantaneous Hamiltonian is given by

$$H_{\tau,\zeta}(\varphi,\pi) = \langle \pi, \dot{\varphi} \rangle - L_{\tau,\zeta}(\varphi,\dot{\varphi})$$

and is defined only on $\mathcal{P}_\tau$.

Thm

Let $(\varphi,\pi) \in \mathcal{P}_\tau$. Then for any holonomic lift σ of (φ,π) to $\mathcal{Z}_\tau$,

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- Observe also that $H_{\tau,\zeta}$ is linear in ζ!

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- Consider a slicing with (gauge) generator

$$\zeta = \zeta^\mu \frac{\partial}{\partial x^\mu} - \left((h_{\sigma\alpha}\zeta^\alpha{}_{,\rho} + h_{\rho\alpha}\zeta^\alpha{}_{,\sigma}) + 2\lambda h_{\sigma\rho} \right) \frac{\partial}{\partial h_{\sigma\rho}}$$

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- The instantaneous Lagrangian is

$$L_{\tau,\zeta}(\varphi, h, \dot{\varphi}, \dot{h}) =$$

$$-\frac{1}{2}\int_{\Sigma_{\tau}} \sqrt{-h}\, g_{AB}\Big(\frac{1}{\zeta^0}h^{00}(\dot{\varphi}^A - \zeta^1 D\varphi^A)(\dot{\varphi}^B - \zeta^1 D\varphi^B)$$

$$+ 2h^{01}(\dot{\varphi}^A - \zeta^1 D\varphi^A)D\varphi^B$$

$$+ \zeta^0 h^{11} D\varphi^A D\varphi^B\Big) d^1x_0, \quad \text{where} \quad D\varphi^A := \varphi^A{}_{,1}.$$

- The instantaneous momenta are

$$\pi_A = -\sqrt{-h}\, g_{AB}\left(\frac{1}{\zeta^0} h^{00}(\dot{\varphi}^B - \zeta^1 D\varphi^B) + h^{01} D\varphi^B\right)$$

$$\varpi^{\sigma\rho} = 0.$$

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- Thus the primary constraint set is

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- The Hamiltonian is

$$H_{\tau,\zeta}(\varphi, h, \pi, \varpi) =$$

$$-\int_{\Sigma_\tau}\left(\frac{1}{2h^{00}\sqrt{-h}}\zeta^0(\pi^2 + D\varphi^2) + \left(\frac{h^{01}}{h^{00}}\zeta^0 - \zeta^1\right)(\pi \cdot D\varphi)\right) d^1x_0$$

where $\pi^2 := g^{AB}\pi_A\pi_B$ and $\pi \cdot D\varphi := \pi_A D\varphi^A$, etc.

- 1+1 split the metric h à la ADM as (with $\gamma = h_{11}$):

$$\begin{pmatrix} h^{00} & h^{01} \\ h^{10} & h^{11} \end{pmatrix} = \begin{pmatrix} -1/N^2 & M/N^2 \\ M/N^2 & \gamma^{-1} - M^2/N^2 \end{pmatrix}.$$

  Furthermore, the metric volume $\sqrt{-h}$ decomposes as

$$\sqrt{-h} = N\sqrt{\gamma}$$

- Then the Hamiltonian reduces to:

$$H_{\tau,\zeta}(\varphi, h, \pi, \varpi) =$$

$$\int_{\Sigma_\tau} \left(\frac{1}{2\sqrt{\gamma}} \zeta^0 N(\pi^2 + D\varphi^2) + (\zeta^0 M + \zeta^1)(\pi \cdot D\varphi) \right) d^1 x_0.$$

- Note the combinations $\zeta^0 N/\sqrt{\gamma}$ and $\zeta^0 M + \zeta^1$ of kinematic fields (components of h) which appear linearly in this expression; these are atlas fields.

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- Also note the factors $(\pi^2 + D\varphi^2)$ and $\pi \cdot D\varphi$; these are the string superhamiltonian and supermomentum, resp.
- We have an induced presymplectic structure on $\mathcal{P}_\tau$ pulled back from $T^*\mathcal{Y}_\tau$. It is:

$$\omega_\tau(\varphi, h, \pi, \varpi) = \int_{\Sigma_\tau} (d\varphi^A \wedge d\pi_A) \otimes d^2x_0.$$

Initial Value Analysis

We will explore this by means of the bosonic string example.

- We must solve Hamilton's equations

$$X_\tau \lrcorner\, \omega_\tau = dH_{\tau,\zeta} \tag{3}$$

for the evolution vector field X_τ on $\mathcal{P}_\tau$.

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  for the evolution vector field X_τ on $\mathcal{P}_\tau$.

- This is not guaranteed, as ω_τ is merely presymplectic; $dH_{\tau,\zeta}$ may not be in the range of the flat map.

Write

$$X_\tau = \left(\frac{d\varphi^A}{d\lambda}\right)\frac{\delta}{\delta\varphi^A} + \left(\frac{d\pi_A}{d\lambda}\right)\frac{\delta}{\delta\pi_A} + \left(\frac{dh_{\sigma\rho}}{d\lambda}\right)\frac{\delta}{\delta h_{\sigma\rho}},$$

- Then

$$X_\tau \lrcorner\, \omega_\tau = \int_{\Sigma_\tau}\left[-\left(\frac{d\varphi^A}{d\lambda}\right)d\pi_A + \left(\frac{d\pi_A}{d\lambda}\right)d\phi^A\right]\otimes d^1x_0 \quad (4)$$

Note there are no terms involving differentials of the metric.

- On the other hand, after several integrations by parts,

$$
\begin{aligned}
dH_{\tau,\zeta} = \int_{\Sigma_\tau} \Bigg[& \frac{\zeta^0 N}{\sqrt{\gamma}} \pi \cdot d\pi - D\left(\frac{\zeta^0 N}{\sqrt{\gamma}} \varphi \right) \cdot d\varphi \\
& + (\zeta^0 M + \zeta^1) d\pi \cdot D\varphi - D\Big((\zeta^0 M + \zeta^1)\pi \Big) \cdot d\varphi) \\
& + \frac{1}{2} \zeta^0 (\pi^2 + D\varphi^2)] \, d\left(\frac{N}{\sqrt{\gamma}} \right) \\
& + \zeta^0 (\pi \cdot D\varphi) dM \Bigg] \otimes d^1 x_0 \qquad (5)
\end{aligned}
$$

- Comparing (4) with (5), we can solve for X_τ provided

$$\pi^2 + D\varphi^2 = 0 \quad \text{and} \quad \pi \cdot D\varphi = 0$$

These are the superhamiltonian and supermomentum initial value constraints. They define the secondary constraint set $\mathcal{P}_2 \subset \mathcal{P}$.

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- So we have on $\mathcal{P}_2$

$$\frac{d\varphi^A}{d\lambda} = \frac{\zeta^0 N}{\sqrt{\gamma}} g^{AB}\pi_B + (\zeta^0 M + \zeta^1) D\varphi^A$$

$$\frac{d\pi_A}{d\lambda} = g_{AB} D\left(\frac{\zeta^0 N}{\sqrt{\gamma}} D\varphi^B\right) + D\left((\zeta^0 M + \zeta^1)\pi_A\right)$$

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- On the instantaneous level, the gauge transformations are generated by the Hamiltonian vector fields

$$
\begin{aligned}
X_{K\mathfrak{H}} &= \frac{K}{\sqrt{\gamma}} g^{AB} \pi_B \frac{\delta}{\delta \varphi^A} + g_{AB} D\left(\frac{K}{\sqrt{\gamma}} D\varphi^B\right) \frac{\delta}{\delta \pi_A} \\
X_{L\mathfrak{I}} &= L D\varphi^A \frac{\delta}{\delta \varphi^A} + D(L\pi_A) \frac{\delta}{\delta \pi_A}
\end{aligned}
\tag{6}
$$

of $K\mathfrak{H}$ and $L\mathfrak{I}$ restricted $\mathcal{P}^2_{\lambda,\zeta}$, respectively, together with the $\delta/\delta h_{\sigma\rho}$

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The Energy-Momentum Map

- Suppose a group $\mathcal{G}$ acts on $K \to X$ by bundle automorphisms. Then $\eta \in \mathcal{G}$ acts on sections σ by

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- CAVEAT: In general, $\mathcal{G}$ will not act on K_τ and $\mathcal{K}_\tau$ if $\mathcal{G}$ acts nontrivially on X.

  This is where our "troubles" begin.

- Let $\mathcal{G}$ act on Z by special covariant canonical transformations with multimomentum map J. Even though $\mathcal{G}$ may not act on $\mathcal{Z}_\tau$, we may still define the energy-momentum map

$$E_\tau : \mathcal{Z}_\tau \to \mathfrak{g}^*$$

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Thm

E_τ drops to the instantaneous energy-momentum map $\mathcal{E}_\tau : \mathcal{P}_\tau \to \mathfrak{g}^*$.

Thm

Let $\xi \in \mathfrak{g}$. If ξ_X is everywhere transverse to Σ_τ, then

$$\langle \mathcal{E}_\tau(\varphi, \pi), \xi \rangle = -H_{\tau,\xi}(\varphi, \pi)$$

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- But $\mathcal{E}_\tau$ makes sense regardless of transversality

Bosonic String

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$$\langle \mathcal{E}_\tau(\varphi, h, \pi), (\xi, \lambda) \rangle$$

$$= - \int_{\Sigma_\tau} \left(\frac{1}{2\sqrt{\gamma}} \xi^0 N(\pi^2 + D\varphi^2) + (\xi^0 M + \xi^1)(\pi \cdot D\varphi) \right) d^1 x_0.$$

- From this one can read off the string superhamiltonian

$$\mathfrak{H} = \frac{1}{2}(\pi^2 + D\varphi^2)$$

  and the string supermomentum

$$\mathfrak{J} = \pi \cdot D\varphi.$$

- Thus as claimed in the Introduction $\mathcal{E} = -(\mathfrak{H}, \mathfrak{J})$, that is, the superhamiltonian and supermomentum are the components of the instantaneous *energy-momentum* map.

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- The supermomentum by itself is a component of the *momentum* map $\mathcal{J}_\tau$ for the stabilizer group $\mathcal{G}_\tau$ of Σ_τ which does act in the instantaneous formalism, unlike $\mathcal{G}$.