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COMMENT

Negative energy states in quantum gravity?

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Abstract. The phenomenon of 'quantum tunnelling through classical constraints' is analysed. I show by example that such tunnelling may lead to physical inconsistencies: systems for which the extended phase space quantisation does not have a well defined classical limit and is wildly incompatible with the reduced phase space quantisation. These pathologies are shown to originate in an inappropriate choice of polarisation, and a quantisation ansatz is proposed which eliminates them. Such results indicate that initial value constraints should form absolute barriers quantum mechanically as well as classically, and consequently that tunnelling does *not* provide a physically viable means of transcending the positive energy theorems in quantum gravity.

Recently, Ashtekar and Horowitz [1] have argued that certain peculiarities in the dynamical structure of general relativity may, upon quantisation, lead to the presence of states with negative energies. Such states are the result of quantum tunnelling into regions which are forbidden by the classical constraints. This phenomenon can occur when one quantises the extended phase space and imposes the constraints as conditions on the quantum wavefunctions. It does not occur if, instead, one first eliminates the constraints classically and then quantises the reduced phase space. This incongruity has far-reaching physical implications and has led Ashtekar and Horowitz to conclude that '... the reduced phase space method is likely to yield an incomplete description of quantum gravity'.

I present here an analysis which renders these assertions suspect. It appears that canonical quantisation—that is, quantisation in the vertical polarisation—is inappropriate for systems with the features considered by Ashtekar and Horowitz. Indeed, I show by example that in such cases canonical quantisation may engender physical absurdities: quantum systems *all* of whose normalisable states are supported entirely in classically inadmissible regions and, moreover, have negative energies—despite the fact that a positive energy condition holds classically! This behaviour clearly indicates that the extended phase space quantisation has gone awry. The underlying cause is that the polarisation has been chosen in a way that is incompatible with the constraints, with the additional consequence that the quantisation of the extended phase space cannot be properly reconciled with that of the reduced phase space which, in contrast, is well behaved.

In the spirit of Ashtekar and Horowitz [1], consider a system with configuration space Q, extended phase space T^*Q and a single initial value constraint $C(q, p) = 0$. Suppose that (i) $C(q, p)$ is quadratic in the momenta; (ii) the constraint is preserved by the equations of motion; (iii) a positive energy condition holds classically; and (iv) the projection Q_+ of the constraint set $C^{-1}(0)$ is a *proper* subset of configuration space Q. Such a system may be thought of as a finite-dimensional dynamical model of general relativity, C playing the role of the super-Hamiltonian constraint. By analogy with the asymptotically flat case, the Hamiltonian is taken to be

$$H(q, p) = C(q, p) + E(q) \tag{1}$$

where E, the 'ADM energy', satisfies

$$E|C^{-1}(0) \geq 0.$$

Of the above, the last property (iv) is the most important and is characteristic of general relativity. Since the canonically quantised states ψ are functions on Q only, this feature gives rise to a novel possibility: need the physically admissible wavefunctions $\psi(q)$ be supported solely in Q_+, or can they tunnel into the classically forbidden region $Q_- = Q - Q_+$? *If* there is appreciable tunnelling into Q_- then, since $E(q)$ may be negative there, states with negative energies *may* appear in the quantum theory. To the extent that such a system accurately models general relativity, one must now—as Ashtekar and Horowitz have emphasised—seriously consider the likelihood of having negative energy states in quantum gravity.

Systems of this type are interesting for another (related) reason. Suppose one first solves the constraints classically and then quantises the resulting reduced phase space. In this framework negative energy states *cannot* emerge in the above fashion because, by construction, the reduced phase space retains no information regarding the classically inaccessible region Q_-. Thus one expects the extended and reduced phase space quantisations of such systems to be genuinely inequivalent.

Ashtekar and Horowitz gave a physically plausible illustration of these phenomena. I now present a similar, but much more bizarre example.

Take $Q = \mathbf{R}^2 - \{(0, 0)\}$ in polar coordinates, set

$$C(r, \theta, p_r, p_\theta) = p_r^2 - \sin\theta \tag{2}$$

and choose $E(\theta)$ such that

$$(\sin\theta)E(\theta) \geq 0. \tag{3}$$

By virtue of (1)-(3) it is clear that conditions (i)-(iv) are satisfied, where

$$Q_+ = \{(r, \theta) | 0 \leq \theta \leq \pi\}.$$

In particular, from (3) it follows that

$$E(\theta)|Q_+ \geq 0 \qquad \text{and} \qquad E(\theta)|Q_- \leq 0. \tag{4}$$

Canonically quantising, the space $\mathscr{H}$ of physically admissible states consists of those ψ which satisfy the quantum constraint equation $\hat{C}\psi = 0$, i.e.

$$\{-\hbar^2[\partial^2/\partial r^2 + (1/r)\partial/\partial r] - \sin\theta\}\psi(r, \theta) = 0.$$

The solution of this zeroth-order Bessel equation is

$$\psi(r, \theta) = \begin{cases} A_+(\theta)J_0(z) + B_+(\theta)Y_0(z) & \text{on } Q_+ \\ A_-(\theta)I_0(z) + B_-(\theta)K_0(z) & \text{on } Q_-\text{'} \end{cases} \tag{5}$$

where $z = \hbar^{-1}|\sin\theta|^{1/2}r$. Such a $\psi \in L^2(Q, r)$ iff $A_+ \equiv 0 \equiv B_+$ and $A_- \equiv 0$ so that the space $\mathscr{H} \cap L^2(Q, r)$ of normalisable physically admissible states contains only wavefunctions of the form

$$\psi(r, \theta) = B_-(\theta)K_0(\hbar^{-1}|\sin\theta|^{1/2}r)$$

with supp $B_-(\theta) \subseteq Q_-$. Thus *every* $\psi \in \mathscr{H} \cap L^2(Q, r)$ is supported *entirely* in the classically forbidden region Q_-. Furthermore, from (1) and (4)

$$\langle \psi, \hat{H}\psi \rangle = \langle \psi, E(\theta)\psi \rangle \leqslant 0$$

for all such ψ. Hence, by choosing $E(\theta)$ appropriately, it is possible to force *each* $\psi \in \mathscr{H} \cap L^2(Q, r)$ to have *negative* energy.

The canonical quantisation of this system is therefore very badly behaved. None of the normalisable physically admissible states have classical analogues and not only is the classical positive energy condition violated quantum mechanically, it is actually reversed! In essence, one may say that this extended phase space quantisation consists of nothing but tunnelling.

On the other hand, one does not expect the reduced phase space quantisation to exhibit any tunnelling at all. For this quantisation, first decompose the constraint set

$$C^{-1}(0) = C_> \cup C_<$$

where $p_r \gtrless 0$ on $C_{\gtrless}$, respectively†. Using the constraint $C = 0$ to solve for p_r and then eliminating r, the reduced phase space becomes $\bar{C}_> \cup \bar{C}_<$, each component of which can be identified with $T^*(0, \pi)$. Canonical coordinates on $\bar{C}_{\gtrless}$ are θ and

$$\bar{p}_\theta = p_\theta \mp \tfrac{1}{2} r (\sin\theta)^{1/2} \cot\theta,$$

and the reduced Hamiltonian is then

$$\bar{H}(\theta, \bar{p}_\theta) = E(\theta). \tag{6}$$

The corresponding quantum Hilbert space is

$$\bar{\mathscr{H}} = L^2(0, \pi) \oplus L^2(0, \pi).$$

Not surprisingly, every $\bar{\psi} = (\bar{\psi}_>(\theta), \bar{\psi}_<(\theta)) \in \bar{\mathscr{H}}$ is supported solely in the classically accessible region $0 < \theta < \pi$ and, since $E(\theta)$ is non-negative there, (6) implies that

$$\langle \bar{\psi}, \hat{\bar{H}}\bar{\psi} \rangle = \langle \bar{\psi}, E(\theta)\bar{\psi} \rangle \geqslant 0$$

always.

It is obvious that these two quantisations, as they stand, are in no sense comparable‡. Both this disparity and the pathological features of the extended phase space quantisation are directly attributable to the presence, in (5), of the terms involving I_0 and K_0. *If* these terms were absent, the association

$$\bar{\psi} \to \bar{\psi}_>(\theta) J_0(z) + \bar{\psi}_<(\theta) Y_0(z) \tag{7}$$

would define an isomorphism of $\bar{\mathscr{H}}$ with $\mathscr{H}$ and the quantisation of the extended phase space, like that of the reduced phase space, would be well behaved with a reasonable classical limit§.

This analysis, I believe, conclusively demonstrates that the extended phase space quantisation of this system is incorrect. What has gone wrong? A careful examination

† For simplicity, I excise the hypersurface in $C^{-1}(0)$ along which $p_r = 0$. These points will be taken into account quantum mechanically by the Cauchy completion process.

‡ In particular, this reduced phase space quantisation *cannot* be viewed as an 'incomplete' version of the extended phase space quantisation, cf [1].

§ Now, however, $\mathscr{H} \cap L^2(Q, r) = \{0\}$ so that (7) must be interpreted in a distributional sense.

shows that the mechanism responsible for this 'runaway' tunnelling is to a large extent an artefact of the canonical quantisation procedure. This is most easily understood using the language of geometric quantisation theory [2].

Recall that geometric quantisation proceeds by introducing a polarisation of the classical phase space. The quantum wavefunctions relative to a chosen polarisation are (essentially) functions on phase space which are constant along the leaves of the polarisation. In particular, canonical quantisation corresponds to geometric quantisation in the 'vertical' polarisation whose leaves are the fibres of the cotangent bundle projection $T^*Q \to Q$.

Now the tunnelling phenomenon in the above example is made possible by two circumstances. The first is condition (iv) which ensures that $Q_+ \subset Q$ strictly, and the second is that the fibres of T^*Q are just the leaves of the polarisation used in canonical quantisation. Putting these together, it follows that this sort of tunnelling is a result of the fact that *not* every leaf of the vertical polarisation intersects the constraint set or, equivalently, that the canonically quantised wavefunctions are *not* uniquely determined by their restrictions to the constraint set. Thus the ultimate source of the bizarre behaviour of the extended phase space quantisation is that *the polarisation has been chosen in a way that is incompatible with the constraints.*

In view of these results, should such 'quantum tunnelling through classical constraints' be considered a realistic possibility? Unfortunately this tunnelling, which was relatively mild and manageable in Ashtekar and Horowitz's system [1], can also rage out of control as in the example presented here. Since there is no *a priori* reason to discount either example and as it is not clear how to distinguish between 'acceptable' and 'extreme' tunnelling, it seems that one must either permit this type of tunnelling whenever it occurs—and occasionally forego a reasonable classical limit—or disallow it altogether.

Moreover, it follows from both the above discussion and reference [1] that there is little chance of any agreement between the extended and reduced phase space quantisations when tunnelling is present in the former. Even when the tunnelling is 'acceptable' this agreement is at best approximate and rigorous equivalence can be expected only when there is no tunnelling whatsoever. Of course, rigorous equivalence is itself a highly desirable objective in any case.

I therefore propose the latter course, namely, outlawing tunnelling through classical initial value constraints completely. This is most easily accomplished by adopting the following 'quantisation ansatz' for constrained classical systems: *quantisation should always be carried out in such a way that all wavefunctions are uniquely determined by their restrictions to the constraint set.* In practice, this places a restriction on the choice of polarisation. From a physical standpoint, this requirement seems reasonable since classically only those states contained in the constraint set are physically admissible and/or relevant, and this fact should be reflected in the quantum formalism [3]. Mathematically, as illustrated previously, this condition plays a crucial role in establishing the equivalence of the extended and reduced phase space quantisations of a constrained classical system [4]. For further discussion of this ansatz, see [3] and [4].

It remains to determine how the above example can be 'repaired'. The simplest way to enforce the quantisation ansatz is by discarding those leaves of the vertical polarisation which do not intersect $C^{-1}(0)$. This amounts to replacing the extended phase space T^*Q by T^*Q_+. Quantisation now yields the first line of (5) *only* and (7) defines a strict equivalence of the extended and reduced phase space quantisations, the latter of course being unchanged from before. Alternatively, one can leave the phase space as is and search for a new polarisation which satisfies the quantisation ansatz†.

To conclude, it appears that the tunnelling phenomenon of Ashtekar and Horowitz does not provide a physically viable means of transcending the classical positive energy theorems. This is not to say that negative energy states cannot arise in quantum gravity via other mechanisms; indeed, there are a number of arguments which strongly suggest their presence (cf reference [1] and the references cited therein). However, to the extent that one can safely generalise, I suggest that initial value constraints should form absolute barriers quantum mechanically as well as classically.

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References

[1] Ashtekar A and Horowitz G T 1982 *Phys. Rev.* D **26** 3342–53
[2] Woodhouse N M J 1980 *Geometric Quantization* (Oxford: Clarendon)
[3] Gotay M J and Sniatycki J 1981 *Commun. Math. Phys.* **82** 377–89
[4] Gotay M J 1986 *Constraints, Reduction and Quantization* to appear

† These do exist, although they are difficult to find. I shall not discuss these alternate quantisations here.