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A multisymplectic framework for classical field theory and the calculus of variations II: Space + time decomposition

Mark J. Gotay
Mathematics Department, United States Naval Academy, Annapolis, MD 21402-5002, U.S.A.

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Abstract: In a previous paper I laid the foundations of a covariant Hamiltonian framework for the calculus of variations in general. The purpose of the present work is to demonstrate, in the context of classical field theory, how this covariant Hamiltonian formalism may be space + time decomposed. It turns out that the resulting "instantaneous" Hamiltonian formalism is an infinite-dimensional version of Ostrogradskiĭ's theory and leads to the standard symplectic formulation of the initial value problem. The salient features of the analysis are: (i) the instantaneous Hamiltonian formalism does not depend upon the choice of Lepagean equivalent; (ii) the space + time decomposition can be performed either before or after the covariant Legendre transformation has been carried out, with equivalent results; (iii) the instantaneous Hamiltonian can be recovered in natural way from the multisymplectic structure inherent in the theory; and (iv) the space + time split symplectic structure lives on the space of Cauchy data for the evolution equations, as opposed to the space of solutions thereof.

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8. Introduction

A long-standing problem in the calculus of variations was to develop a Hamiltonian formalism which covariantly generalizes the basic notions of Hamiltonian mechanics. In a previous paper [6, hereafter referred to as I], I laid the foundations of such a framework in general, predicated upon the De Donder-Weyl approach to the calculus of variations. There I (i) proposed a new candidate for the covariant phase space, (ii) showed that it carries a canonical multisymplectic structure, (iii) constructed and characterized corresponding covariant Legendre transformations, and (iv) gave a suitable notion of regularity. This formalism enables one to deal directly with higher order Lagrangians as well as multiple integrals in much the same way as one treats ordinary mechanics.

But for various reasons (e.g., to perform an initial value analysis of a given set of evolution equations), it is often necessary to break manifest covariance. In this sequel I demonstrate, in the context of classical field theory, how this covariant Hamiltonian formalism may be space + time decomposed. The essential idea is to show how the multisymplectic structure on the covariant level "descends" to a symplectic structure on the instantaneous level once a Cauchy surface for the dynamics has been chosen. It turns out that the resulting "instantaneous" Hamiltonian formalism is an infinite-dimensional version of Ostrogradskiĭ's theory [10,13] and leads to the standard symplectic formulation of the initial value problem [7,15]. Thus this paper in conjunction with **I** provides a general and systematic means of canonically analyzing any Lagrangian variational principle.

Some noteworthy corollaries of the analysis are:

(a) The space + time split Hamiltonian and Lagrangian formalisms do not depend upon the choice of Lepagean equivalent. This is despite the fact that each such choice—corresponding to a particular approach to the calculus of variations—leads to markedly differing results in the covariant context. Nonetheless, the De Donder-Weyl approach (which is based upon the so-called "Cartan form") is in some sense the most natural for classical field theory.

(b) The space + time decomposition can be performed either before or after the covariant Legendre transformation has been carried out, with equivalent results. In other words, the following diagram commutes:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\text{Covariant}\\ \text{Lagrangian}\\ \text{formalism}\end{array}} \arrow[r, "\begin{array}{c}\text{Covariant}\\ \text{Legendre transformation}\end{array}"] \arrow[d, "\begin{array}{c}\text{space}\\ +\\ \text{time}\\ \text{split}\end{array}"'] & \boxed{\begin{array}{c}\text{Covariant}\\ \text{Hamiltonian}\\ \text{formalism}\end{array}} \arrow[d, "\begin{array}{c}\text{space}\\ +\\ \text{time}\\ \text{split}\end{array}"] \\
\boxed{\begin{array}{c}\text{Instantaneous}\\ \text{Lagrangian}\\ \text{formalism}\end{array}} \arrow[r, "\begin{array}{c}\text{Instantaneous}\\ \text{Legendre transformation}\end{array}"'] & \boxed{\begin{array}{c}\text{Instantaneous}\\ \text{Hamiltonian}\\ \text{formalism}\end{array}}
\end{tikzcd}$$

Diagram 1. Space + time decomposition

(c) The instantaneous Hamiltonian can be recovered in a natural way from the multisymplectic structure inherent in the theory, once one has fixed an "infinitesimal slicing" of the configuration bundle of the system at hand. This generalizes a result of [7], which states that the instantaneous Hamiltonian can be obtained as a "component" of the energy-momentum mapping associated to the gauge group of the system.

(d) The space + time split symplectic structure obtained here lives on the space of Cauchy data for the evolution equations, and *not* on the space of solutions thereof (as in, e.g., [3,12]). This is an important distinction, as the former is typically better behaved than the latter. It also obviates the problem of constructing a differentiable structure on the space of solutions as was attempted in [9].

Breaking manifest covariance is relatively trivial in mechanics, and is also well-understood in first order field theories, at least in the De Donder-Weyl framework (cf. [7] after which this paper is patterned). But the space + time decomposition does not seem to have been systematically studied in the context of alternative approaches to the calculus of variations, even in the first order case. Furthermore for higher order field theories several new features emerge. One has to do with the fact that there is additional ambiguity in the choice of Lepagean equivalent. Another is that a second symplectic reduction must be performed to eliminate the spatial derivatives of the fields which appear in the covariant formalism, but which are redundant in the instantaneous formalism. (The first symplectic reduction eliminates the spatial "multimomenta" which are irrelevant for the initial value problem.)

The present paper is the fourth in a series devoted to a new multisymplectic approach to classical field theory and the calculus of variations. The companion article **I** covers the basic ideas and develops the covariant Hamiltonian formalism. Previously I worked out the details of both this formalism and its space + time decomposition for the KdV equation as a test case [4]; it provides a simple yet interesting illustration of a (constrained) second order field theory. In a related paper [5], I present a new theory of Lepagean equivalents based upon the exterior differential systems approach to the variational calculus.

I begin in Section 9 with some preliminaries on Cauchy surfaces, spaces of fields and their tangent and cotangent bundles. The plan is then to construct each of the vertical arrows in Diagram 1 in turn. In Section 10 I space + time decompose the covariant Lagrangian formalism and use the instantaneous Legendre transformation to set up the initial value problem. I attack the Lagrangian formalism first, as it is the traditional way to proceed; moreover, the Lagrangian formalism is somewhat simpler in that the ambiguities which appear in the Hamiltonian context are not as severe here. The emphasis in this section is on those aspects of the space + time decomposition which pertain to higher order Lagrangians. Section 11 is concerned with the space + time decomposition of the covariant Hamiltonian formalism à la De Donder-Weyl. In particular, Section 11 describes the symplectic reductions necessary to obtain the space + time split symplectic structure on the space of Cauchy data for the evolution equations, and shows how to construct the instantaneous Hamiltonian in a "covariant" fashion. I also prove here that Diagram 1 commutes. The final section briefly examines how the space + time decomposition proceeds for field theories which are based on other approaches to the calculus of variations.

Notation and terminology are established in **I**, to which the reader is referred for the relevant background. Except for references, all numbering (section, equation, etc.) is consecutive with **I**.

9. Cauchy surfaces and spaces of fields

In the space + time decomposed ("instantaneous") formalism, dynamics is played out on infinite-dimensional spaces of fields at a given instant of time. To describe these spaces it is convenient to introduce some notation. If $K \to X$ is a fibration, then the space of its smooth sections will be denoted by the corresponding script letter, in this case $\mathcal{K}$. Occasionally, when this might be confusing, I will resort to the notation $\Gamma(K)$.

Let $\Sigma \subset X$ be an n-dimensional Cauchy (i.e., noncharacteristic) surface for the field equations. In what follows I assume either that Σ is compact without boundary or that asymptotic conditions are imposed on all fields which allow one to neglect surface terms. I choose coordinates x^μ, $\mu = 0, \dots, n$ on X which are *adapted* to Σ in the sense that Σ is locally given by $x^0 = 0$. Latin indices will always refer to spatial coordinates; viz. coordinates on Σ. Define $d^{n+1}x = dx^0 \wedge \dots \wedge dx^n$ and set $d^n x_\nu = i(\partial/\partial x^\nu)d^{n+1}x$, etc. Let i_Σ be the inclusion $\Sigma \hookrightarrow X$.

If $\Sigma \subset X$, then K_Σ signifies the restriction of K to Σ, and so $\mathcal{K}_\Sigma$ is the space of all sections κ of $K_\Sigma \to \Sigma$. Equivalently, $\mathcal{K}_\Sigma$ consists of restrictions of sections of $K \to X$ to Σ. If k^A are coordinates along the fibers of $K \to X$, then induced "coordinates" on $\mathcal{K}_\Sigma$ are $\kappa^A = k^A \circ \kappa$. It will not be necessary to topologize the spaces $\mathcal{K}_\Sigma$ in this paper; however, note that when completed in appropriate C^k or Sobolev topologies, they are known to be smooth manifolds [1, 14].

The tangent space to $\mathcal{K}_\Sigma$ at a section κ is

$$T_\kappa \mathcal{K}_\Sigma = \{\mathcal{V} : \Sigma \to VK_\Sigma \mid \mathcal{V} \text{ covers } \kappa\}.$$

Similarly, the smooth (or L^2) cotangent space to $\mathcal{K}_\Sigma$ at κ is

$$T^*_\kappa \mathcal{K}_\Sigma = \{\pi : \Sigma \to L(VK_\Sigma, \Lambda^n\Sigma) \mid \pi \text{ covers } \kappa\},$$

where $L(VK_\Sigma, \Lambda^n\Sigma)$ is the vector bundle over K_Σ whose fiber above $k \in K_x$ is the set of linear maps from $V_k K_\Sigma$ to $\Lambda^n_x\Sigma$. In adapted coordinates an element $\pi \in T^*_\kappa \mathcal{K}_\Sigma$ is expressible as

$$\pi = \pi_A dk^A \otimes d^n x_0.$$

The natural pairing of $T_\kappa \mathcal{K}_\Sigma$ with $T^*_\kappa \mathcal{K}_\Sigma$ is given by integration:

$$\pi \cdot \mathcal{V} = \int_\Sigma i_\mathcal{V} \pi.$$

The standard symplectic structure ω_Σ on $T^*\mathcal{K}_\Sigma$ is given in the usual fashion: $\omega_\Sigma = -d\theta_\Sigma$, where

$$\theta_\Sigma(\kappa, \pi) \cdot \mathcal{V} = \int_\Sigma i(T\tau \cdot \mathcal{V})\pi$$

is the canonical 1-from, $\mathcal{V} \in T_\pi(T^*\mathcal{K}_\Sigma)$, and τ is the cotangent projection. These have the local representations

$$\theta_\Sigma(\kappa, \pi) = \int_\Sigma \pi_A d\kappa^A \otimes d^n x_0$$

and

$$\omega_\Sigma(\kappa, \pi) = \int_\Sigma (d\kappa^A \wedge d\pi_A) \otimes d^n x_0,$$

and hence (κ^A, π_A) are canonical coordinates on $T^*\mathcal{K}_\Sigma$.

Finally, I need some special notation for jet bundles. Let $J^r_\Sigma K := (J^r K)_\Sigma$ be the restriction of $J^r K \to X$ to $\Sigma \subset X$. This must be distinguished from $J^r K_\Sigma$, which is the rth jet bundle of the fibration $K_\Sigma \to \Sigma$. Also, denote by $j^r\mathcal{K}$ the space of holonomic sections of $J^r K \to X$, so that $j^r_\Sigma \mathcal{K} := (j^r\mathcal{K})_\Sigma$ consists of restrictions of holonomic sections of $J^r K \to X$ to Σ. Again, this space should not be confused with the set $j^r\mathcal{K}_\Sigma$ of holonomic sections of $J^r K_\Sigma \to \Sigma$.

10. The instantaneous Lagrangian formalism

To space + time decompose a covariant theory it is necessary to choose an *infinitesimal slicing* of the covariant configuration bundle $\pi : Y \to X$. This consists of a Cauchy surface $\Sigma \subset X$ along with a π-projectable vector field ζ on Y with the property that $\zeta \mid Y_\Sigma$ is everywhere transverse to Y_Σ. (Actually ζ need only be defined in a neighborhood of Y_Σ.) The vector field ζ should be thought of as an "evolution direction" on Y; it governs the propagation of coordinates off Y_Σ to Y. Let ζ_X be its projection to X; in relativity, this is the "lapse/shift" vector field.

The covariant Lagrangian formalism lives on jet bundles $J^r Y$ whereas its instantaneous counterpart lives on space of fields and their time derivatives over the Cauchy surface Σ. To get from the former to the latter one first considers the spaces $\Gamma(J^r_\Sigma Y)$. Given $\gamma \in \Gamma(J^r_\Sigma Y)$ set $\gamma^A_{\mu_1\ldots\mu_s} = y^A_{\mu_1\ldots\mu_s} \circ \gamma$, so that $\Gamma(J^r_\Sigma Y)$ is parametrized by the quantities $\gamma^A, \gamma^A_\mu, \ldots, \gamma_{\mu_1\ldots\mu_r}$. Insofar as the initial value problem is concerned, only the temporal derivatives $\gamma^A_{0\ldots0}$ are important; the spatial derivatives $\gamma^A_{\mu_1\ldots\mu_s i}$ are irrelevant. These spatial derivatives can be eliminated by restricting to the subset $j^r_\Sigma\mathcal{Y} \subset \Gamma(J^r_\Sigma Y)$. Elements $\gamma \in j^r_\Sigma\mathcal{Y}$ are *spatially* holonomic in the sense that $\gamma^A_{\mu_1\ldots\mu_s i} = \gamma^A_{\mu_1\ldots\mu_s, i}$ along Σ for $0 \leqslant s \leqslant r-1$. It follows that $j^r_\Sigma\mathcal{Y}$ is parametrized by γ^A, $\gamma^A_0, \ldots, \gamma^A_{0_r} := \gamma^A_{0\ldots0}$ (r times).

Proposition 9. *A choice of evolution direction ζ on Y gives rise to an identification $j^r_\Sigma\mathcal{Y} \approx T^r\mathcal{Y}_\Sigma$.*

Here $T^r\mathcal{Y}_\Sigma := J^r_0(\mathbb{R}, \mathcal{Y}_\Sigma)$ is the rth order tangent bundle of $\mathcal{Y}_\Sigma$. The proof proceeds in two stages.

Lemma 10. *A choice of ζ induces an injection $j^r_\Sigma\mathcal{Y} \to Tj^{r-1}_\Sigma\mathcal{Y}$.*

Proof. Let $\gamma \in J^r Y$ so that $\gamma = j^r\phi(x)$ for some $\phi \in \mathcal{Y}$, where $x = \pi^r(\gamma)$. Recall that γ induces a map $\hat\gamma : T_x X \to T_{\pi^r_{r-1}(\gamma)} J^{r-1} Y$ according to $\hat\gamma(\xi) = T_x(j^{r-1}\phi) \cdot \xi$. In coordinates

$$\hat\gamma\left(\frac{\partial}{\partial x^\nu}\right) = \frac{\partial}{\partial x^\nu} + \gamma^A_\nu \frac{\partial}{\partial y^A} + \cdots + \gamma^A_{\mu_1\ldots\mu_{r-1}\nu}\, \frac{\partial}{\partial y^A_{\mu_1\ldots\mu_{r-1}}}.$$

Now suppose that γ is actually a section. Fixing the evolution direction ζ then gives rise to the map $\gamma_\zeta : X \to VJ^{r-1}Y$ defined by

$$\gamma_\zeta(x) = \hat{\gamma}(\zeta_X(x)) - j^{r-1}\zeta(\pi^r_{r-1}(\gamma(x))),$$

where $j^{r-1}\zeta$ is the prolongation of ζ on Y to a vector field on $J^{r-1}Y$. (The second term compensates for the fact that $\hat{\gamma}(\zeta_X)$ is not π^{r-1}-vertical.) Upon writing $j^{r-1}\zeta = (\zeta^\mu, \zeta^A, \ldots, \zeta^A_{\mu_1\ldots\mu_{r-1}})$ the local expression for γ_ζ becomes

$$\gamma_\zeta = \left(\zeta^\nu \gamma^A_\nu - \zeta^A\right)\frac{\partial}{\partial y^A} + \cdots + \left(\zeta^\nu \gamma^A_{\mu_1\ldots\mu_{r-1}\nu} - \zeta^A_{\mu_1\ldots\mu_{r-1}}\right)\frac{\partial}{\partial y^A_{\mu_1\ldots\mu_{r-1}}}.$$

Since γ_ζ covers $\pi^r_{r-1}\circ\gamma : X \to J^{r-1}Y$, it follows that $\gamma_\zeta \in T_{\pi^r_{r-1}(\gamma)}\Gamma(J^{r-1}Y)$.

This works equally well if one restricts to Σ. Thus the association $\gamma \rightsquigarrow \gamma_\zeta$ yields a map $\Gamma(J^r_\Sigma Y) \to T\Gamma(J^{r-1}_\Sigma Y)$. Note that both $\Gamma(J^r_\Sigma Y)$ and $T\Gamma(J^{r-1}_\Sigma Y)$ fiber over $\Gamma(J^{r-1}_\Sigma Y)$ and that this map is fiber preserving.

Next, suppose $\gamma \in j^r_\Sigma\mathcal{Y}$; then I claim that γ_ζ actually lies in $Tj^{r-1}_\Sigma\mathcal{Y} \subset T\Gamma(J^{r-1}_\Sigma Y)$. For this, it suffices to show that the components

$$(\gamma_\zeta)^A_{\mu_1\ldots\mu_s} = \zeta^\nu\gamma^A_{\mu_1\ldots\mu_s\nu} - \zeta^A_{\mu_1\ldots\mu_s} \tag{10.1}$$

of γ_ζ satisfy

$$(\gamma_\zeta)^A_{\mu_1\ldots\mu_s i} = (\gamma_\zeta)^A_{\mu_1\ldots\mu_s,i} \tag{10.2}$$

for all spatial indices i. As $\zeta \mid Y_\Sigma$ is everywhere transverse to Y_Σ, there exists an adapted chart on a neighborhood of Y_Σ in Y in which $\partial/\partial x^0 = \zeta$. (I refer to such a chart as "ζ-adapted".) Then locally $j^{r-1}\zeta = \partial/\partial x^0$ as well, and hence (10.2) follows immediately from (10.1) and the fact that γ is spatially holonomic.

Finally, it remains to verify that the induced map $j^r_\Sigma\mathcal{Y} \to Tj^{r-1}_\Sigma\mathcal{Y}$ is one-to-one. But in ζ-adapted charts, the association $\gamma \rightsquigarrow \gamma_\zeta$ is

$$(\gamma^A, \ldots, \gamma^A_{0_r}) \rightsquigarrow \left((\gamma^A, \ldots, \gamma^A_{0_{r-1}}), \gamma^A_0\frac{\partial}{\partial y^A} + \cdots + \gamma^A_{0_r}\frac{\partial}{\partial y^A_{0_{r-1}}}\right) \tag{10.3}$$

and the result in obvious. □

Proof of Proposition 9. I use induction on r. For $r = 1$, the lemma gives an injection $j^1_\Sigma\mathcal{Y} \to T\mathcal{Y}_\Sigma$. From (10.3) it follows that this is in fact a surjection, since γ^A and γ^A_0 are independently specifiable along Σ. Now suppose there is an isomorphism $j^{r-1}_\Sigma\mathcal{Y} \approx T^{r-1}\mathcal{Y}_\Sigma$. Then the lemma gives an injection $j^r_\Sigma\mathcal{Y} \to Tj^{r-1}_\Sigma\mathcal{Y} \approx T(T^{r-1}\mathcal{Y}_\Sigma)$, the image of which clearly lies on the "diagonal" $T^r\mathcal{Y}_\Sigma \subset T(T^{r-1}\mathcal{Y}_\Sigma)$. In a ζ-adapted chart this map is thus

$$(\gamma^A, \ldots, \gamma^A_{0_r}) \rightsquigarrow \left(\gamma^A, \gamma^A_0\frac{\partial}{\partial y^A} + \cdots + \gamma^A_{0_r}\frac{\partial}{\partial y^A_{0_{r-1}}}\right).$$

As before, since $\gamma^A, \ldots, \gamma^A_{0_r}$ are independently specifiable along Σ, this map is surjective. This proves the proposition. □

If $\gamma^A, \gamma^A_{(1)}, \ldots, \gamma^A_{(r)}$ are tangent coordinates on $T^r\mathcal{Y}_\Sigma$, then the isomorphism $t^r_{\Sigma,\zeta} : j^r_\Sigma \mathcal{Y} \to T^r\mathcal{Y}_\Sigma$ takes the form

$$\gamma^A_{(s)} = (\gamma_\zeta)^s[\gamma^A]$$

for $0 \leqslant s \leqslant r$, where $(\gamma_\zeta)^s$ denotes s-fold differentiation by γ_ζ. Aesthetically it is perhaps preferable to work on $T^r\mathcal{Y}_\Sigma$, but computationally $j^r_\Sigma\mathcal{Y}$ is simpler. In any case I will henceforth suppress the isomorphism and use ζ-adapted coordinates to identify these two spaces.

Using Proposition 9 it is now possible to construct the instantaneous Lagrangian formalism for a kth order field theory. The *instantaneous Lagrangian* $L_{\Sigma,\zeta} : T^k\mathcal{Y}_\Sigma \to \mathbb{R}$ is

$$L_{\Sigma,\zeta}(\gamma) = \int_\Sigma \gamma^*(i_{j^k\zeta}\mathcal{L}),$$

where $\mathcal{L}$ is the Lagrangian density, viewed as a π^k-horizontal $(n+1)$-form on J^kY. Locally $\mathcal{L} = Ld^{n+1}x$ and so the Lagrangian becomes

$$L_{\Sigma,\zeta}(\gamma) = \int_\Sigma (L \circ \gamma) d^n x_0.$$

In this expression the integrand has the functional form

$$L \circ \gamma = L\left(j^k\gamma^A, j^{k-1}\gamma^A_0, \ldots, \gamma^A_{0_k}\right). \tag{10.4}$$

Here I am viewing the $\gamma^A_{0_s}$ as locally defined functions on Σ, and so the jets involved consist only of spatial derivatives.

The velocity phase space is $T^{2k-1}\mathcal{Y}_\Sigma$ and comes equipped with the Poincaré-Cartan 1-form $\theta_{L,\Sigma}$. This form can be constructed directly from the Lagrangian in the usual fashion [10], but here I examine how it arises via the space + time decomposition. Let $\Theta_\mathcal{L}$ be a *strict* Lepagean equivalent of $\mathcal{L}$. (One could consider more general Lepagean equivalents with the same results, but I restrict attention to strict ones for simplicity. For further details on Lepagean equivalents see Section 3 of **I** and [5].) Thus $\Theta_\mathcal{L}$ is a (π^{2k-1}_{k-1})-horizontal $(n+1)$-form on $J^{2k-1}Y$ which in jet charts looks like

$$\Theta_\mathcal{L} = Ld^{n+1}x + \left(p_A^{\mu_1\ldots\mu_{k-1}\nu}\psi^A_{\mu_1\ldots\mu_{k-1}} + \cdots + p_A^\nu\psi^A\right) \wedge d^n x_\nu + \chi, \tag{10.5}$$

where χ is at least quadratic in the generators $\psi^A_{\mu_1\ldots\mu_s} = dy^A_{\mu_1\ldots\mu_s} - y^A_{\mu_1\ldots\mu_s\nu}dx^\nu$ of the contact ideal on $J^{2k-1}Y$, and the coefficients $p_A^{\mu_1\ldots\mu_s}$ are given by

$$p_A^{\mu_1\ldots\mu_s} = \begin{cases} \dfrac{\partial L}{\partial y^A_{\mu_1\ldots\mu_k}} + c_A^{\mu_1\ldots\mu_k}, & s = k \\[2ex] \dfrac{\partial L}{\partial y^A_{\mu_1\ldots\mu_s}} - D_\nu p_A^{\mu_1\ldots\mu_s\nu} + c_A^{\mu_1\ldots\mu_s}, & 1 \leqslant s < k. \end{cases} \tag{10.6}$$

Here the functions $c_A^{\mu_1\ldots\mu_s}$, of order at most $2k-2$, satisfy $c_A^{\mu_1\ldots(\mu_{s-1}\mu_s)} = 0$ and $c_A^{\mu} = 0$, but are otherwise undetermined.

Define the 1-form $\Theta_{\mathcal{L},\Sigma}$ on $\Gamma(J_\Sigma^{2k-1}Y)$ by

$$\Theta_{\mathcal{L},\Sigma}(\gamma)\cdot\mathcal{V} = \int_\Sigma \gamma^*(i_\mathcal{V}\Theta_\mathcal{L}),$$

where $\mathcal{V} \in T_\gamma\Gamma(J_\Sigma^{2k-1}Y)$. Now suppose $\gamma \in j_\Sigma^{2k-1}\mathcal{Y}$ so that $\gamma = j^{2k-1}\phi \circ i_\Sigma$ for some $\phi \in \mathcal{Y}$. Then

$$\gamma^*(i_\mathcal{V}\chi) = i_\Sigma^*(j^{2k-1}\phi)^*(i_\mathcal{V}\chi) = 0$$

as χ is at least 2-contact. Thus the term χ in $\Theta_\mathcal{L}$ effectively disappears once the formalism is space + time split. In adapted charts, (10.5) and some integrations by parts then yield

$$\begin{aligned}\Theta_{\mathcal{L},\Sigma}(\gamma) &= \int_\Sigma \left(\varrho_A^{\mu_1\ldots\mu_{k-1}0} d\gamma^A_{\mu_1\ldots\mu_{k-1}} + \cdots + \varrho_A^0 d\gamma^A\right)\otimes d^nx_0 \\ &= \int_\Sigma \left(\pi_A^{0_{k-1}} d\gamma^A_{0_{k-1}} + \cdots + \pi_A d\gamma^A\right)\otimes d^nx_0 \end{aligned} \tag{10.7}$$

as a 1-form on $j_\Sigma^{2k-1}\mathcal{Y}$, where $\varrho_A^{\mu_1\ldots\mu_s} = p_A^{\mu_1\ldots\mu_s}\circ\gamma$ and

$$\pi_A^{0_s} = \sum_{r=0}^{k-s-1}(-1)^r\binom{s+r}{s}\left(\varrho_A^{i_1\ldots i_r 0_{s+1}}\right)_{,i_1\ldots i_r}. \tag{10.8}$$

Finally, under the identification $j_\Sigma^{2k-1}\mathcal{Y} \approx T^{2k-1}\mathcal{Y}_\Sigma$, $\Theta_{\mathcal{L},\Sigma}$ induces the sought after form $\theta_{L,\Sigma}$ on $T^{2k-1}\mathcal{Y}_\Sigma$ whose expression in ζ-adapted charts is just (10.7).

With $L_{\Sigma,\zeta}$ and $\theta_{L,\Sigma}$ in hand, the construction of the left hand arrow in Diagram 1 is finished. From here on the instantaneous formalism reduces to higher order mechanics, albeit on infinite-dimensional spaces of sections. (See for example [10, 13] and Section 6 of **I**.) But there is one important observation to be made; namely, that the instantaneous Lagrangian formalism obtained in this manner is independent of all the ambiguities which are present on the covariant level:

Proposition 11. *The instantaneous Lagrangian formalism is independent of the choice of Lepagean equivalent* $\Theta_\mathcal{L}$.

Proof. It is only necessary to show that the Poincaré-Cartan form $\theta_{L,\Sigma}$ does not depend upon the specific choice of $\Theta_\mathcal{L}$. I have already indicated that the at least 2-contact term χ in (10.5) is irrelevant, so it remains to verify that the antisymmetric components $c_A^{\mu_1\ldots\mu_s}$ of the multimomenta $p_A^{\mu_1\ldots\mu_s}$ play no role in $\theta_{L,\Sigma}$ as well.

To see this, eliminate the $\varrho_A^{\mu_1\ldots\mu_s}$ in (10.8) by repeatedly substituting from (10.6) while keeping the functional form (10.4) in mind. Since the $c_A^{\mu_1\ldots\mu_s}$ are antisymmetric in their last two indices while symmetric in their first $s-1$ indices, some algebra yields

$$\pi_A^{0_s} = \frac{\delta L}{\delta y^A_{0_{s+1}}}(\gamma) \tag{10.9}$$

as might be expected. But this expression involves only L. □

Remark 12. Of course Proposition 11 is hardly surprising since, as remarked previously, the instantaneous Lagrangian formalism effectively amounts to a higher order single integral variational problem, and in this circumstance there is a unique Lepagean equivalent (or Poincaré-Cartan form).

Remark 13. When computing the instantaneous momenta $\pi_A^{0_s}$ via (10.9), one must be careful to take the spatial dependence of the fields into account. Explicitly, the space + time split variational derivative is

$$\frac{\delta L}{\delta y^A_{0_s}} = \sum_{r=0}^{k-s}\left[(-1)^r D_0^r\left(\frac{\partial L}{\partial y^A_{0_{s+r}}}\right)\right] + \sum_{q=0}^{k-s-1}\left[\sum_{p=1}^{k-s-q}\left[(-1)^{q+p}\binom{q+p}{p}D_{i_1}\cdots D_{i_p}D_0^q\left(\frac{\partial L}{\partial y^A_{0_{s+q}i_1\ldots i_p}}\right)\right]\right]$$

where $D_0^r = D_0\cdots D_0$ (r times). The first term in the above is exactly what one would get in mechanics, while the second term reflects the fact that one is space + time decomposing a *field* theory.

Continuing with the instantaneous formalism, the phase space is $T^*(T^{k-1}\mathcal{Y}_\Sigma)$ with its standard symplectic structure $\omega_\Sigma = -d\theta_\Sigma$. Elements of this space can be written

$$\pi = (\pi_A^{0_{k-1}}\,dy^A_{0_{k-1}} + \cdots + \pi_A\,dy^A)\otimes d^nx_0.$$

Note that $T^{2k-1}\mathcal{Y}_\Sigma$ and $T^*(T^{k-1}\mathcal{Y}_\Sigma)$ are vector bundles over $T^{k-1}\mathcal{Y}_\Sigma$ of equal rank. This fact and the expression for π indicate that, for higher order systems, the instantaneous configuration space is in some sense really $T^{k-1}\mathcal{Y}_\Sigma$ and *not* $\mathcal{Y}_\Sigma$, as might be naively thought.

The instantaneous Lagrangian and Hamiltonian formalisms are connected by the *instantaneous Legendre transformation*, which is defined analogously to its covariant alter ego (cf. Section 4 of **I**). In view of (10.7) one may consider $\theta_{L,\Sigma}$ as a 1-form on $T^{k-1}\mathcal{Y}_\Sigma$ with coefficients in $C^\infty(T^{2k-1}\mathcal{Y}_\Sigma)$; i.e., as a $(T^{k-1}\mathcal{Y}_\Sigma)$-bundle map $\sigma_{L,\Sigma}: T^{2k-1}\mathcal{Y}_\Sigma \to T^*(T^{k-1}\mathcal{Y}_\Sigma)$. This "tautologous" map has the coordinate expression

$$\gamma^A,\ldots,\gamma^A_{0_{2k-1}} \rightsquigarrow \gamma^A,\ldots,\gamma^A_{0_{k-1}};\pi_A,\ldots,\pi_A^{0_{k-1}}.$$

Clearly $\theta_{L,\Sigma} = \sigma^*_{L,\Sigma}\theta_\Sigma$. Let Π^{k-1}_Σ be the image of $\sigma_{L,\Sigma}$ in $T^*(T^{k-1}\mathcal{Y}_\Sigma)$; Π^{k-1}_Σ is the *instantaneous primary constraint set.* It inherits a (generically) presymplectic structure from $T^*(T^{k-1}\mathcal{Y}_\Sigma)$ which I also denote by ω_Σ.

Finally, it is necessary to construct the instantaneous Hamiltonian. To this end, define the *energy* $E_{\Sigma,\zeta}: T^{2k-1}\mathcal{Y}_\Sigma \to \mathbb{R}$ by [2]

$$E_{\Sigma,\zeta}(\gamma) = -\int_\Sigma \gamma^*(i_{j^{2k-1}\zeta}\Theta_{\mathcal{L}}). \tag{10.10}$$

A short computation in ζ-adapted charts similar to that leading to (10.7) gives

$$E_{\Sigma,\zeta}(\gamma) = \int_\Sigma \left(\pi_A^{0_{k-1}}\gamma_{0_k}^A + \cdots + \pi_A\gamma_0^A - L(\gamma)\right) d^n x_0.$$

Moreover one checks that $(\ker T\sigma_{L,\Sigma})[E_{\Sigma,\zeta}] = 0$. Thus, provided the fibers of $\sigma_{L,\Sigma}$ are connected, it is clear that the relation $H_{\Sigma,\zeta} \circ \sigma_{L,\Sigma} = E_{\Sigma,\zeta}$ implicitly defines the *instantaneous Hamiltonian* $H_{\Sigma,\zeta}$ on Π_Σ^{k-1}. Hamilton's equations on Π_Σ^{k-1} are then

$$i_\xi \omega_\Sigma = dH_{\Sigma,\zeta},$$

which is to be solved for (the flow of) the evolution vector field ξ, etc.

As with its Lagrangian counterpart, the instantaneous Hamiltonian formalism so constructed is completely fixed, independent of any covariant ambiguities. The stage is now set to proceed with the initial value analysis as is explained, e.g., in [7, 15]. Note that—computational complexity aside—the initial value analysis proceeds as always regardless of the order of the field theory. Discussions of these and related matters in the specific case of higher order systems (along with numerous references) can be found in [10, 8].

11. The instantaneous Hamiltonian formalism

In the preceding section I obtained the instantaneous Hamiltonian formalism by first space + time decomposing the covariant Lagrangian formalism, and then performing the instantaneous Legendre transformation. Here I show how it arises by directly space + time decomposing the covariant Hamiltonian formalism of **I**.

This covariant Hamiltonian formalism is based specifically on the De Donder-Weyl approach to the calculus of variations. Throughout this section I therefore assume that the chosen Lepagean equivalent $\Theta_\mathcal{L}$ is in fact a Cartan form in the sense of Section 3. (In other words, $\Theta_\mathcal{L}$ is a strict Lepagean equivalent for which the at least 2-contact term χ vanishes.) I will indicate what happens in a more general setting in the next section.

Since the covariant phase space is Z^{k-1}, one is led to study the space $\mathcal{Z}_\Sigma^{k-1} = \Gamma(Z_\Sigma^{k-1})$. Recalling that Z^{k-1} fibers over $J^{k-1}Y$ with projection λ, one obtains the induced fibration $\mathcal{Z}_\Sigma^{k-1} \to \Gamma(J_\Sigma^{k-1}Y)$.

The first stage of the space + time decomposition consists of integrating the canonical $(n+2)$- and $(n+1)$-forms Ω and Θ on Z^{k-1} over Σ thereby obtaining 2- and 1-forms Ω_Σ and Θ_Σ on $\mathcal{Z}_\Sigma^{k-1}$, as follows. For $\varrho \in \mathcal{Z}_\Sigma^{k-1}$ and $\mathcal{V}, \mathcal{W} \in T_\varrho \mathcal{Z}_\Sigma^{k-1}$, set

$$\Theta_\Sigma(\varrho)\cdot\mathcal{V} = \int_\Sigma \varrho^*(i_\mathcal{V}\Theta)$$

and

$$\Omega_\Sigma(\varrho)\cdot(\mathcal{V},\mathcal{W}) = \int_\Sigma \varrho^*(i_\mathcal{W} i_\mathcal{V}\Omega).$$

A straightforward calculation using Stokes' theorem shows that $\Omega_\Sigma = -d\Theta_\Sigma$. In adapted coordinates (4.3) gives

$$\Theta_\Sigma(\varrho) = \int_\Sigma \left(\varrho_A^{\mu_1\ldots\mu_{k-1}0} d\varphi^A_{\mu_1\ldots\mu_{k-1}} + \cdots + \varrho_A^0 d\varphi^A\right) \otimes d^n x_0$$

and

$$\Omega_\Sigma(\varrho) = \int_\Sigma \left(d\varphi^A_{\mu_1\ldots\mu_{k-1}} \wedge d\varrho_A^{\mu_1\ldots\mu_{k-1}0} + \cdots + d\varphi^A \wedge d\varrho_A^0\right) \otimes d^n x_0,$$

where $\varphi = \lambda \circ \varrho \in \Gamma(J^{k-1}_\Sigma Y)$, $\varphi^A_{\mu_1\ldots\mu_s} = y^A_{\mu_1\ldots\mu_s} \circ \varphi$ and $\varrho_A^{\mu_1\ldots\mu_s} = p_A^{\mu_1\ldots\mu_s} \circ \varrho$.

However, the form Ω_Σ on $\mathcal{Z}^{k-1}_\Sigma$ is merely *pre*symplectic, as it has a nontrivial kernel. In fact, from the formula above for Ω_Σ it is apparent that

$$\ker\, \Omega_\Sigma = \text{span}\left\{\frac{\partial}{\partial p_A^{\mu_1\ldots\mu_{k-1}i}}, \ldots, \frac{\partial}{\partial p_A^i}, \frac{\partial}{\partial p}\right\}.$$

These characteristics can be eliminated by a symplectic reduction.

Consider the vector bundle map $R_\Sigma : \mathcal{Z}^{k-1}_\Sigma \to T^*\Gamma(J^{k-1}_\Sigma Y)$ over $\Gamma(J^{k-1}_\Sigma Y)$ defined by

$$R_\Sigma(\varrho) \cdot \mathcal{V} = \int_\Sigma \varphi^*(i_\mathcal{V} \varrho)$$

for $\mathcal{V} \in T_\varphi \Gamma(J^{k-1}_\Sigma Y)$. Using canonical coordinates

$$\varphi^A, \ldots, \varphi^A_{\mu_1\ldots\mu_{k-1}}, \bar{\pi}_A, \ldots, \bar{\pi}_A^{\mu_1\ldots\mu_{k-1}}$$

on $T^*\Gamma(J^{k-1}_\Sigma Y)$, the local expression for R_Σ becomes simply

$$\bar{\pi}_A^{\mu_1\ldots\mu_s} = \varrho_A^{\mu_1\ldots\mu_s 0} \tag{11.1}$$

for $0 \leqslant s \leqslant k-1$. A moment's reflection shows that R_Σ is a surjective submersion.

Now let $\bar{\theta}_\Sigma$ and $\bar{\omega}_\Sigma$ be the canonical 1- and 2-forms on $T^*\Gamma(J^{k-1}_\Sigma Y)$, respectively. Another routine calculation using the universal property of Θ establishes

Lemma 12. $R^*_\Sigma \bar{\theta}_\Sigma = \Theta_\Sigma$.

Thus $R^*_\Sigma \bar{\omega}_\Sigma = \Omega_\Sigma$. Since $\bar{\omega}_\Sigma$ is (weakly) nondegenerate and R_Σ is a submersion it follows that $\ker\, \Omega_\Sigma = \ker\, TR_\Sigma$. From all this one obtains:

Proposition 13. *The reduced symplectic manifold* $\mathcal{Z}^{k-1}_\Sigma/(\ker \Omega_\Sigma)$ *is canonically isomorphic to* $T^*\Gamma(J^{k-1}_\Sigma Y)$ *with its standard symplectic structure.*

This reduction therefore splits the "temporal" multimomenta $p_A^{\mu_1\ldots\mu_s 0}$ off from the "spatial" multimomenta $p_A^{\mu_1\ldots\mu_s i}$ and eliminates the latter as well as the "covariant Hamiltonian" p. Clearly only the temporal components are relevant for the initial value problem.

However, this is not all, since the space $\Gamma(J^{k-1}_\Sigma Y)$ still contains the superfluous spatial derivatives $\varphi^A_{\mu_1\ldots\mu_s i}$. These can be eliminated as in the preceding section via a further symplectic reduction. (Note that this is unnecessary when $k = 1$.)

For this, again restrict to the set $j^{k-1}_\Sigma\mathcal{Y}$ of spatially holonomic sections. One may view the inclusion $j_\Sigma : j^{k-1}_\Sigma\mathcal{Y} \hookrightarrow \Gamma(J^{k-1}_\Sigma Y)$ as defining a holonomic constraint $\tau^{-1}(j^{k-1}_\Sigma\mathcal{Y}) \subset T^*\Gamma(J^{k-1}_\Sigma Y)$ in the sense of mechanics [11]; reduction then yields the cotangent bundle $T^*(j^{k-1}_\Sigma\mathcal{Y})$ with its standard symplectic structure. Repeatedly integrating by parts, the corresponding projection $j^*_\Sigma : \tau^{-1}(j^{k-1}_\Sigma\mathcal{Y}) \to T^*(j^{k-1}_\Sigma\mathcal{Y})$ is found to be

$$\pi^{0_s}_A = \sum_{r=0}^{k-s-1}(-1)^r\binom{s+r}{s}\left(\bar\pi^{i_1\ldots i_r 0_s}_A\right)_{,i_1\ldots i_r}, \tag{11.2}$$

where $\varphi^A_{0_s}$, $\pi^{0_s}_A$, $0 \leqslant s \leqslant k-1$, are canonical coordinates on $T^*(j^{k-1}_\Sigma\mathcal{Y})$. Using Proposition 9, one can then identify $T^*(j^{k-1}_\Sigma\mathcal{Y}) \approx T^*(T^{k-1}\mathcal{Y}_\Sigma)$. Thus in this manner one recovers the instantaneous phase space of the preceding section.

Now let $\sigma_{\mathcal{L}} : J^{2k-1}Y \to Z^{k-1}$ be the covariant Legendre transformation. Recall from Section 4 that the "covariant primary constraint set" is

$$P^{k-1} = \sigma_{\mathcal{L}}(J^{2k-1}Y) \subset Z^{k-1}.$$

With a slight abuse of notation, set

$$\mathcal{P}^{k-1}_\Sigma = \sigma_{\mathcal{L}}(j^{2k-1}_\Sigma\mathcal{Y}) \subset \mathcal{Z}^{k-1}_\Sigma.$$

Proposition 14. *The following diagram commutes*:

$$\begin{array}{ccc} j^{2k-1}_\Sigma\mathcal{Y} & \xrightarrow{\sigma_{\mathcal{L}}} & \mathcal{P}^{k-1}_\Sigma \\ {\scriptstyle t^{2k-1}_{\Sigma,\zeta}}\downarrow & & \downarrow{\scriptstyle j^*_\Sigma\circ R_\Sigma} \\ T^{2k-1}\mathcal{Y}_\Sigma & \xrightarrow[\sigma_{L,\Sigma}]{} & T^*(T^{k-1}\mathcal{Y}_\Sigma) \end{array}$$

Proof. Work in ζ-adapted coordinates. Recursively substituting (4.5) into (11.2) while taking (11.1) into account yields exactly (10.9). □

Corollary 15. $j^*_\Sigma\big(R_\Sigma(\mathcal{P}^{k-1}_\Sigma)\big) = \Pi^{k-1}_\Sigma$.

Proof. That $j^*_\Sigma\big(R_\Sigma(\mathcal{P}^{k-1}_\Sigma)\big) \subseteq \Pi^{k-1}_\Sigma$ is a consequence of Proposition 14. To prove surjectivity, fix $(\varphi,\pi) \in \Pi^{k-1}_\Sigma$. Then by Proposition 9 and the definition of the instantaneous primary constraint set, there is some $\phi \in \mathcal{Y}$ such that $\gamma = t^{2k-1}_{\Sigma,\zeta}\left(j^{2k-1}\phi \circ i_\Sigma\right)$ satisfies $\sigma_{L,\Sigma}\circ\gamma = (\varphi,\pi)$. But then

$$(\varphi,\pi) = j^*_\Sigma\big(R_\Sigma(\sigma_{\mathcal{L}}\circ j^{2k-1}\phi\circ i_\Sigma)\big)$$

again by Proposition 14. □

Remark 14. A section $\varrho \in \mathcal{P}_\Sigma^{k-1}$ which projects onto $(\varphi, \pi) \in \Pi_\Sigma^{k-1}$ is called a *holonomic lift* of (φ, π). This last result then says in particular that holonomic lifts always exist.

It remains to "covariantly" recover the instantaneous Hamiltonian. For this, define $K_{\Sigma,\zeta} : \mathcal{Z}_\Sigma^{k-1} \to \mathbb{R}$ by

$$K_{\Sigma,\zeta}(\varrho) = -\int_\Sigma \varphi^*(i_{j^{k-1}\zeta}\varrho) \tag{11.3}$$

where, as usual, $\varphi = \lambda \circ \varrho$.

Proposition 16. *The instantaneous Hamiltonian* $H_{\Sigma,\zeta} : \Pi_\Sigma^{k-1} \to \mathbb{R}$ *is given by*

$$H_{\Sigma,\zeta}(\varphi, \pi) = K_{\Sigma,\zeta}(\varrho) \tag{11.4}$$

where ϱ *is any holonomic lift of* (φ, π) *to* $\mathcal{P}_\Sigma^{k-1}$.

Proof. Using the definition (4.1) of Θ, $K_{\Sigma,\zeta}$ can be rewritten

$$K_{\Sigma,\zeta}(\varrho) = -\int_\Sigma \varrho^*\left(i_{\overline{j^{k-1}\zeta}}\Theta\right) \tag{11.5}$$

where $\overline{j^{k-1}\zeta}$ is the canonical lift of the vector field $j^{k-1}\zeta$ on $J^{k-1}Y$ to Z^{k-1}. Now, if $\varrho \in \mathcal{P}_\Sigma^{k-1}$, then $\varrho = \sigma_\mathcal{L} \circ \gamma$ for some $\gamma \in j_\Sigma^{2k-1}\mathcal{Y}$. Then from (4.6), Proposition 14 and (10.10),

$$\begin{aligned} K_{\Sigma,\zeta}(\varrho) = K_{\Sigma,\zeta}(\sigma_\mathcal{L} \circ \gamma) &= -\int_\Sigma (\sigma_\mathcal{L} \circ \gamma)^*\left(i_{\overline{j^{k-1}\zeta}}\Theta\right) \\ &= -\int_\Sigma \gamma^*\sigma_\mathcal{L}^*\left(i_{\overline{j^{k-1}\zeta}}\Theta\right) = -\int_\Sigma \gamma^*(i_{j^{2k-1}\zeta}\Theta_\mathcal{L}) \\ &= E_{\Sigma,\zeta}(\gamma) = H_{\Sigma,\zeta}(\sigma_{L,\Sigma} \circ \gamma) = H_{\Sigma,\zeta}(\varphi, \pi) \end{aligned}$$

where I have used the facts that Θ is λ-horizontal and

$$T\sigma_\mathcal{L} \cdot j^{2k-1}\zeta = \overline{j^{k-1}\zeta} \mod V\lambda.$$

Provided the fibers of $\sigma_{L,\Sigma}$ are connected, the desired result now follows from Corollary 15. □

In summary, (11.3) represents a "purely Hamiltonian" definition of the instantaneous Hamiltonian. (Of course, $K_{\Sigma,\zeta}$ only "becomes" the Hamiltonian when restricted to $\mathcal{P}_\Sigma^{k-1}$, and here—in the definition of the covariant primary constraint set—is where the Lagrangian comes into play. In this regard, note that $K_{\Sigma,\zeta}$ does *not* directly project through R_Σ to $T^*\Gamma(J_\Sigma^{k-1}Y)$; one must restrict to $\mathcal{P}_\Sigma^{k-1}$ first.) Actually, from (11.5) one sees that this definition is "multisymplectic." Other results along these lines can be found in [2, 7].

The construction of Diagram 1 is now complete, and from Propositions 14 and 16 one sees that it commutes. In particular, it follows that the instantaneous Hamiltonian formalism obtained this way is independent of the choice of Cartan form.

12. Other approaches

The space + time decomposition described in the previous section assumes the De Donder-Weyl approach to the calculus of variations. Naturally, the question arises as to happens in other contexts. (The reader is referred to Section 6 of **I** and [5] for background on alternative approaches to the calculus of variations as well as references.) As shown in Section 10, the instantaneous Lagrangian formalism is completely insensitive to the choice of Lepagean equivalent. This should not be surprising: after all, the initial value analysis—which ultimately yields the symplectic structure on the space of admissible Cauchy data for the Euler-Lagrange equations—is concerned only with these equations and their properties; Lepagean equivalents play no role here except as a means to the end.

But the problem is tricker on the Hamiltonian side. This is because the entire structure of the covariant Hamiltonian formalism is dictated by the choice of Lepagean equivalent: different such choices give rise to disparate multiphase spaces, multisymplectic structures, and even notions of regularity, cf. Sections 5–6. Nonetheless, one expects that the dependence on the choice of $\Theta_{\mathcal{L}}$ in the covariant Hamiltonian formalism will disappear once a space + time decomposition is performed. Certainly the instantaneous Hamiltonian formalism obtained by traversing Diagram 1 counterclockwise contains no remnants of this choice. The problem is to determine what happens going clockwise. In view of the form of the general (strict) Lepagean equivalent (10.5), one must ascertain the effects of the higher degree contact term χ upon the Hamiltonian formalism. (In this regard, note that the ambiguity encoded in the undetermined components $c_A^{\mu_1\ldots\mu_{s-1}\mu_s} = p_A^{\mu_1\ldots[\mu_{s-1}\mu_s]}$ of the first degree multimomenta is purely a higher order phenomenon, and will be present regardless of which approach to the calculus of variations one takes. As illustrated in the previous section, the quantities which are ultimately relevant—viz. the instantaneous momenta $\pi_A^{0_s}$—turn out to be independent of the $c_A^{\mu_1\ldots\mu_s}$. Thus although the $c_A^{\mu_1\ldots\mu_s}$ appear in the covariant Hamiltonian formalism, they will drop out during the space + time split.)

These effects are striking on the covariant level. To appreciate them without introducing extraneous complications, consider first order field theory "à la Lepage" (cf. Section 6 of **I**). This formalism is based upon the most general strict Lepagean equivalent

$$\Theta_{\mathcal{L}} = Ld^{n+1}x + \frac{\partial L}{\partial y^A_\mu}\psi^A \wedge d^n x_\mu + \sum_{s=2}^{n+1} \lambda^{|\mu_1\ldots\mu_s|}_{|A_1\ldots A_s|}\psi^{A_1} \wedge \ldots \wedge \psi^{A_s} \wedge d^{n+1-s}x_{\mu_1\ldots\mu_s}$$

on J^1Y, where the $\lambda^{\mu_1\ldots\mu_s}_{A_1\ldots A_s}$ are fixed but arbitrary functions and the vertical bars indicate that the summation extends only over increasing sequences of indices.

The covariant phase space in this instance is all of $\Lambda^{n+1}Y$ (and *not* just its subbundle Z^0!) with the canonical $(n+1)$-form

$$\Theta = pd^{n+1}x + \sum_{s=1}^{n+1} p^{|\mu_1\ldots\mu_s|}_{|A_1\ldots A_s|} dy^{A_1} \wedge \ldots \wedge dy^{A_s} \wedge d^{n+1-s}x_{\mu_1\ldots\mu_s}.$$

As before, one considers the space $\Gamma(\Lambda^{n+1}_\Sigma Y)$. Integration over Σ yields the 1-form

$$\Theta_\Sigma(\varrho) = \int_\Sigma \tilde{\varrho}^0_A d\varphi^A \otimes d^n x_0$$

on $\Gamma(\Lambda^{n+1}_\Sigma Y)$, where

$$\tilde{\varrho}^0_A = \varrho^0_A + \varrho^{0\nu}_{AB}\varphi^B_{,\nu} + \varrho^{0|\nu\eta|}_{ABC}\varphi^B_{,\nu}\varphi^C_{,\eta} + \cdots .$$

The analysis now proceeds exactly as in the De Donder-Weyl case: the 2-form $\Omega_\Sigma = -d\Theta_\Sigma$ is presymplectic, and reduction yields $T^*\mathcal{Y}_\Sigma$ with its standard symplectic structure. (The second reduction of Section 11 is not necessary here as the theory is only first order.) Thus one ends up with the same *instantaneous* phase space as in the De Donder-Weyl approach, despite the fact that the *covariant* phase spaces are entirely different.

One also finds that (11.3) and (11.4) together still induce the same instantaneous Hamiltonian. It is instructive to verify this explicitly, although in fact Proposition 16 and its proof remain valid as stated. For $\varrho = \sigma_{\mathcal{L}} \circ \gamma$ with $\gamma \in j^1_\Sigma\mathcal{Y}$, a straightforward computation in ζ-adapted charts using the expression (6.8) for the covariant Legendre transformation in this context gives

$$\begin{aligned} K_{\Sigma,\zeta}(\varrho) &= \int_\Sigma \Big((\varrho^0_A + \varrho^{0\nu}_{AB}\gamma^B_{,\nu} + \varrho^{0|\nu\eta|}_{ABC}\gamma^B_{,\nu}\gamma^C_{,\eta} + \cdots)\gamma^A_0 - L(\gamma)\Big) d^n x_0 \\ &= \int_\Sigma \left(\bar{\pi}_A\gamma^A_0 - L(\gamma)\right) d^n x_0 \end{aligned}$$

with $\bar{\pi}_A = \partial L/\partial y^A_0$, consistent with what one gets via the instantaneous Legendre transformation.

This analysis shows that the only real effect of the higher degree contact term χ is hence to "shift" the reduction map R_Σ from $\bar{\pi}_A = \varrho^0_A$ to $\bar{\pi}_A = \tilde{\varrho}^0_A$. Thus the De Donder-Weyl and Lepagean versions of first order field theory both give rise to the same instantaneous Hamiltonian formalism. It is moreover clear that an analogous result will hold for higher order theories, so that the instantaneous Hamiltonian formalism is independent of the choice of strict Lepagean equivalent.

Even so, the De Donder-Weyl formalism is in a sense the most natural for classical field theory, at least insofar as the space + plus time decomposition and the subsequent initial value analysis are concerned. Certainly it is algebraically simpler than other approaches, and is the most straightforward to space + time split. (This reflects the fact that the De Donder-Weyl framework for field theory most closely mimics mechanics.) And, as the above computations illustrate, other approaches effectively "shift" to De Donder-Weyl once one integrates over the Cauchy surface.

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