

# A Multisymplectic Approach to the KdV Equation

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## Abstract

A canonical multisymplectic formulation of the KdV equation is presented which, when space + time decomposed, gives rise to the usual symplectic description of dynamics on the appropriate space of Cauchy data. In addition to allowing one to treat the KdV equation covariantly, this formalism enables one to derive the Gardner symplectic structure for the KdV equation in a completely systematic way.

## I. Introduction

Oftentimes the study of evolution equations begins with these equations presented in the specialized form

$$\underline{u}_t = \mathcal{D} \cdot \delta \mathcal{H}[\underline{u}], \quad (1.1)$$

where  $\underline{u}$  is a matrix of fields,  $\mathcal{D}$  is a matrix differential operator,  $\delta$  is the variational derivative with respect to  $\underline{u}$  and  $\mathcal{H}$  is the Hamiltonian density [0]. While ultimately very useful, this form is somewhat unsatisfactory from the point of view of classical field theory, since (i) it essentially ignores the variational aspects of the theory, (ii) it is intrinsically noncovariant, (iii) it presupposes that an initial value analysis has already been carried out (at least formally), and (iv) the underlying symplectic geometry of the theory is not readily apparent. Ideally, one would like to be able to *derive* (1.1) in a "canonical" way, that is, from a Lagrangian variational principle followed by a space + time decomposition of the corresponding Hamiltonian formalism. But this program has been hampered by the lack of an adequate *covariant* Hamiltonian formalism in the calculus of variations [dLR]. Such a theory has now been developed [Go]; the key ingredient is the notion of a "multisymplectic" structure [Ki].

The principal advantage of this canonical approach is that it is both general and systematic. Consequently, one can deal *directly* with higher order Lagrangians as well as multiple integrals in the calculus of variations in much the same way as one treats ordinary mechanics. It is also straightforward to handle systems which have initial value constraints. Moreover, after a space + time decomposition, this multisymplectic framework gives rise to the standard symplectic description of dynamics on the appropriate space of Cauchy data, and (1.1)

effectively reappears as the complete set of dynamic field equations thereupon.

Here I apply this formalism to the KdV equation

$$u_t = 6uu_x - u_{xxx}. \quad (1.2)$$

The analysis has two interesting features. First, the standard KdV Lagrangian is second order, so this provides a simple illustration of how to canonically analyze higher order field theories *without* having to reduce the order of the Lagrangian. Second, it enables one to systematically *derive* the somewhat "mysterious" Gardner symplectic structure [Ga] for the KdV equation from first principles.

The reader is referred to [Go] for the relevant definitions and the details of the formalism; notation is explained in the Appendix. Throughout, I work formally in the  $C^\infty$  category.

## II. Covariant Formulation

To obtain a Lagrangian for the KdV equation it is necessary to introduce the potential  $v_x = u$ . The corresponding covariant configuration space  $Y$  is the trivial bundle  $\mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$  with coordinates  $x^\mu = (x, t)$  on "spacetime"  $\mathbb{R}^2$  and  $v$  on the fiber. Then the Lagrangian density, viewed as a map  $\mathcal{L}: J^2Y \rightarrow \wedge^2(\mathbb{R}^2)$ , is

$$\mathcal{L}[v] = \left[ \frac{1}{2} v_t v_x - v_x^3 - \frac{1}{2} v_{xx}^2 \right] dx \wedge dt.$$

The first step is to construct a Cartan form  $\theta_{\mathcal{L}}$  on  $J^2Y$ ; the formalism requires it to be "quasi-symmetric" [Ko]. Since  $\mathcal{L}$  is only second order, there a unique such  $\theta_{\mathcal{L}}$ ; it is

$$\begin{aligned}
\Theta_{\mathcal{L}} &= -v_{XX} dv_X \wedge dt - \frac{1}{2} v_X dv \wedge dx \\
&+ \left[ \frac{1}{2} v_t - 3v_X^2 + v_{XXX} \right] dv \wedge dt \\
&+ \left[ 2v_X^3 + \frac{1}{2} v_{XX}^2 - \frac{1}{2} v_t v_X - v_X v_{XXX} \right] dx \wedge dt.
\end{aligned} \tag{2.1}$$

In terms of the Cartan form, a section  $\gamma$  of  $Y$  satisfies the KdV equation iff

$$(j^3 \gamma)^* \left[ i(\xi) d\Theta_{\mathcal{L}} \right] = 0 \tag{2.2}$$

for all vector fields  $\xi$  on  $J^3 Y$ .

The covariant phase space for the KdV equation is  $J^{1\star}(J^1 Y)$ , where the star denotes an affine  $\Lambda^2(\mathbb{R}^2)$ -valued dual. This space can be canonically identified with the subbundle

$$Z = \left\{ z \in \Lambda^2(J^1 Y) \mid i(\xi) i(\eta) z = 0 \text{ for all } \xi, \eta \in \mathcal{V}\pi^1 \right\}.$$

of  $\Lambda^2(J^1 Y)$ . Coordinates on  $Z$  are  $(x^\mu, v, v_\mu; p^\nu, p^{\mu\nu}, p)$ . The space  $Z$  carries a canonical 2-form given in these coordinates by

$$\Theta = \left[ p^{\mu\nu} dv_\mu + p^\nu dv \right] \wedge dx_\nu + p d^2 x. \tag{2.3}$$

The *multisymplectic form* is the 3-form  $\Omega = -d\Theta$ .

The connection between the variational and multisymplectic formalisms is provided by the covariant Legendre transformation  $\sigma_{\mathcal{L}}: J^3 Y \rightarrow Z$ . It is the the  $J^1 Y$ -bundle map defined by the factorization

$$\theta_{\mathcal{L}} = \sigma_{\mathcal{L}} \circ \wedge^2 \left[ T\pi_1^3 \right]$$

of the Cartan form. Using (2.1) this yields the coordinate expression for  $\sigma_{\mathcal{L}}$ :

$$\begin{aligned} p^{tt} &= 0, \quad p^{tx} = p^{xt} = 0, \quad p^{xx} = -v_{xx} \\ p^t &= \frac{1}{2}v_x, \quad p^x = \frac{1}{2}v_t - 3v_x^2 + v_{xxx} \\ p &= 2v_x^3 + \frac{1}{2}v_{xx}^2 - \frac{1}{2}v_tv_x - v_x v_{xxx}. \end{aligned} \quad (2.4)$$

This transformation is clearly degenerate; its image is the "covariant primary constraint set"  $P \subset Z$  defined by equations (2.4). However,  $\sigma_{\mathcal{L}}$  is a submersion onto its image  $P$ .

Noting that  $\sigma_{\mathcal{L}}^* \theta = \theta_{\mathcal{L}}$ , it then follows from (2.2) that  $\gamma \in \mathcal{Y}$  is a solution of the KdV equation iff  $\rho = \sigma_{\mathcal{L}} \circ j^3 \gamma$  satisfies

$$\rho^* \left[ i(\xi) \Omega \right] = 0$$

for all vector fields  $\xi$  on  $P$ . This is the multisymplectic version of the KdV equation.

### III. Space + Time Decomposition

I now 1+1 decompose the covariant formalism of the previous section. Fix a Cauchy surface  $\Sigma \subset \mathbb{R}^2$  and choose spacetime coordinates such that  $\Sigma$  is the level set  $t = 0$ . Take the evolution direction  $\zeta$  on  $\mathcal{Y}$  to be  $\partial_t$ . Assume all fields vanish sufficiently rapidly at spatial infinity that boundary terms may be ignored. The decomposition will proceed in two stages.

Consider the space  $\mathcal{I}_\Sigma$  of sections of  $Z|\Sigma$ . The canonical 2-form  $\theta$  on  $Z$  induces a 1-form  $\theta_\Sigma$  on  $\mathcal{I}_\Sigma$  by integration:

$$\theta_\Sigma(\rho) \cdot \xi = \int_\Sigma \rho^* [i(\xi)\theta]$$

where  $\rho \in \mathcal{I}_\Sigma$  and  $\xi \in T_\rho \mathcal{I}_\Sigma$ . Set  $\Omega_\Sigma = -d\theta_\Sigma$ ; this defines a closed 2-form on  $\mathcal{I}_\Sigma$ . However,  $(\mathcal{I}_\Sigma, \Omega_\Sigma)$  is not symplectic; using (2.3) it is straightforward to check that

$$\ker \Omega_\Sigma = \text{span} \left\{ \frac{\delta}{\delta p^{tx}}, \frac{\delta}{\delta p^{xx}}, \frac{\delta}{\delta p^x}, \frac{\delta}{\delta p} \right\}.$$

Then reduction yields  $(\mathcal{I}_\Sigma, \Omega_\Sigma) / (\ker \Omega_\Sigma) \approx (T^*(\Gamma(J^1 Y)_\Sigma), \bar{\Omega}_\Sigma)$ , where  $\bar{\Omega}_\Sigma = -d\bar{\theta}_\Sigma$  is the canonical symplectic form on the cotangent bundle. Canonical "coordinates" on  $T^*(\Gamma(J^1 Y)_\Sigma)$  are  $(v, v_x, v_t; p^t, \bar{p}^t, \bar{p}^{tt})$  and the canonical 1-form is

$$\bar{\theta}_\Sigma(\bar{\rho}) = \int_\Sigma \left[ \bar{\rho}^* [p^{tt} \delta v_t + p^{xt} \delta v_x + p^t \delta v] \right] \otimes dx \quad (3.1)$$

where  $\bar{\rho} \in T^*(\Gamma(J^1 Y)_\Sigma)$ . Note that this reduction essentially amounts to eliminating the *spatial* "multimomenta"  $p^{tx}$ ,  $p^{xx}$ ,  $p^x$  and  $p$  from the formalism, these being irrelevant for the initial value problem.

The second stage of the decomposition is to eliminate redundant *spatial* derivatives (in this case, just  $v_x$ ). This is accomplished by requiring sections of  $(J^1 Y)_\Sigma$  to be *spatially* holonomic, i.e., to lie in the space  $(j^1 Y)_\Sigma \subset \Gamma(J^1 Y)_\Sigma$ . The corresponding inclusion  $i$  induces a symplectic projection  $i^*$ :

$$(T^*(\Gamma(J^1 Y)_\Sigma), \bar{\Omega}_\Sigma) \rightarrow (T^*(j^1 Y)_\Sigma, \omega_\Sigma), \text{ where again } \omega_\Sigma = -d\theta_\Sigma \text{ is the}$$

canonical symplectic form on the cotangent bundle.

Moreover, one may use the evolution direction  $\zeta$  to identify  $(j^1\mathcal{Y})_\Sigma$  with  $\mathcal{Y}_\Sigma$ . Thus, finally, the 1+1 decomposed phase space is  $(T^*\mathcal{Y}_\Sigma, \omega_\Sigma)$ . In canonical "coordinates"  $(v, v_t; \pi, \pi^t)$  on this space the canonical 1-form is

$$\theta_\Sigma(\varphi) = \int_\Sigma \varphi^* \left[ \pi \delta v + \pi^t \delta v_t \right] \otimes dx \quad (3.2)$$

for  $\varphi \in T^*\mathcal{Y}_\Sigma$ , and the projection  $i^*$  is given by  $(v, v_x, v_t; p^t, p^{xt}, p^{tt}) \rightarrow (v, v_t; \pi, \pi^t)$ , where

$$\pi = \left[ p^t - D_x p^{xt} \right], \quad \pi^t = p^{tt}, \quad (3.3)$$

cf. (3.1) and (3.2).

#### IV. Initial Value Analysis

Turn next to a (formal) initial value analysis of the KdV equation. According to general principles, the Hamiltonian - defined as a functional on  $\mathcal{P}_\Sigma$  - is given by

$$H_\Sigma(\rho) = \int_\Sigma \rho^* \left[ i(j^1\zeta)\theta \right],$$

where  $j^1\zeta$  is the first jet prolongation of the evolution direction  $\zeta$ . Taking (2.3) and (2.4) into account, this becomes

$$H_\Sigma(\rho) = - \int_\Sigma \rho^* \left[ p^{xx} D_x v_x + p^x D_x v + p \right] dx. \quad (4.1)$$

Now  $\mathcal{P}_\Sigma$  projects through the two reductions of the previous section to a submanifold  $\mathcal{E}_\Sigma$  of  $T^*TY_\Sigma$ , the "1+1 primary constraint set." From (2.4) and (3.3),  $\mathcal{E}_\Sigma$  is defined by the constraints

$$\pi^t = 0, \quad \pi = \frac{1}{2} D_X v. \quad (4.2)$$

It turns out that the first of these constraints is first class in the sense of Dirac, while the second is second class  $[N, K_e]$ .

It is possible to show that the Hamiltonian (4.1) on  $\mathcal{P}_\Sigma$  induces the function  $h_\Sigma$  on  $\mathcal{E}_\Sigma$  given by

$$h_\Sigma(\varphi) = \int_\Sigma \varphi^* \left[ (D_X v)^3 + \frac{1}{2} (D_X D_X v)^2 \right] dx, \quad (4.3)$$

while  $\omega_\Sigma$  pulls back to a closed 2-form  $\eta_\Sigma$  on  $\mathcal{E}_\Sigma$  which, in view of (3.2) and (4.2), is

$$\eta_\Sigma(\varphi) = \frac{1}{2} \int_\Sigma \varphi^* \left[ \delta v \wedge \delta D_X v \right] \otimes dx. \quad (4.4)$$

Hamilton's equations on  $\mathcal{E}_\Sigma$  are

$$i(\xi)\eta_\Sigma = dh_\Sigma, \quad (4.5)$$

which are to be solved for the evolution vector field  $\xi$ . One checks that there are no initial value constraints other than (4.2)  $[N, K_e]$ , so  $\mathcal{E}_\Sigma$  is in fact the *final* constraint set. Thus, writing  $\xi = v\delta/\delta v + v_t\delta/\delta v_t$ , (4.5) may be solved in the form

$$v = 3(D_X v)^2 - (D_X^3 v), \quad v_t = \text{arbitrary}. \quad (4.6)$$

Differentiating the first of these and eliminating the potential  $v$  yields (1.2). On the other hand, (4.6)(ii) reflects the gauge freedom of the system, corresponding to the presence of the first class constraint (4.2)(i).

One may eliminate this trivial ambiguity by reducing  $(\mathcal{E}_\Sigma, \eta_\Sigma)$  by  $\ker \eta_\Sigma = \text{span}\{\delta/\delta v_t\}$ . The reduced space can be identified with  $\mathcal{Y}_\Sigma$  (now with "coordinate"  $u = D_x v$ ), and the reduced symplectic form is, from (4.4),

$$\bar{\eta}_\Sigma(\bar{\varphi}) = \frac{1}{2} \int_\Sigma \int_{-\infty}^x \bar{\varphi}^* [\delta u(y) \wedge \delta u(x)] \otimes dy dx \quad (4.7)$$

where  $\bar{\varphi} \in \mathcal{Y}_\Sigma$ . This is the KdV symplectic structure as presented in [ZF].

Thus, ultimately one has the KdV equation realized as a Hamiltonian system on  $\mathcal{Y}_\Sigma$ :

$$i(\bar{\xi})\bar{\eta}_\Sigma = d\bar{h}_\Sigma, \quad (4.8)$$

where the reduced Hamiltonian is

$$\bar{h}_\Sigma(\bar{\varphi}) = \int_\Sigma \bar{\varphi}^* [u^3 + \frac{1}{2}(D_x u)^2] dx, \quad (4.9)$$

cf. (4.3). Solving (4.8), one obtains immediately from (4.7) the KdV equation in the form (1.1), where  $\mathcal{D} = D_x$  and  $\mathcal{H}[u]$  is the Hamiltonian density corresponding to (4.9).

**Remarks:** 1. This treatment of the KdV equation should be compared with those given by Nutku [N] and Kentwell [Ke], who reduce the Lagrangian to first order. Their method has the

disadvantage of introducing spurious *secondary* constraints. Yet other approaches can be found in [BdK] and [S].

2. Actually, Gardner presented not the symplectic form (4.7), but rather its associated Poisson bracket [Ga]. The connection between these is given in [BdK]; this paper also examines in some detail symplectic structures which generalize (4.7).

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#### Appendix

Let  $\pi: K \rightarrow X$  be a fibration. Adapted coordinates on  $K$  are  $v^A$  along the fibers and  $x^\mu$  on the base. The space of sections of  $\pi$  will be denoted by either  $\Gamma K$  or the script letter  $\mathcal{X}$ . Similarly, for  $\Sigma \subset X$ , the space of sections of  $K|_\Sigma \rightarrow \Sigma$  will be denoted by either  $\Gamma K_\Sigma$  or  $\mathcal{X}_\Sigma$ .

The volume form on  $X$  is  $d^n x = dx^1 \wedge \cdots \wedge dx^n$ , where  $n = \dim X$ . Then  $d^{n-1} x_\mu := i(\partial_\mu) d^n x$ .

For  $f \in C^\infty(K)$ , the formal partial derivative  $D_\mu f$  of  $f$  in the direction of  $x^\mu$  is defined by  $\varphi^*(D_\mu f) = \partial_\mu (f \circ \varphi)$  for all  $\varphi \in \mathcal{X}$ .

The vertical bundle  $V\pi$  is the subbundle  $\ker T\pi$  of  $TK$ .

The  $r^{\text{th}}$  jet bundle of  $K$  is  $J^r K$  with coordinates  $x^\mu, v^A, v^A_{\mu_1}, \dots, v^A_{\mu_1 \dots \mu_r}$ . The projections  $J^r K \rightarrow J^s K$  for  $r \geq s$  and  $J^r K \rightarrow X$  are denoted  $\pi_s^r$  and  $\pi^r$ , respectively. Set  $J^0 K = K$  and  $\pi^0 = \pi$ . The  $r^{\text{th}}$  jet prolongation is denoted  $j^r$ .

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