CAN QUANTUM EFFECTS PREVENT SPACETIME COLLAPSE?

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Introduction

One of the outstanding problems in general relativistic cosmology is whether or not quantum effects can modify the character of -- or perhaps prevent entirely -- the classical final singularity. Although this question was extensively studied during the late 1960's and early 1970's [1], no consensus has emerged. However, much of the work done to date tends to support Misner's 1969 assertion [2] that "quantum effects do not change the nature of the singularity." More recently, Wheeler [3] has proposed a "rule of unanimity" which, if valid, implies that quantum theory can provide no escape from gravitationally induced spacetime collapse.

There are three main obstacles to settling this issue: (*i*) The absence of a complete consistent quantum treatment of the gravitational field and its interactions. (*ii*) Ambiguities inherent in the canonical quantization procedure. (*iii*) The lack of precise general criteria for determining whether or not the quantized system in fact collapses.

These problems, although formidable, can to some extent be circumvented. In particular, it is possible to include the quantum effects of both gravity and matter in the analysis -- while keeping the latter mathematically tractable -- by "freezing out" all but a finite number of degrees of freedom; one is then left with a typical minisuperspace problem [2,4]. Furthermore, many of the difficulties associated with the application of canonical quantization to Hamiltonian cosmology [1,4-8] can be overcome by using instead the geometric quantization procedure of Kostant & Souriau [9]. Indeed, geometric quantization -- essentially a rigorous global generalization of canonical quantization -- has proven to be an effective computational tool in quantum cosmology [5-7]. Finally, in simple cases the last problem (cf. [1,6,8]) can be avoided by directly calculating the asymptotic temporal behavior of certain relevant matrix elements [6,7].

The simplest dynamically nontrivial homogeneous cosmologies are Robertson-Walker universes containing a Klein-Gordon scalar field ("$RW\phi$" models). Recently [6,7] we have geometrically quantized one of the classically collapsing $RW\phi$ universes and have rigorously shown that the quantized model collapses as well. However, there are several important issues that remain to be elucidated, such as the effects of different choices of time and polarization upon the quantum dynamics of these models. To this end, we geometrically quantize here an $RW\phi$ model in a different choice of time and with a topologically different type of polarization than have been used previously. The quantum dynamics so obtained is essentially equivalent to that

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found in [6,7], as well as that resulting from canonical quantization [8].

There are two primary inferences to be drawn from this work. First, at least in some (highly symmetric) cosmological models, quantum effects do *not* prevent spacetime collapse. Second, whereas different choices of time and polarization may lead to *quantitative* changes in the quantum dynamics of these simple models, such differing choices do not seem to significantly affect the quantized models' *qualitative* behavior.

Classical $RW\phi$ Models [1,6,8]

The homogeneous and isotropic $RW\phi$ universes are described by the metric

$$ds^2 = -N^2(t)dt \otimes dt + R^2(t)\, g_{ij} dx^i \otimes dx^j \;,$$

where $N(t)$ is the "lapse," $R(t)$ is the "radius" and g is the standard metric on S^3. The minisuperphase space for these collapsing models is a 4-manifold with global coordinates R, π_R, ϕ, π_ϕ satisfying $R > 0$ and $\pi_\phi \neq 0$ (the values $R = 0$ and $\pi_\phi = 0$ corresponding to singular states). The Hamiltonian is

$$-NK := -\, N\left(\frac{1}{24R}\pi_R{}^2 - \frac{1}{2R^3}\pi_\phi{}^2 + 6R\right) \;,$$

and is constrained to vanish: $K = 0$. For convenience, we have chosen the scalar field $\phi(t)$ to be massless.

The vanishing Hamiltonian indicates that the system is in parametrized form and therefore admits a reduction via "choice of time" [1,6,8]. Here, we choose the "extrinsic" time $t = \pi_R$, since (i) π_R-time smoothly covers the entire classical evolution of the model [$\pi_R \in (-\infty, 0)$ is the expansion phase, while $\pi_R \in (0,\infty)$ is the contraction phase], and (ii) with extrinsic time, we may directly quantize the radius R and monitor the asymptotic temporal behavior of its expectation value as a test for collapse.

The unconstrained phase space resulting from this reduction is the disjoint union $\mathbb{R}^2_+ \oplus \mathbb{R}^2_-$ with the standard symplectic structure $d\pi_\phi \wedge d\phi$, where $\mathbb{R}^2_\pm := \{(\phi, \pi_\phi) \mid \pi_\phi \gtrless 0\}$. The choice $t = \pi_R$ has the attractive simplifying feature that the reduced Hamiltonian π_t is just the radius $R(t)$:

$$R(t) = \alpha\left\{\sqrt{\beta^2 t^4 + \gamma \pi_\phi{}^2} - \beta t^2\right\}^{\frac{1}{2}} \;, \tag{1}$$

where $\alpha = (24)^{-\frac{1}{2}}$, $\beta = 1/12$ and $\gamma = 48$. Since $\pi_\phi \neq 0$, $R(t)$ is positive definite.

Geometric Quantization of the Extrinsic-Time Model [5,6,9]

The simplest choice of polarization F is the "horizontal" one spanned by the vector field $\partial/\partial\phi$. Since $\mathbb{R}^2_+ \oplus \mathbb{R}^2_-$ is topologically trivial, the various geometric quantization structures (the prequantization line bundle L, the metaplectic frame bundle, and the bundle $\sqrt{\wedge^1 F}$ of half-forms relative to F) are unique and trivial.

Let λ denote a trivializing section of L which is normalized to unity, and ν the appropriate half-form. The quantum Hilbert space H_F defined by the polarization F is the completion of the space of smooth, compactly supported (modulo F) sections of $L \otimes \sqrt{\wedge^1 F}$ of the form

(2) $$\psi = \begin{cases} f_+(\pi_\phi) exp[i\pi_\phi \phi]\ \lambda \otimes \nu \ , \text{ on } \mathbb{R}^2_+ \\ f_-(\pi_\phi) exp[i\pi_\phi \phi]\ \lambda \otimes \nu \ , \text{ on } \mathbb{R}^2_- \end{cases}$$

(where $f_\pm$ are arbitrary, and $\hbar \equiv 1$) with respect to the inner product

(3) $$<\psi'|\psi>_F = \int_{-\infty}^{0} f'_-(\pi_\phi)\overline{f_-(\pi_\phi)}d\pi_\phi + \int_{0}^{\infty} f'_+(\pi_\phi)\overline{f_+(\pi_\phi)}d\pi_\phi \quad .$$

Setting $f_* := \chi_- f_- + \chi_+ f_+$, where $\chi_\pm$ are the characteristic functions of $\mathbb{R}^2_\pm$, we can rewrite (2) and (3) as

(4) $$\psi = f_*(\pi_\phi) exp[i\pi_\phi \phi]\ \lambda \otimes \nu$$

and

(5) $$<\psi'|\psi>_F = \int_{-\infty}^{+\infty} f'_*(\pi_\phi)\overline{f_*(\pi_\phi)}d\pi_\phi$$

respectively. The association $f_*(\pi_\phi) exp[i\pi_\phi \phi]\ \lambda \otimes \nu \to f_*(\pi_\phi)$ defines a unitary isomorphism of H_F with $L^2(\mathbb{R})$.

Quantum Dynamics

Since the polarization F diagonalizes $R(t)$, the Hamiltonian/radius operator $\mathcal{R}(t)$ acts on H_F by multiplication:

(6) $$\mathcal{R}(t)[\psi] = R(t)\psi \quad .$$

This positive self-adjoint operator commutes for different times, so that we can solve the Schrödinger equation by expanding in an evolving complete set of states $\{\psi_E\}$ which are simultaneous eigenfunctions of $\mathcal{R}(t)$ at all times. Thus, if

(7) $$\mathcal{R}(t_0)[\psi_E] = E\psi_E$$

at some reference time t_0, then there will exist numbers $E(t)$ [with $E(t_0) = E$] such that

(8) $$\mathcal{R}(t)[\psi_E] = E(t)\psi_E \quad .$$

Consequently (cf. [8]), the states ψ_E evolve according to

(9) $$\psi_E(t) = exp\left(-i\int_{t_0}^{t} E(s)ds\right)\psi_E(t_0) \quad .$$

From (6) and (4) we find that (7) has the distributional solutions [10]

$$\psi_E \sim exp[i\pi_\phi \phi]\delta\big(R(t_0) - E\big)\ \lambda \otimes \nu \quad ,$$

from which it follows that $E > 0$. Employing (1) and manipulating the δ-function in the above expression, we obtain

(10) $$\psi_E = \left(A_+ e^{i\eta\phi}\delta(\pi_\phi - \eta) + A_- e^{-i\eta\phi}\delta(\pi_\phi + \eta)\right)\ \lambda \otimes \nu \quad ,$$

where

(11) $$\eta = \gamma^{-\frac{1}{2}}\left(\left(\alpha^{-2}E^2 + \beta t_0^2\right)^2 - \beta^2 t_0^4\right)^{\frac{1}{2}}$$

and $A_\pm$ are normalization constants.

Substituting (10) into (8) and making use of (1) and (11), we calculate

(12) $$E(t) = \alpha\left(\sqrt{\left(\alpha^{-2}E^2 + \beta t_0^2\right)^2 - \beta^2(t_0^4 - t^4)} - \beta t^2\right)^{\frac{1}{2}} \quad .$$

Since $E > 0$, this implies that $\mathcal{R}(t)$ has a purely continuous spectrum of $(0,\infty)$ for all t. Finally, substituting (12) into (9), we obtain the evolution (cf. [8])

$$\psi_E(t) = \left[exp\left\{-i\left\{\tfrac{1}{2}[tE(t) - t_0E] + \tfrac{1}{4}E(t_0{}^2 + 144E^2)^{\frac{1}{2}} \times \right.\right.\right.$$

$$(13) \qquad \left.\left.\left. \times \ln\left\{\frac{[tE(t) - E(t_0{}^2 + 144E^2)^{\frac{1}{2}}][t_0 + (t_0{}^2 + 144E^2)^{\frac{1}{2}}]}{[tE(t) + E(t_0{}^2 + 144E^2)^{\frac{1}{2}}][t_0 - (t_0{}^2 + 144E^2)^{\frac{1}{2}}]}\right\}\right\}\right\}\right]\psi_E(t_0)$$

Gravitational Collapse

We claim that the quantized extrinsic-time $RW\phi$ model, like its classical counterpart, collapses to a singularity in the sense that

$$(14) \qquad \lim_{t\to\infty} \langle\psi(t)|\mathcal{R}(t)|\psi(t)\rangle_F = 0$$

for any evolving state $\psi(t)$ in the domain of $\mathcal{R}(t)$ [11].

Since $\mathcal{R}(t)$ commutes for different times we have, upon expanding $\psi(t)$ in terms of the eigenstates $\psi_E(t)$ and applying (13),

$$\langle\psi(t)|\mathcal{R}(t)|\psi(t)\rangle_F = \langle\psi(t_0)|\mathcal{R}(t)|\psi(t_0)\rangle_F$$

for some initial time t_0. From (6), (1) and (5), then,

$$(15) \qquad \langle\psi(t_0)|\mathcal{R}(t)|\psi(t_0)\rangle_F = \alpha\int_{-\infty}^{\infty} f_*(\pi_\phi)\overline{f_*(\pi_\phi)}\left(\sqrt{\beta^2t^4 + \gamma\pi_\phi{}^2} - \beta t^2\right)^{\frac{1}{2}}d\pi_\phi$$

for $\psi(t_0)$ of the general form (4). Now, consider the one-parameter family of functions $\{I_t(\pi_\phi)\}$, where I_t is the integrand of (15) at time t. The family $\{I_t(\pi_\phi)\}$ converges pointwise to zero, and $I_t(\pi_\phi) < I_s(\pi_\phi)$ for each π_ϕ if $t > s$. Since in addition $I_t \in L^1(\mathbb{R})$ for each t, the dominated convergence theorem implies that

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} I_t(\pi_\phi)d\pi_\phi = \int_{-\infty}^{\infty}[\lim_{t\to\infty} I_t(\pi_\phi)]d\pi_\phi$$

vanishes. Thus (14) follows.

Since the Hamiltonian $\mathcal{R}(t)$ is self-adjoint, the model *must* evolve to the $t \to \infty$ limit. Consequently, (14) implies that *all physically well-defined states of the extrinsic-time RWϕ model collapse* [12]. Note furthermore that, for large t, the dominant terms in (15) tend to zero as t^{-1}. Since classically $R(t) \to 0$ also as t^{-1}, it follows that the quantum collapse rate matches the classical collapse rate.

Comparison with Canonical Quantization [9]

Blyth & Isham [8], using canonical techniques, have also quantized this extrinsic-time $RW\phi$ model. Since, roughly speaking, canonical quantization is just geometric quantization in the vertical polarization, we may use the Blattner-Kostant-Sternberg ("BKS") transform [9] to compare our respective results.

The "vertical" polarization V is spanned by the vector field $\partial/\partial\pi_\phi$; wave functions relative to V have the general form

$$(16) \qquad \sigma = g_*(\phi)\ \lambda \otimes \mu \quad ,$$

where μ is the appropriate half-form. Since $\left(\mathbb{R}^2_+ \oplus \mathbb{R}^2_-\right)/V \approx \mathbb{R} \oplus \mathbb{R}$, the Hilbert space H_V defined by the vertical polarization is unitarily isomorphic to the direct sum $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ with the inner product

(17) $$<\sigma'|\sigma>_V = \int_{-\infty}^{\infty} g_+'(\phi)\overline{g_+(\phi)}d\phi + \int_{-\infty}^{\infty} g_-'(\phi)\overline{g_-(\phi)}d\phi \quad .$$

The BKS kernel K: $H_V \times H_F \to \mathbb{C}$ is given by

(18) $$K(\sigma,\psi) = (2\pi)^{-\frac{1}{2}}e^{+i\pi/4}\iint_{-\infty}^{+\infty} g_*(\phi)\overline{f_*(\pi_\phi)}e^{-i\pi_\phi\phi}d\pi_\phi d\phi$$

for $\psi \in H_F$ and $\sigma \in H_V$ of the form (4) and (16) respectively. The kernel K defines a linear operator U: $H_F \to H_V$ via $K(\sigma,\psi) = <\sigma|U\psi>_V$. Comparing this with (18) and (17), we compute

(19) $$U\psi = \left\{(2\pi)^{-\frac{1}{2}}e^{-\frac{i\pi}{4}} \sum_{\pm} \left[\chi_\pm \int_{-\infty}^{\infty} \chi_\pm f_\pm(\pi_\phi)e^{i\pi_\phi\phi}d\pi_\phi\right]\right\} \lambda \otimes \mu \quad .$$

Thus, U is "essentially" the Fourier transform.

Applying U to the energy eigendistributions (10), (19) gives

(20) $$U\psi_E = (2\pi)^{-\frac{1}{2}}e^{-i\pi/4}(\chi_+A_+e^{in\phi} + \chi_-A_-e^{-in\phi})\ \lambda \otimes \mu \quad .$$

This is just Blyth & Isham's result [8], with one major difference [13]: B&I find that the spectrum of the Hamiltonian is not $(0,\infty)$ for all times, but rather $(-\infty,0) \cup (0,\infty)$. Within the canonical framework, the presence of these (unphysical) negative eigenvalues is due to B&I's use of a Klein-Gordon type equation to generate the quantum evolution rather than a true Schrödinger equation. From the standpoint of geometric quantization, on the other hand, this anomaly apparently can be traced to the incompleteness [14] of the vertical polarization [15] -- indeed, such spurious negative eigenvalues do not appear when one quantizes using a complete polarization (such as the horizontal polarization employed in this paper).

Although the BKS transform U: $H_F \to H_V$ is an isometry, i.e., $<U\psi'|U\psi>_V = <\psi'|\psi>_F$, a straightforward calculation shows that U is not invertible. Thus, the BKS transform does *not* unitarily intertwine the quantizations in the horizontal and vertical polarizations. Of course, this is not really surprising since the two polarizations are of topologically different types.

Discussion

In [6,7] we have geometrically quantized a classically collapsing $RW\phi$ model using the "matter-time" $t = \phi$ and a "radial" polarization. The results of that quantization compare favorably with our findings here as well as with Blyth & Isham's work. Indeed, in the quantizations of the positive curvature $RW\phi$ universes (with massless scalar field) studied so far, one finds not only that the quantized models unquestionably collapse, but also that the quantum collapse rate is always exactly the same as the classical collapse rate [16].

Changes in the choices of time and polarization may, as illustrated in the last section, result in modifications to the finer details of the quantum dynamics of these models (e.g., alterations in the spectra of observables). However, such variations do not seem to influence the system's overall characteristics. Hence, it may be that these diverse quantizations will yield at least *qualitatively* equivalent -- if not strictly equivalent -- results. In any case, our experience with the $RW\phi$ models supports the contention that the final singularity cannot be avoided -- quantum mechanically or otherwise -- and hence [2] must "be treated as an essential element of cosmological theory."

The $RW\phi$ models considered here furnish a handy laboratory for studying quantum gravity, and the Kostant/Souriau theory allows one to quantize these models in a rather exact fashion. Even so, it is clear that much remains to be done before any agreement can be reached regarding the issue of quantum collapse. One should therefore, as MacCallum [1] points out, properly regard the present work (and, indeed, quantum cosmology in general) as "suggesting possibilities rather than providing definite answers."

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References & Notes

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10. Note that we have imposed no boundary conditions.
11. Since the domain of the operator $\mathcal{R}(t)$ is time-independent, the expectation value (15) and the limit (14) are both well-defined.
12. Technically, we must temper our enthusiasm somewhat here, for the following reason. Since $\mathcal{R}(t)$ is an unbounded operator (with domain $\mathcal{D}$, say -- cf. [11]), we have really only shown that those states $\psi(t) \in \mathcal{D}$ undergo quantum collapse. However, as $\mathcal{D}$ is dense in $L^2(\mathbb{R})$, this is probably "good enough". It is not clear what can be said regarding those states $\psi(t) \notin \mathcal{D}$, since such states have no well-defined "radius".
13. A minor discrepancy concerns the presence of the characteristic functions in (20). They appear here because the classical phase space is disconnected, but not in B&I's treatment where the phase space is assumed connected.
14. A polarization is *complete* if the canonical vectorfields spanning it are complete (cf. [9]).
15. J. Śniatycki & B. Kostant (private communications).
16. This remark can be shown to hold true also for the canonically quantized models of Blyth & Isham.