An Exterior Differential Systems Approach to the Cartan Form

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Dedicated to Jean-Marie Souriau

Abstract

The notion of a "Lepagean equivalent" of a given variational problem is defined, and the basic properties of these objects are sketched. Using some ideas of Bryant, Dedecker and Griffiths, I show that every constant rank variational problem has a canonical Lepagean equivalent and that, as a consequence, to every such variational problem there is assigned a canonical "generalized Cartan form." These observations rely crucially upon the "generalized Cartan form." These observations rely crucially upon the exterior differential systems approach to the calculus of variations. Then I prove that this generalized Cartan form is *universal* in the sense that every Cartan form for a classical variational problem can be obtained from it by pullback upon sectioning a certain bundle. These results lead to a simple new proof of the existence of Cartan forms for classical variational problems, and explain in intrinsic terms why and to what extent classical Cartan forms are (typically) not uniquely determined by a Lagrangian.

1. Introduction

The "Cartan form" is the basic geometric object in the calculus of variations [13, 14, 16, 17, 19].[1] It generalizes the Poincaré-Cartan 1-form

$$\Theta = L\,dt + \frac{\partial L}{\partial \dot q^i}\left(dq^i - \dot q^i\,dt\right)$$

familiar from mechanics. However, in field theory this form is not uniquely defined, even for first order Lagrangians. Indeed, consider a Lagrangian density $\mathcal{L} = L\omega$ on the first jet bundle J^1Y of a fibration $Y \to X$. (Notation and terminology are explained in the Appendix.) The "standard" Cartan form corresponding to $\mathcal{L}$ [14, 16] is

$$\Theta = L\omega + \frac{\partial L}{\partial y^A_\nu}\psi^A \wedge \omega_\nu. \tag{1.1}$$

[1]There is substantial variance in terminology regarding Cartan forms in the literature. For the moment, I will use the term loosely. More precise nomenclature will be introduced later.

But if L is nowhere zero, then

$$\Theta = \frac{1}{L^{n-1}}\eta^1 \wedge \ldots \wedge \eta^n, \tag{1.2}$$

where $\eta^\mu = L\,dx^\mu + \frac{\partial L}{\partial y^A_\mu}\psi^A$ and $n = \dim X$, is another Cartan form [6, 35]. Yet a third is [2, 26, 35]:

$$\Theta = L\omega + \sum_{s=1}^{n}\frac{1}{(s!)^2}\left(\frac{\partial^s L}{\partial y^{A_1}_{\nu_1}\ldots\partial y^{A_s}_{\nu_s}}\right)\psi^{A_1}\wedge\ldots\wedge\psi^{A_s}\wedge\omega_{\nu_1\ldots\nu_s}. \tag{1.3}$$

These Cartan forms are designed with specific purposes in mind. The first one seems well suited to studying the Hamiltonian aspects of a theory and the initial value problem [18]. Carathéodory's form (1.2) has the property of being invariant under contact transformations of the Lagrangian $\mathcal{L}$, as opposed to (1.1), which is invariant only under (prolongations of) bundle automorphisms [4]. The form (1.3) has the virtue that it is closed iff $\mathcal{L}$ is variationally trivial. See [1, 3, 11, 23] for further discussion.

These three examples do not exhaust all the possibilities. In fact, they are merely special instances of the Cartan form

$$\Theta = L\omega + \frac{\partial L}{\partial y^A_\nu}\psi^A\wedge\omega_\nu + \sum_{s=2}^{n}\lambda^{\nu_1\ldots\nu_s}_{A_1\ldots A_s}\psi^{A_1}\wedge\ldots\wedge\psi^{A_s}\wedge\omega_{\nu_1\ldots\nu_s}, \tag{1.4}$$

where the coefficients $\lambda^{\nu_1\ldots\nu_s}_{A_1\ldots A_s}$ are arbitrarily specifiable functions on J^1Y. This is the most general Cartan form for a first order theory which is semi-basic with respect to the projection $J^1Y \to Y$ [30, 31]. The significance of such forms is in general unclear.

The situation is correspondingly more complex for higher order theories [1, 25, 27].

This peculiarity has been extensively studied in recent years [1, 2, 8, 12, 15, 20, 21, 24-29, 32, 33, 35-38]. As a result, one now knows that Cartan forms can be associated with any "classical" variational problem – that is, one defined on an appropriate jet bundle of some fibration – and the Cartan forms corresponding to a given Lagrangian have to a large extent been characterized. Several explicit methods of constructing Cartan forms have also been developed. Thus the situation is somewhat under control, at least from a practical standpoint.

But there is more to be done. In particular, it is not altogether clear – in intrinsic, global terms – exactly *why* the Cartan form is nonunique or, more precisely, *why* there is no preferred choice in general. Part of the reason is that the techniques used to construct Cartan forms are themselves

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ambiguous (in that they rely upon the specification of elements extrinsic to the variational problem under consideration, such as connections and partitions of unity). Put another way, it is difficult to see what the geometry behind these constructions is. Related aspects of the problem are that it is not obvious how to choose a Cartan form in specific cases, nor is it apparent what the consequences of different such choices are.

Anderson has studied these issues using the variational bicomplex [1]. Here I present a different technique for constructing Cartan forms which also sheds some light upon these questions. It is based on the exterior differential systems approach to the calculus of variations [7, 10, 19]. In this framework, I define the general notion of a "Lepagean equivalent" of a given variational problem and sketch some of the basic properties of these objects. Then, following ideas of Bryant, Dedecker and Griffiths [5, 10, 11, 19], I show that every constant rank variational problem has a canonical Lepagean equivalent and that, as a consequence, to every such variational problem there is *canonically* associated a "generalized Cartan form." This is not, however, a Cartan form in the usual sense of the classical calculus of variations, as it is not defined directly on a jet bundle. Nonetheless, this canonical object can be used to induce "classical" Cartan forms on the jets. In fact I show that this generalized Cartan form is *universal* inasmuch as every Cartan form for a classical variational problem can be obtained from it by pullback upon sectioning a certain bundle. These results lead to a new proof of the existence of Cartan forms for classical variational problems, and enable one to pinpoint exactly what the nonuniqueness in the classical Cartan form is and how it arises. The method is relatively simple and algorithmic, and so is possibly less "mysterious" than some other approaches.

The constructions in this paper also yield a new and interesting connection between the Hamiltonian and Lagrangian formalisms in the classical calculus of variations. I will show that one can "interpolate" between these two pictures by means of a series of Lepagean equivalents, and that the Hamiltonian formalism is closely related to the canonical Lepagean equivalent. In this way one can recover Dedecker's theory [11] as well as Griffiths' formalism [19] as special cases.

The paper is laid out as follows. In §2 I introduce and discuss Lepagean equivalents. §3 is devoted to demonstrating the existence of the canonical Lepagean equivalent of any constant rank variational problem, and to determining the relationship between their extremals. The key result here states that every extremal of the equivalent problem projects to an extremal of the original problem. In the next section I study the canonical Lepagean equivalent of a classical variational problem, and prove the converse of the preceding result in this context. Then in §5 I review the theory of classical Cartan forms on jet bundles following [25, 27]. §6 adapts the

proof of the main theorem in §4 to prove the universality of the canonical generalized Cartan form in the sense explained above. I conclude with some speculations and open problems.

2. Lepagean Equivalents

The basic data in the calculus of variations consists of a fibration $\pi : M \to X^n$, a differential ideal $\mathcal{I}$ in the exterior algebra $C^\infty(\wedge^* M)$ and an n-form $\mathcal{L}$ on M. Usually the Lagrangian density $\mathcal{L}$ is required to be semibasic, but this is not essential. In this context an *integral* of $\mathcal{I}$ is a section $\phi \in \Gamma(\pi)$ such that $\phi^*\mathcal{I} = 0$. The space of all sections of $\pi : M \to X$ which are integrals of $\mathcal{I}$ is denoted $\Gamma(\pi, \mathcal{I})$.

The *variational problem* $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$ is to extremize the action functional

$$\mathcal{A}_{\mathcal{L}}(\phi) = \int_X \phi^*\mathcal{L} \tag{2.1}$$

over all $\phi \in \Gamma(\pi, \mathcal{I})$.[2] This can be viewed as a "constrained" variational problem, inasmuch as one is not free to extremize over all sections of π, but rather only those which are integrals of $\mathcal{I}$. It is in principle simpler to consider "free" problems, in which one is allowed to extremize over arbitrary sections of a fibration.[3] To accomplish this is the *raison d'être* of "Lepagean equivalents."

Definition 1. A *Lepagean equivalent* of a variational problem $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$ is another variational problem $(W \xrightarrow{\rho} X, \{0\}, \Theta)$, together with a surjective submersion $\nu : W \to M$, such that: (i) $\rho = \pi \circ \nu$, and (ii) if $\gamma \in \Gamma(\rho)$ satisfies $\nu \circ \gamma \in \Gamma(\pi, \mathcal{I})$, then

$$\gamma^*\Theta = (\nu \circ \gamma)^*\mathcal{L}. \tag{2.2}$$

The problem now is to extremize

$$\mathcal{A}_\Theta(\gamma) = \int_X \gamma^*\Theta \tag{2.3}$$

over *arbitrary* sections γ of $W \to X$. Thus one replaces the constrained variational problem on M with a free problem on W. In effect the *generalized*

[2] Of course $\mathcal{A}_{\mathcal{L}}$ may not be everywhere defined. Throughout this paper I ignore such technicalities and work formally. In particular I suppose that $\partial X = \emptyset$ and neglect surface terms.

[3] In the literature "free" classical variational problems are often referred to as "zeroth order" problems.

Cartan form Θ functions as a new Lagrangian density. By a slight abuse of terminology, I sometimes refer to such a form as a "Lepagean equivalent" of the Lagrangian density. In particular, Θ is a Lepagean equivalent of $\mathcal{L}$ if

$$\Theta \equiv \nu^* \mathcal{L} \bmod \nu^* \mathcal{I};$$

thus (2.2) generalizes the classical "first congruence of Lepage" [30, 31]. A standard argument shows that the variational equations corresponding to $(W \xrightarrow{\rho} X, \{0\}, \Theta)$ take the form: $\gamma \in \Gamma(\rho)$ is extremal iff

$$\gamma^* (i_\zeta d\Theta) = 0 \tag{2.4}$$

for all vector fields ζ on W. I refer to these as the *Cartan equations.*

The relation (2.2) guarantees that the actions $A_\mathcal{L}$ and A_Θ are the same. But it is important to observe that (2.2) does *not* imply any relationship between the extremals of the variational problems $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$ and $(W \xrightarrow{\rho} X, \{0\}, \Theta)$. For instance, $(M \xrightarrow{\pi} X, \{0\}, \mathcal{L})$ is a Lepagean equivalent of $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$ according to the definition, but there is no *a priori* correspondence between their extremals. This trivial example indicates that some Lepagean equivalents are less useful than others. On the other hand, it would not do to impose such a relation *ab initio.* For one thing, as I shall demonstrate later, when a relation between extremals does exist, it may vary depending upon the "type" of Lepagean equivalent being considered (even of the same original variational problem)! Another point is that requiring a variational problem and its Lepagean equivalents to have the *same* extremals is too strong a condition to make the notion of Lepagean equivalent useful, since this is often true only subject to a great deal of regularity (§7). Moreover, such a requirement entails solving the Euler-Lagrange equations, at least to the extent of reducing the problem of computing the extremals of $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$ to that of computing the integral sections of the Cartan system of $d\Theta$ on W, *cf.* Remark 3 below and §1.e of [19]. But the idea behind the introduction of Lepagean equivalents is merely to *simplify* the original variational problem.[4]

In view of these comments it is useful to introduce some terminology. Let $\mathcal{E}(\mathcal{L})$ denote the set of all extremals of $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$, etc. Then say that $(W \xrightarrow{\rho} X, \{0\}, \Theta)$ is a *covariant* Lepagean equivalent if the projection $\nu : W \to M$ induces a map $\mathcal{E}(\Theta) \to \mathcal{E}(\mathcal{L})$. Similarly, it is *contravariant* if every extremal of $\mathcal{L}$ is the projection of some extremal of Θ. As the preceding discussion shows, a given Lepagean equivalent need be neither of these

[4]Thus the appelation Lepagean *equivalent* is somewhat misleading (but is nonetheless prevalent).

types; on the other hand, it might simultaneously be both ("bivariant"). In the latter eventuality it does not follow that $\mathcal{E}(\Theta) \approx \mathcal{E}(\mathcal{L})$, but rather only that $\mathcal{E}(\Theta) \to \mathcal{E}(\mathcal{L})$ is surjective.

Remark 1. The classical notion of a "Lepagean equivalent" of a given Lagrangian is due to Lepage [30, 31] and was later reinterpreted and extended by Dedecker [9-11] and Krupka [27]. Especially relevant is Dedecker's 1957 memoir [10], which contains a number of ideas and results intimately related to those presented here. In particular, the concepts of covariant and bivariant Lepagean equivalent are akin to his "*semi-relèvement d'une structure variationnelle*" and "*relèvement*," respectively. Dedecker also proves, under suitable assumptions, that one can construct a (global) *semi-relèvement* (*cf.* §3 and Theorem 3 *ff.*) And in the classical case, he shows that the *semi-relèvement* so constructed is in fact a *relèvement* (*cf.* Theorem 4 *ff.*).

Remark 2. There are several conceivable alternatives to this definition of Lepagean equivalent. Firstly, if one only demands that the *actions* (2.1) and (2.3) be the same, then (2.2) need only hold modulo a divergence. Such considerations might be important when considering the divergence equivalence of variational problems [5]. Secondly, as I have defined it, a Lepagean equivalent is "larger" than the original problem in the sense that $\nu : W \to M$ is a projection. On occasion – and this happens in general relativity [27] – one encounters Lepagean equivalent variational problems which are strictly "smaller"; that is to say, there is a submersion $M \to W$. I do not consider either of these possibilities here.

Remark 3. It is possible to give a somewhat different description of Lepagean equivalent as follows. Given a form α on a manifold N, define the *Cartan system* $\mathcal{C}(\alpha)$ to be the differential ideal generated by the forms $i_\xi \alpha$ for $\xi \in C^\infty(TN)$. Applying this construction to $d\Theta$ on W, it follows from (2.4) that $(W \xrightarrow{\rho} X, \mathcal{C}(d\Theta), 0)$ and $(W \xrightarrow{\rho} X, \{0\}, \Theta)$ have the same extremals. Thus the extremals of the latter can be computed "simply" by integrating $\mathcal{C}(d\Theta)$. This implies that one could equally well define a Lepagean equivalent by trivializing the Lagrangian rather than the exterior ideal.

3. The Canonical Lepagean Equivalent

It is not obvious that every variational problem $(M \xrightarrow{\pi} X^n, \mathcal{I}, \mathcal{L})$ has a nontrivial Lepagean equivalent. But Dedeckeer [9-11], and subsequently Griffiths [19] and Bryant [5], have shown that this is the case; in fact, they have shown that every variational problem has a *canonical* Lepagean equivalent,[5] provided only that the differential ideal $\mathcal{I}$ has "constant rank" (which I shall assume henceforth). By this I mean that it is the differential ideal generated by $C^\infty(I)$ for some graded subbundle $I \subseteq \wedge^* M$.

The motivation for their construction comes from the Lagrange multiplier trick. Suppose for the moment that $n = 1$ and $\mathcal{I}$ is a Pfaffian ideal, generated by a collection $\{\theta^a\}$ of 1-forms on M. One can rephrase the variational problem $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$ as wanting to extremize the action (2.1) subject to the "side conditions" $\theta^a = 0$. The Lagrange multiplier trick states that one can accomplish this by adjoining these side conditions to the Lagrangian $\mathcal{L}$ with arbitrary multipliers λ_a, obtaining

$$\Theta = \mathcal{L} + \lambda_a \theta^a.$$

Let W be the space M enlarged by considering the λ_a as new variables. Then $(W \to X, \{0\}, \Theta)$ is a free problem.

Here is how this construction is performed intrinsically. Let $I^p = I \cap \wedge^p M$ for each $p \geq 0$ and build the affine subbundle W of $\wedge^n M$ whose fiber above $m \in M$ is

$$W_m = \{\mathcal{L}(m) + \beta_m \mid \beta_m \in I^n_m\}.$$

I denote this by

$$W = \mathcal{L} + I^n.$$

Since $\wedge^n M$ is a bundle of n-forms, it carries a canonical n-form defined in the usual manner. Let $\Theta_\mathcal{L}$ be the pullback of this form to W, let ν be the restriction of the projection $\wedge^n M \to M$ to W and set $\rho = \pi \circ \nu$.

Proposition 2. *$(W \xrightarrow{\rho} X, \{0\}, \Theta_\mathcal{L})$ so defined is a Lepagean equivalent of $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$.*

Proof. Suppose $\gamma \in \Gamma(\rho)$ with $\phi = \nu \circ \gamma \in \Gamma(\pi, \mathcal{I})$. Then there exists $w \in \Gamma(\nu)$ such that $\gamma = w \circ \phi$. By the universal property of $\Theta_\mathcal{L}$ (*viz.* $w^*\Theta_\mathcal{L} = w$ for all $w \in \Gamma(\nu)$),

$$\gamma^*\Theta_\mathcal{L} = (w \circ \phi)^*\Theta_\mathcal{L} = \phi^* w^* \Theta_\mathcal{L} = \phi^* w = \phi^*\mathcal{L} = (\nu \circ \gamma)^*\mathcal{L}$$

where in the second to last equality I have used the definition of W and the fact that ϕ is an integral of $\mathcal{I}$. ∎

Thus one is justified in calling $(W \xrightarrow{\rho} X, \{0\}, \Theta_\mathcal{L})$ the *canonical Lepagean equivalent* of $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$. It is obvious that the assignment

[5]The rudiments of this idea can be traced (at least) as far back as 1941, to a remark in a paper by Pâquet [34].

$(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L}) \rightsquigarrow (W \xrightarrow{\rho} X, \{0\}, \Theta_\mathcal{L})$ is an (affine) functor from the category of constrained variational problems to the category of free ones. This is a welcome feature of the canonical Lepagean equivalent, as classical Lepagean equivalents are not functorial in general.

Next is the fundamental relation between the extremals of a variational problem and its canonical Lepagean equivalent; to some extent it reflects the fact that W, being a bundle of forms, is a covariant object. Although I will not use this result I include it for completeness.

Theorem 3. *The canonical Lepagean equivalent is covariant.*

Proof. (Bryant [5]). Let $\gamma \in \mathcal{E}(\Theta_\mathcal{L})$ and set $\phi = \nu \circ \gamma$. There are two things to prove: (i) $\phi^*\mathcal{I} = 0$ and (ii) $\mathcal{A}_\mathcal{L}(\phi)$ is a critical value.

For (i), let $\beta \in C^\infty(I^n)$. Then the translations $\alpha_m \rightsquigarrow \alpha_m + t\beta(m)$ define a flow on W whose generating vector field X_β satisfies

$$i(X_\beta)\, d\Theta_\mathcal{L} = \nu^*\beta. \tag{3.1}$$

(This works exactly as in the cotangent bundle case.) But as γ is an extremal,

$$\phi^*\beta = \gamma^*\nu^*\beta = \gamma^*\left(i(X_\beta)\, d\Theta_\mathcal{L}\right) = 0.$$

Thus $\phi^* C^\infty(I^n) = 0$. Of course $\phi^* C^\infty(I^p) = 0$ for $p > n$. Finally, suppose $\alpha \in C^\infty(I^p)$ for $p < n$. Choose any $(n-p)$-form η; then $\alpha \wedge \eta \in C^\infty(I^n)$ and so $\phi^*(\alpha \wedge \eta) = \phi^*\alpha \wedge \theta^*\eta = 0$. Since this is true for all η, $\phi^*\alpha = 0$. Thus (i) follows.

For (ii), let ϕ_t be any variation of $\phi = \phi_0$ through integrals of $\mathcal{I}$. Let γ_t be any lift of ϕ_t to W with $\gamma_0 = \gamma$. Then by (2.2) $\mathcal{A}_\mathcal{L}(\phi_t) = \mathcal{A}_{\Theta_\mathcal{L}}(\gamma_t)$. Hence

$$\frac{d}{dt}\mathcal{A}_\mathcal{L}(\phi_t)\big|_{t=0} = \frac{d}{dt}\mathcal{A}_{\Theta_\mathcal{L}}(\gamma_t)\big|_{t=0} = 0$$

as γ_0 extremizes $\mathcal{A}_{\Theta_\mathcal{L}}$. ∎

It is unclear to what extent the canonical Lepagean equivalent is also contravariant. In complete generality one has formal recourse to the Lagrange multiplier theorem (see, *e.g.*, §43.9 of [40]), but the assumptions there are rather severe. For single integral problems Hsu [22] has shown that this is so, provided $\mathcal{I}$ is Pfaffian and has no completely integrable subsystems (modulo some nontrivial technicalities concerning the existence of compactly supported variations). On the other hand, Dedecker [11] has shown that the canonical Lepagean equivalent of the classical variational problem $(J^1Y \xrightarrow{\pi^1} X, \mathcal{J}_1, \mathcal{L})$ is contravariant. He has also proven a number of results implying that the same is true for higher order classical problems in a semi-holonomic setting [10]. In the next section I will extend these last results to fully holonomic classical problems of any order.

I want to dwell for a moment on the case when $\mathcal{I}$ is Pfaffian, so that $C^\infty(I)$ is locally generated by a collection $\{\theta^a\}$ of 1-forms. Then every element w of W can be locally written

$$\begin{aligned} w &= \mathcal{L} + \lambda^i_a \theta^a \wedge \tau_i + \lambda^{ij}_{ab}\theta^a \wedge \theta^b \wedge \tau_{ij} + \dots \\ &= \mathcal{L} + \sum_r \lambda^{i_1\dots i_r}_{a_1\dots a_r}\theta^{a_1}\wedge\dots\wedge\theta^{a_r}\wedge\tau_{i_1\dots i_r}, \end{aligned} \tag{3.2}$$

where the forms $\{\tau_{i_1\dots i_r}\}$ constitute a basis for a complement of I^{n-r} in $\wedge^{n-r}M$. Thus the fibers of $W \to M$ are parametrized by the coefficients $\lambda^{i_1\dots i_r}_{a_1\dots a_r}$. Since $\Theta_\mathcal{L}$ is a tautological form, its coordinate expression at w is the same as that of w itself, *viz.*,

$$\Theta_\mathcal{L} = \mathcal{L} + \sum_r \lambda^{i_1\dots i_r}_{a_1\dots a_r}\theta^{a_1}\wedge\dots\wedge\theta^{a_r}\wedge\tau_{i_1\dots i_r}. \tag{3.3}$$

But when $\mathcal{I}$ is Pfaffian there are other constructions which also give rise to canonical Lepagean equivalents. For instance, let K^n be the subspace of I^n consisting of forms which are of degree one in the generators $\{\theta^a\}$, and set

$$V = \mathcal{L} + K^n.$$

In a similar manner one obtains a canonical n-form on V, etc. (I use the same notation as before.) But now $\Theta_\mathcal{L}$ looks like

$$\Theta_\mathcal{L} = \mathcal{L} + \lambda^i_a\theta^a \wedge \tau_i.$$

Clearly one can perform a similar construction using forms of degree at most r, $1 \le r \le n$, in the $\{\theta^a\}$. It is straightforward to check that all of these possibilities define canonical Lepagean equivalents with the same properties as the one based on I^n. Later in §6 I will show that W is intimately related to "classical Lepagean equivalents" in the sense of Krupka [27], while V corresponds to the special case of "classical Cartan forms" as defined in §5.

4. Classical Variational Problems

Let $\pi : Y \to X^n$ be a fibration and let $\mathcal{L} = L\omega$ be a k^{th}-order Lagrangian density, now presumed to be a semi-basic n-form on J^kY. The corresponding *classical variational problem* is $(J^kY \xrightarrow{\pi^k} X, \mathcal{J}_k, \mathcal{L})$ where

$\mathcal{J}_k$ is the contact differential ideal on J^kY. As is well-known, a section ϕ of π^k is an extremal iff $\phi = j^k\varphi$ where φ satisfies the Euler-Lagrange equations

$$\frac{\delta L}{\delta y^A}\left(j^{2k}\varphi\right) = \sum_{|\underline{\mu}|=0}^{k}(-1)^{|\underline{\mu}|}\partial_{\underline{\mu}}\left(\frac{\partial L}{\partial y^A_{\underline{\mu}}}\left(j^k\varphi\right)\right) = 0.$$

Let $(W \xrightarrow{\rho} X, \{0\}, \Theta_\mathcal{L})$ be the canonical Lepagean equivalent of $(J^kY \xrightarrow{\pi^k} X, \mathcal{J}_k, \mathcal{L})$, where $W = \mathcal{L} + J^n_k$. Theorem 3 states that it is covariant. This section will be devoted to proving:

Theorem 4. *The canonical Lepagean equivalent of a classical variational problem is contravariant.*

Let $\phi \in \mathcal{E}(\mathcal{L})$. The plan is to show that ϕ can be lifted to a section γ of ρ satisfying (2.4).

The problem can be simplified as follows. Let γ be any lift of ϕ to W; I claim that $\gamma^*(i_\zeta d\Theta_\mathcal{L}) = 0$ for all ν-vertical vector fields ζ on W. Indeed, referring to part (i) of the proof of Theorem 3, one sees that the translational vector fields X_β span $C^\infty(V\nu)$, where $\beta \in C^\infty(J^n_k)$. Since $\phi \in \Gamma(\pi^k, \mathcal{J}_k)$, (3.1) yields

$$\gamma^*\left(i\left(X_\beta\right)d\Theta_\mathcal{L}\right) = \gamma^*\nu^*\beta = \phi^*\beta = 0.$$

Now γ can be decomposed as $\gamma = w \circ \phi$ for some $w \in \Gamma(\nu)$. Since

$$TW|w\left(J^kY\right) = Tw\left(TJ^kY\right) \oplus V\nu,$$

by the above one may suppose in (2.4) that $\zeta|w\left(J^kY\right) = Tw \cdot \xi$ for some $\xi \in C^\infty(TJ^kY)$. Then γ is an extremal iff

$$0 = \gamma^*\left(i_\zeta d\Theta_\mathcal{L}\right) = \phi^*w^*\left(i_{Tw\cdot\xi}d\Theta_\mathcal{L}\right) = \phi^*\left(i_\xi dw^*\Theta_\mathcal{L}\right) = \phi^*\left(i_\xi dw\right).$$

Thus to construct the extremal γ it suffices to find a section $w : J^kY \to W$ satisfying

$$\phi^*\left(i_\xi dw\right) = 0 \tag{4.1}$$

for all vector fields ξ on J^kY. Actually, slightly less will do: w and dw need only be defined along $\phi(X)$ in J^kY. Observe also that in (4.1) one has $\phi = j^k\varphi$.

What must such a form w on J^kY look like? Since J_k is Pfaffian and $\mathcal{L}$ is semi-basic, (3.2) gives

$$w = L\omega + \sum_{|\underline{\mu}|=0}^{k-1} \lambda_A^{\underline{\mu}\nu} \psi_{\underline{\mu}}^A \wedge \omega_\nu + \mathcal{O}\left(J_k \wedge dy^A_{\mu_1 \ldots \mu_k}\right) + \mathcal{O}\left(J_k \wedge J_k\right).$$

The presence of terms involving the top differentials $dy^A_{\mu_1\ldots\mu_k}$ is computationally awkward. However, it suffices to assume that w is (π^k_{k-1})-horizontal, in which case

$$w = L\omega + \sum_{|\underline{\mu}|=0}^{k-1} \lambda_A^{\underline{\mu}\nu} \psi_{\underline{\mu}}^A \wedge \omega_\nu + \mathcal{O}\left(J_k \wedge J_k\right). \tag{4.2}$$

The case of general w can then be handled by pulling w back to $J^{k+1}Y$ and arguing as below.

Now from (4.2)

$$\begin{aligned} dw = &\sum_{|\underline{\mu}|=0}^{k} \frac{\partial L}{\partial y^A_{\underline{\mu}}} dy^A_{\underline{\mu}} \wedge \omega + \sum_{|\underline{\mu}|=0}^{k-1} d\lambda_A^{\underline{\mu}\nu} \wedge \psi^A_{\underline{\mu}} \wedge \omega_\nu \\ &- \sum_{|\underline{\mu}|=0}^{k-1} \lambda_A^{\underline{\mu}\nu} dy^A_{\underline{\mu}\nu} \wedge \omega + \mathcal{O}\left(J_k \wedge \mathcal{J}_k\right). \end{aligned} \tag{4.3}$$

To investigate the conditions that (4.1) places on w as ξ ranges over $C^\infty(TJ^kY)$, set $\xi = \frac{\partial}{\partial y^A_{\underline{\mu}}}$ with $|\underline{\mu}| = k, \ldots, 1$ in turn. Then with $\phi = j^k\varphi$, (4.3) yields:

$$\lambda_A^{(\underline{\mu})} \circ j^k\varphi = \begin{cases} \dfrac{\partial L}{\partial y^A_{\underline{\mu}}} \circ j^k\varphi, |\underline{\mu}| = k \\[2ex] \dfrac{\partial L}{\partial y^A_{\underline{\mu}}} \circ j^k\varphi - \partial_\nu\left(\lambda_A^{\underline{\mu}\nu} \circ j^k\varphi\right), 1 \le |\underline{\mu}| < k, \end{cases} \tag{4.4}$$

where the parentheses denotes symmetrization. Setting $\xi = \frac{\partial}{\partial y^A}$ in (4.1) and using (4.4) recursively, one computes

$$\left(j^k\varphi\right)^* \left(i_{\partial/\partial y^A} dw\right) = \frac{\delta L}{\delta y^A}\left(j^{2k}\varphi\right)\omega = 0$$

as $j^k\varphi = \phi$ is an Euler-Lagrange extremal. Finally, setting $\xi = \frac{\partial}{\partial x^\mu}$ in (4.1) gives nothing new. Note that (4.1) places no restrictions whatsoever

on the higher degree contact terms in w; only the "principal part" of w is relevant for contravariance.

To construct such a form w I use a standard argument [27]. Introduce a cover $\{U_i\}$ of J^kY by π^k-bundle charts, and let $\{\sigma_i\}$ be a partition of unity subordinate to this cover. Set $\widetilde{U}_i = (\pi_k^{2k-1})^{-1}(U_i)$ and $\tilde{\sigma}_i = (\pi_k^{2k-1})^*\sigma_i$. Define forms

$$\tilde{w}_i = \tilde{\sigma}_i L\omega + \sum_{|\underline{\mu}|=0}^{k-1} \tilde{\lambda}_{A,i}^{\underline{\mu}\nu} \psi^A_{\underline{\mu}} \wedge \omega_\nu$$

on $J^{2k-1}Y$ with *supp* $\tilde{w}_i \subset \widetilde{U}_i$ by setting

$$\tilde{\lambda}^{\underline{\mu}}_{A,i} = \begin{cases} \dfrac{\partial(\tilde{\sigma}_i L)}{\partial y^A_{\underline{\mu}}}, |\underline{\mu}| = k \\[2ex] \dfrac{\partial(\tilde{\sigma}_i L)}{\partial y^A_{\underline{\mu}}} - D_\nu \tilde{\lambda}^{\underline{\mu}\nu}_{A,i}, 1 \le |\underline{\mu}| < k \end{cases} \tag{4.5}$$

where D_ν is the total derivative. Let $\tilde{w} = \Sigma_i \tilde{w}_i$; then $\tilde{w}$ is a globally defined form on $J^{2k-1}Y$ whose components satisfy (4.4) in each chart $\widetilde{U}_i$. But then the equation $w|_{j^k\varphi(X)} = \tilde{w}|_{j^{2k-1}\varphi(X)}$ implicitly defines a form w along the image of $j^k\varphi$ whose differential, by construction, satisfies (4.1). This finishes the proof of Theorem 4.

Remark 4. The form $\tilde{w}$ is defined on $J^{2k-1}Y$, *not* J^kY, because of the total divergence in the second of equations (4.5). These terms force the order of each successive coefficient $\tilde{\lambda}^{\underline{\mu}}_{A,i}$, $|\underline{\mu}| = k, \ldots, 1$, to increase by one (so that $\tilde{\lambda}^{\underline{\mu}}_{A,i}$ is actually a locally defined function on $J^{2k-|\underline{\mu}|}Y$). All in all J^kY must be prolonged $k-1$ times to $J^{2k-1}Y$. This observation will be exploited in §6.

Remark 5. The construction of the form $\tilde{w}$ is somewhat subtle in one regard. In general (*i.e.*, $k > 2$) there is no globally defined form on $J^{2k-1}Y$ whose first degree contact components $\tilde{\lambda}^{\underline{\mu}}_A$ satisfy

$$\tilde{\lambda}^{\underline{\mu}}_A = \begin{cases} \dfrac{\partial L}{\partial y^A_{\underline{\mu}}}, |\underline{\mu}| = k \\[2ex] \dfrac{\partial L}{\partial y^A_{\underline{\mu}}} - D_\nu \tilde{\lambda}^{\underline{\mu}\nu}_A, 1 \le |\underline{\mu}| < k \end{cases} \tag{4.6}$$

in *every* bundle chart, as one may verify directly from the transformation properties of such a form. (In other words, the vanishing of the $\tilde{\lambda}^{[\underline{\mu}]}_A$ is not an invariant condition.) But (4.6), with $\tilde{\lambda}^{(\underline{\mu})}_A$ on the left hand side instead of $\tilde{\lambda}^{\underline{\mu}}_A$, *is* an invariant condition, and this explains why $\tilde{\omega}$ as defined above satisfies (4.4). See [27] for an extensive discussion of this point.

Although this proof basically consists of a straightforward calculation, it should be noted that, at least *formally*, Theorem 4 is a direct consequence of the Lagrange multiplier theorem, *cf.* Proposition 43.21 in [40]. The reason I have proceeded in this fashion is that the proof itself is interesting, insofar as a slight reinterpretation of it leads to the main construction and results of the paper in §6.

5. Classical Lepagean Equivalents

In this section I reprise the classical theory of Lepagean equivalents as developed by Krupka [27] and Kolář [24]. Along the way I will make the connection with the theory of §2. Throughout I am concerned solely with the classical variational problem $(J^kY \xrightarrow{\pi^k} X, \mathcal{J}_k, \mathcal{L})$. Here is the technical definition.

Definition 5. An n-form Θ on $J^{2k-1}Y$ is a *classical Lepagean equivalent* of $\mathcal{L}$ provided:

(LE1) $\Theta - \left(\pi^{2k-1}_k\right)^*\mathcal{L}$ is contact, and

(LE2) $i_\xi d\Theta$ is contact for all $\xi \in V\pi^{2k-1}_0$.

These are generalizations of the "congruences of Lepage" extended to the higher order case. Every Lagrangian density has a classical Lepagean equivalent on $J^{2k-1}Y$, but the latter is not uniquely determined unless $n = 1$ [1, 25, 27, 32]. Classical Lepagean equivalents may exist on jet bundles of lower order in particular instances, but this cannot be guaranteed in general.[6] (The reason will become apparent in the next section.) In jet charts, any classical Lepagean equivalent Θ of $\mathcal{L}$ can be expressed as follows:

$$\left(\pi^{2k}_{2k-1}\right)^*\Theta = L\omega + \sum_{|\underline{\mu}|=0}^{2k-1} \lambda^{\underline{\mu}\nu}_A \psi^A_{\underline{\mu}} \wedge \omega_\nu + \chi,$$

where χ is at least quadratic in the generators of J_{2k} and the coefficients

[6] On occasion one sees classical Lepagean equivalents defined on the semi-holonomic jet bundle $\bar{J}^{2k-1}Y \subset J^1(J^{2k-2}Y)$. This approach is useful when discussing the so-called "Poincaré-Cartan forms" [37].

$\lambda^{\underline{\mu}}_A$ are given according to

$$\lambda^{\underline{\mu}}_A = \begin{cases} c^{\underline{\mu}}_A, \ |\underline{\mu}| = 2k \\ -D_\nu\lambda^{\underline{\mu}\nu}_A + c^{\underline{\mu}}_A, \ k < |\underline{\mu}| < 2k \\ \dfrac{\partial L}{\partial y^A_{\underline{\mu}}} - D_\nu\lambda^{\underline{\mu}\nu}_A + c^{\underline{\mu}}_A, \ 1 \le |\underline{\mu}| \le k. \end{cases} \tag{5.1}$$

Here the functions $c^{\underline{\mu}}_A$, of order at most $2k-2$, satisfy $c^{(\underline{\mu})}_A = 0$ and $c^{\underline{\mu}1}_A = 0$ but are otherwise arbitrary. The nonuniqueness of Θ is reflected in the arbitrariness of both the $c^{\underline{\mu}}_A$ and the higher degree contact term χ.

However, *not* every classical Lepagean equivalent so defined is a Lepagean equivalent in the sense of §2. The discrepancy stems from a slight difference between (LE1) and (2.2): the former can be rewritten

$$\left(j^{2k-1}\varphi\right)^*\Theta = \left(j^k\varphi\right)^*\mathcal{L}, \tag{5.2}$$

while the latter reduces to

$$\gamma^*\Theta = \left(j^k\varphi\right)^*\mathcal{L}. \tag{5.3}$$

But γ in (5.3) need not be $(2k-1)$-holonomic, as is required in (5.2).

This indicates that the above definition of classical Lepagean equivalent is too broad. Moreover, one sees from (5.1) that the top components $\lambda^{\underline{\mu}}_A$, $k < |\underline{\mu}| \le 2k$, carry no essential information. It is therefore useful to restrict the arbitrariness in Θ by requiring it to satisfy certain additional invariant conditions.

Definition 6. A classical Lepagean equivalent Θ is *strict* if it is (π^{2k-1}_{k-1})-horizontal.

This condition kills the above-mentioned components. Combined with (LE1) it implies that, as forms on $J^{2k-1}Y$,

$$\Theta \equiv \mathcal{L} \bmod J_k. \tag{5.4}$$

This is Lepage's first congruence for a higher order variational problem. It follows from (5.3) and (5.4) that a strict classical Lepagean equivalent is a genuine Lepagean equivalent according to Definition 1.

Every strict classical Lepagean equivalent can be locally written

$$\Theta = L\omega + \sum_{|\underline{\mu}|=0}^{k-1} \lambda^{\underline{\mu}\nu}_A \psi^A_{\underline{\mu}} \wedge \omega_\nu + \chi, \tag{5.5}$$

where now χ is at least quadratic in the generators of J_k and

$$\lambda_A^{\underline{\mu}} = \begin{cases} \dfrac{\partial L}{\partial y^A_{\underline{\mu}}} + c_A^{\underline{\mu}}, \ |\underline{\mu}| = k \\ \dfrac{\partial L}{\partial y^A_{\underline{\mu}}} - D_\nu \lambda_A^{\underline{\mu}\nu} + c_A^{\underline{\mu}}, \ 1 \leq |\underline{\mu}| < k. \end{cases} \tag{5.6}$$

Strict classical Lepagean equivalents always exist (*cf.* [27]; I will give another proof in §6); they are unique iff $n = 1$ or $k = 1$ and $N = 1$, where N is the fiber dimension of Y.

Remark 6. On the other hand, it is clear that *every* classical Lepagean equivalent, strict or not, is a Lepagean equivalent in the sense of Definition 1 for the prolonged system $(J^{2k-1}Y \overset{\pi^{2k-1}}{\longrightarrow} X, \mathcal{J}_{2k-1}, (\pi_k^{2k-1})^*\mathcal{L})$. So nothing is really lost.

Remark 7. To my knowledge, every classical Lepagean equivalent which arises in specific applications is strict. In this connection note that the form (1.4) – originally considered by Lepage – is the most general strict classical Lepagean equivalent for a first order field theory.

As Definitions 5 and 6 do not fix the higher degree contact term χ, it is useful to specialize further.

Definition 7. A strict classical Lepagean equivalent such that

$$\text{(C)} \quad i_\xi i_\eta \Theta = 0 \quad \text{for all} \quad \xi, \eta \in V\pi^{2k-1}$$

will be called a *classical Cartan form.*

Condition (C) forces χ in (5.5) to vanish. One may therefore think of a classical Cartan form as being the (invariantly defined) "principal part" of a strict classical Lepagean equivalent. As with the latter, the former always exist [12, 21, 36] and are typically nonunique (but not so much so; they are uniquely determined when $n = 1$ or $k = 1$).

Since jets are contravariant objects, one might expect that classical Lepagean equivalents are contravariant in the sense of §2. Indeed this is the case:

Proposition 8. *(Strict) classical Lepagean equivalents are contravariant.*

Proof. Let φ satisfy the Euler-Lagrange equations; I will show that

$$\left(j^{2k-1}\varphi\right)^* (i_\zeta d\Theta) = 0 \tag{5.7}$$

for all vector fields ζ on $J^{2k-1}Y$, so that every extremal $j^k\varphi$ of $\mathcal{L}$ is the projection $\pi_k^{2k-1} \circ j^{2k-1}\varphi$ of an extremal of Θ.

By (LE2) this is true for all $\zeta \in C^\infty(V\pi_0^{2k-1})$. Next, just as in the proof of Theorem 4, one finds that (5.6) in conjunction with the fact that $j^k\varphi$ is an Euler-Lagrange extremal imply that (5.7) is true for $\zeta = \frac{\partial}{\partial y^A}$ as well. Finally, also as before, (5.7) is then automatically satisfied for $\zeta = \frac{\partial}{\partial x^\mu}$. ■

Now recall that the *canonical* Lepagean equivalent of a classical variational problem is both covariant *and* contravariant. Unhappily, this is not so for *classical* Lepagean equivalents, at least not without additional regularity assumptions.[7] The problem is that $\gamma \in \mathcal{E}(\Theta)$ need not be k-holonomic. Moreover, even under appropriate regularity conditions, $\mathcal{E}(\Theta) \not\approx \mathcal{E}(\mathcal{L})$ unless n or $k = 1$. This is because a Cartan extremal γ need not be $(2k-1)$-holonomic, even when regularity guarantees that it is k-holonomic. I will return to this point in §7.

Thus the canonical Lepagean equivalent of a classical variational problem is really rather special in that one gets both covariance and contravariance without any regularity whatsoever.

6. Universality

I now establish the main result of this paper, *viz.*, the canonical Lepagean equivalent of a classical variational problem is universal in the sense that every strict classical Lepagean equivalent can be obtained from it by "pullback."

To this end let $\overline{W}$ be the subbundle of $W = \mathcal{L} + J_k^n$ which consists of (π_{k-1}^k)-horizontal forms; $\overline{W}$ inherits a canonical form $\bar{\Theta}_\mathcal{L}$ from W. According to (3.3) this form has the local expression

$$\bar{\Theta}_\mathcal{L} = L\omega + \sum_{|\underline{\mu}|=0}^{k-1} \lambda_A^{\underline{\mu}\nu} \psi_{\underline{\mu}}^A \wedge \omega_\nu + \mathcal{O}\left(J_k \wedge J_k\right). \tag{6.1}$$

[7] See [17] for a review and detailed references.

Comparing the forms (6.1) on $\overline{W}$ and (5.5) on $J^{2k-1}Y$, it is not surprising that one can recover the latter from the former; the problem is to elucidate the precise mechanism which accomplishes this.

The key is contained in the proof of Theorem 4, which I now reprise from a slightly different angle. The basic idea is to regard (4.1) as a first order differential condition on w – now viewed as a section of $\overline{W} \to J^kY$ – and to describe the corresponding differential relation in $J^1(\overline{W})$. Set $\overline{W}^r = (\pi^r_k)^{\#}\overline{W}$.

Now for each $s = 1, \ldots, k$, the first s equations of (4.4), starting with $|\underline{\mu}| = k$ and continuing down, can be rewritten:

$$\lambda^{(\mu)}_A = \begin{cases} \dfrac{\partial L}{\partial y^A_{\underline{\mu}}}, & |\underline{\mu}| = k \\ \dfrac{\partial L}{\partial y^A_{\underline{\mu}}} - D_\nu \lambda^{\underline{\mu}\nu}_A, & k - s + 1 \le |\underline{\mu}| < k. \end{cases} \tag{6.2}$$

In this expression I regard the $\lambda^{\underline{\mu}}_A$ as coordinates along the fibers of $\overline{W}^{k+s-1} \to J^{k+s-1}Y$ and use the abbreviation

$$D_\nu \lambda^{\underline{\mu}\nu}_A := \left(\lambda^{\underline{\mu}\nu}_A\right)_\nu + \sum_{|\underline{\rho}|=0}^{2k-|\underline{\mu}|-1} \left(\lambda^{\underline{\mu}\nu}_A\right)^{\underline{\rho}}_B y^B_{\underline{\rho}\nu}, \tag{6.3}$$

where $(\lambda^{\underline{\mu}\nu}_A)_\sigma$ and $(\lambda^{\underline{\mu}\nu}_A)^{\underline{\rho}}_B$ are jet coordinates along the fibers of $J^1(\overline{W}^{k+s-1}) \to \overline{W}^{k+s-1}$.

Equations (6.2) thus specify the $\lambda^{(\mu)}_A$ in terms of the derivatives of L and the jet coordinates $(\lambda^{\underline{\mu}\nu}_A)_\sigma$ and $(\lambda^{\underline{\mu}\nu}_A)^{\underline{\rho}}_B$. Consequently for each s these conditions define an affine subbundle $R^{k+s-1} \subset J^1(\overline{W}^{k+s-1})$.[8] Two points are worth noting (*cf.* Remark 4): (i) Each time s increases by one, it is necessary to prolong the base once. This shows up explicitly in (6.3), since $|\underline{\mu}|$ decreases as s increases. (ii) The process terminates after k steps, so that $R^{2k-1} \to J^{2k-1}Y$ encodes the totality of equations (4.4). This construction may be summarized by the diagram:

[8] Secretly lurking here is the fact that the derived flag of the contact system J_k has constant rank.

$$\begin{array}{ccccccc} R^k & & & & & & \\ \downarrow & \ddots & R^{k+1} & & & & \\ \downarrow & & \downarrow & \ddots & & & \\ \downarrow & & \downarrow & & \ddots & & R^{2k-1} \\ \downarrow & & \downarrow & & & & \downarrow \\ J^kY & \leftarrow & J^{k+1}Y & \leftarrow & \ldots & \leftarrow & J^{2k-1}Y \end{array} \tag{6.4}$$

The goal is to induce classical Lepagean equivalents on $J^{2k-1}Y$, so consider the bundle $R^{2k-1} \to J^{2k-1}Y$. Observe that the canonical form $\bar{\Theta}_{\mathcal{L}}$ on $\overline{W}$ can be pulled back to $\overline{W}^{2k-1}$, then to $J^1(\overline{W}^{2k-1})$, and then finally to R^{2k-1}. I use the same symbol for this induced form; note that its coordinate expression remains (6.1). To obtain classical Lepagean equivalents on the jets, then, it suffices to section this bundle.

Theorem 9. (i) *Let r be any section of $R^{2k-1} \to J^{2k-1}Y$. Then $r^*\bar{\Theta}_{\mathcal{L}}$ is a Lepagean equivalent of $\mathcal{L}$ in the sense of* §2.

(ii) *If r is holonomic, then $r^*\bar{\Theta}_{\mathcal{L}}$ is a strict classical Lepagean equivalent of $\mathcal{L}$ in the sense of* §5.

(iii) *Every strict classical Lepagean equivalent on $J^{2k-1}Y$ is $r^*\bar{\Theta}_{\mathcal{L}}$ for some holonomic section r of $R^{2k-1} \to J^{2k-1}Y$.*

Proof. (i) Suppose $\gamma \in \Gamma(\pi^{2k-1})$ is such that $\pi^{2k-1}_k \circ \gamma = j^k\varphi$ for some $\varphi \in \Gamma(\pi)$. It must be shown that $\gamma^*(r^*\bar{\Theta}_{\mathcal{L}}) = (j^k\phi)^*\mathcal{L}$. But this follows immediately from (6.1).

(ii) It is necessary to verify that $r^*\bar{\Theta}_{\mathcal{L}}$ satisfies (LE1), (LE2) and Definition 6. Now (LE1) follows from part (i). Next, $r^*\bar{\Theta}_{\mathcal{L}}$ satisfies (LE2) by virtue of the construction of R^{2k-1} and the assumption that r is holonomic. Finally, Definition 6 is a consequence of the very definition of $\overline{W}$.

(iii) Consider a strict classical Lepagean equivalent Θ on $J^{2k-1}Y$. Then Θ is (π^{2k-1}_{k-1})-horizontal, and hence from (5.6) it follows that Θ defines a section of $\overline{W}^{2k-1}$ with the property that $j^1\Theta$ maps into R^{2k-1}. But as $\bar{\Theta}_{\mathcal{L}}$ is a tautological form, $(j^1\Theta)^*\bar{\Theta}_{\mathcal{L}} = \Theta$. ■

This construction provides an alternate proof of the existence of strict classical Lepagean equivalents as well as a characterization of the ambiguity therein. In particular the proof of Theorem 4 shows that there exists a global section, call it λ, of $\overline{W}^{2k-1} \to J^{2k-1}Y$ with the property that $j^1\lambda$ maps into R^{2k-1}. Combined with part (ii) of Theorem 9, this yields:

Corollary 10. *Strict classical Lepagean equivalents always exist.*

Regarding uniqueness, Theorem 9 and a comparison of (5.5) and (5.6) with (6.1) and (6.2), respectively, immediately give:

Corollary 11. *The set of strict classical Lepagean equivalents on $J^{2k-1}Y$ is in one-to-one correspondence with the set of holonomic sections of $R^{2k-1} \to J^{2k-1}Y$.*

The algorithm (6.4) is entirely canonical except for the last step, where an extrinsic element – the section r – enters. *This* is why (strict) classical Lepagean equivalents are not uniquely determined in general; the ambiguity therein is a reflection of the freedom in the choice of holonomic section of $R^{2k-1} \to J^{2k-1}Y$. When $n = 1$ or $k = 1$ and $N = 1$ (where N is the fiber dimension of Y), this bundle "collapses" in the sense that it then has only one holonomic section. Thus in particular there is a unique strict classical Lepagean equivalent in mechanics (of any order); it is [15, 28, 38]:

$$\Theta = L\,dt + \sum_{s=0}^{k-1}\left[\sum_{r=s}^{k-1}\left[(-1)^{r-s}\frac{d^{r-s}}{dt^{r-s}}\left(\frac{\partial L}{\partial q^A_{(r+1)}}\right)\right]\right]\psi^A_{(s)}. \tag{6.5}$$

Here I have set $t = x^1$, $q^A_{(s)} = y^A_{\mu_1\cdots\mu_1}$ (s times), and $D_t = \frac{d}{dt}$.

Remark 8. The results of this section also explain why (strict) classical Lepagean equivalents usually cannot be defined on jet bundles of order less than $2k-1$: it is because generically the algorithm for determining the λ^{μ}_{A} only terminates with R^{2k-1}. Put another way, Theorem 9 is not necessarily valid for $R^q \to J^qY$ with $q < 2k-1$.

Remark 9. When constructing a strict classical Lepagean equivalent by directly sectioning $R^{2k-1} \to J^{2k-1}Y$, one arbitrarily specifies the $c^{\mu}_{A} = \lambda^{[\mu]}_{A}$ as functions on $J^{2k-2}Y$. But if instead one constructs such a section in stages; *i.e.*, by sectioning each bundle $R^{k+s-1} \longrightarrow J^{k+s-1}Y$, $s = 1,\ldots,k$ in turn, one actually obtains the c^{μ}_{A} as functions on $J^{2k-|\mu|}Y$, *cf.* [21].

The construction presented here can be both specialized and extended. For instance, one can recover all classical Cartan forms in much the same way as all strict classical Lepagean equivalents. For this one need only replace the bundle $\overline{W}$ in the above by its subbundle $\overline{V}$ which consists of all (π^k_{k-1})-horizontal forms on J^kY which in addition are of at most degree one in the generators of J_k (*cf.* §3). One obtains analogues of Theorem 9 and Corollaries 10 and 11 in this context, where the classical Cartan form is unique provided only that $n = 1$ or $k = 1$. (The condition on the fiber dimension of Y is no longer necessary, since now $\chi = 0$). This unique classical Cartan form for a first order field theory is just (1.1). It is straightforward to specialize even further to the so-called "Poincaré-Cartan forms" defined in [12, 17, 21, 24].

Likewise, it is possible to consider the entire bundle W on J^kY as the starting point. Then the analogues of parts (i) and (ii) of Theorem 9 are still valid, but the classical Lepagean equivalents $r^*\Theta_{\mathcal{L}}$ so obtained have no special designation. In fact all classical Lepagean equivalents on $J^{2k-1}Y$ which satisfy $\Theta \equiv \mathcal{L} \bmod J_{2k-1}$ can be gotten by this general procedure provided, in view of Remark 6, one now uses the canonical Lepagean equivalent of the prolonged system $(J^{2k-1}Y \xrightarrow{\pi^{2k-1}} X, \mathcal{J}_{2k-1}, (\pi^{2k-1}_k)^*\mathcal{L})$. Here no prolongation of the base is required as in (6.4), as it is "built in" from the beginning.

Finally, I point out that there are Lepagean equivalents of $\mathcal{L}$ on $J^{2k-1}Y$ which are not classical in the sense of §5. For instance, if $c^{\mu_1\cdots\mu_{k+1}}_{A}(x)$ are arbitrary functions on X antisymmetric in their last two indices, the form $\Theta = c^{\mu_1\cdots\mu_k\nu}_{A}(x)\,dy^A_{\mu_1\cdots\mu_k} \wedge \omega_\nu$ is contact and hence is a (local) Lepagean equivalent of the zero Lagrangian. But this Θ does not satisfy (5.1) and hence is not classical. This example also shows that not every Lepagean equivalent on $J^{2k-1}Y$ is the pullback of $\Theta_{\mathcal{L}}$ on W. Thus, roughly speaking, the *canonical* Lepagean equivalent is universal for *classical* Lepagean equivalents, but not for all Lepagean equivalents.

Remark 10. Muñoz has defined a "universal Poincaré-Cartan form" associated to a Lagrangian density [33]. It functions in much the same way that the universal Cartan form $\bar{\Theta}_{\mathcal{L}}$ on $\overline{V}$ does in that any classical Poincaré-Cartan form can be obtained from his universal one by sectioning a certain bundle. But Muñoz's approach differs from mine in two respects: his universal form is not itself a Lepagean equivalent as is $\bar{\Theta}_{\mathcal{L}}$, and it is unclear if his construction extends to classical Lepagean equivalents more general than Poincaré-Cartan forms.

7. Discussion

For a given variational problem $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$, it is obviously of some interest to construct a bivariant Lepagean equivalent $(W \xrightarrow{\rho} X, \{0\}, \Theta)$.

Under what circumstances this can be accomplished is problematical. In the classical case, as I have shown, the canonical Lepagean equivalent *is* bivariant, due to the fact that the contact ideal is so "nice" in many respects. But this makes it difficult to isolate those features of a classical variational problem which may be relevant for bivariance in a more general setting. A straightforward way to attack the problem would be to extend the algorithm of §6 – which in §4 (implicitly) appeared as a set of necessary and sufficient conditions for contravariance – to nonclassical situations. In §1.e of [19] Griffiths describes a similar procedure in the general single integral case. To carry this through, however, one might have to require that the derived systems of I have constant rank (but see [22]).

Naturally, the "ultimate" goal is to construct a Lepagean equivalent which has the stronger property that

$$\mathcal{E}(\Theta) \approx \mathcal{E}(\mathcal{L}). \tag{7.1}$$

This is certainly too ambitious, even classically, except in special instances. In mechanics of any order k, the canonical Lepagean equivalent satisfies (7.1) by virtue of the fact that any extremal of $\mathcal{L}$ can be *uniquely* lifted to an extremal of $\Theta_{\mathcal{L}}$.[9] The same is true for the (unique) classical Lepagean equivalent (6.5) provided L is regular in the sense that

$$\det\left(\frac{\partial^2 L}{\partial q^i_{(k)}\partial q^j_{(k)}}\right) \neq 0.$$

(This condition forces every extremal of Θ on $J^{2k-1}Y$ to actually be $(2k-1)$-holonomic, not just k-holonomic, *cf.* [21, 28].) Likewise, for first order field theories, the generalized Cartan form on $\overline{V}$ satisfies (7.1), as does the classical Cartan form (1.1) provided

$$\det\left(\frac{\partial^2 L}{\partial y^A_\mu \partial y^B_\nu}\right) \neq 0, \tag{7.2}$$

as is well-known. But the situation is more complicated than these results would indicate. For instance, there is no such isomorphism for the canonical Lepagean equivalent on $\overline{W}$ since now the lifting $\mathcal{E}(\mathcal{L}) \to \mathcal{E}(\bar{\Theta}_{\mathcal{L}})$ constructed in the proof of Theorem 4 is not uniquely determined. This is because nothing in the construction fixes the higher degree contact terms in (4.2). On the other hand, for more general classical Lepagean equivalents such as (1.4), and in particular (1.2) and (1.3), (7.2) is no longer the appropriate

[9] The proof is identical to that which gives the unicity of the classical Lepagean equivalent in this context.

regularity condition. In fact, a sufficient condition for "regularity" in this context is [17, 23]

$$\det\left(\frac{\partial^2 L}{\partial y^A_\mu \partial y^B_\nu} - \lambda^{\mu\nu}_{AB} - \frac{\partial \lambda^{\nu\rho}_{BC}}{\partial y^A_\mu} y^C_\rho + \ldots\right) \neq 0.$$

But this is not sufficient to guarantee that an extremal of such a Θ is holonomic; global considerations intervene. In both the classical and canonical cases, then, difficulties arise from the presence of higher degree contact terms.

A more tractable problem this would be to determine conditions under which *every* lift of an extremal of $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$ is an extremal of $(W \xrightarrow{\rho} X, \{0\}, \Theta)$. (This is related to Dedecker's problem [10] of constructing a *"relèvement libre et complet."*) The reader is invited to study the above examples in this light. Further discussion of these matters in a variety of contexts may be found in [10, 11, 19, 22].

Regarding the classical theory, it is natural to ask if there is (something like) a functor which assigns to each classical variational problem on J^kY a classical Lepagean equivalent problem on $J^{2k-1}Y$. Unfortunately, for $k > 2$ Anderson has shown that no such global correspondence can exist, at least if it is to be *linear* in $\mathcal{L}$ (*cf.* Proposition 5.53 in [1]). The basic problem is that there is no "natural" way to fix the arbitrary terms c^μ_A in (5.1).[10] Very likely a similar result is true even if the correspondence is to be nonlinear, as is the case for Carathéodory's theory, *cf.* (1.2). Even so, one can ask for less. Consider a strict classical Lepagean equivalent Θ of the form (5.5). The algorithm of §6 fixes – insofar as is possible – the coefficients $\lambda^{\mu\nu}_A$ of the first degree contact terms in Θ via (5.6). But as observed above it says nothing at all about the higher degree contact term χ. Is there any procedure analogous to that presented here which can be used to specify this term, other than to force it to vanish à la (C)? If such procedures exist, what is their significance? For example, in the first order case Betounes [2] iteratively determined the coefficients $\lambda^{\mu_1\ldots\mu_s}_{A_1\ldots A_s}$ in (1.4) and thereby obtained (1.3) by requiring that $d\Theta = 0$ iff $\mathcal{L}$ is a divergence. His procedure extends to higher order field theories producing local classical Lepagean equivalents which have this property. But it does not work globally unless $k = 1$ [3].

The algorithm of §6 leads to some intriguing speculations on the relation between the Hamiltonian and Lagrangian formalisms in the classical calculus of variations. As one moves left to right in the diagram (6.4), one

[10] That is, without introducing some element extrinsic to the problem, such as a connection on X.

may think of the formalism as becoming "more and more Lagrangian": the process starts on the "nonclassical" bundle $\overline{W} \to J^kY$ and ends on the "nearly classical" bundle $R^{2k-1} \to J^{2k-1}Y$. Once the latter bundle is sectioned, one recovers the usual classical Lagrangian formalism on $J^{2k-1}Y$, which one can think of as the covariant analogue of the tangent bundle. So it might be expected that as one goes toward the *left*, the formalism becomes progressively "more Hamiltonian." This is indeed the case, as I now explain.

First recall the setup for strict classical Lepagean equivalents, which is based upon the subbundle $\overline{W} \subset W \subset \Lambda^n(J^kY)$ defined in §6. Each element $\overline{w}$ of $\overline{W}$ can be written

$$\overline{w} = L\omega + \sum_{|\underline{\mu}|=0}^{k-1} \lambda_A^{\underline{\mu}\nu} \psi_{\underline{\mu}}^A \wedge \omega_\nu + \mathcal{O}\left(J_k \wedge J_k\right). \tag{7.3}$$

As just observed, the algorithm of §6 successively determines the coefficients of the first degree contact terms in such a form. But the horizontal term – *viz.* $L\omega$ – was fixed *ab initio* by the very definition of the bundle $\overline{W}$. To incorporate this into the above general scheme, consider the bundle $\Lambda^n(J^{k-1}Y)$. Every element $z \in \Lambda^n(J^{k-1}Y)$ takes the form

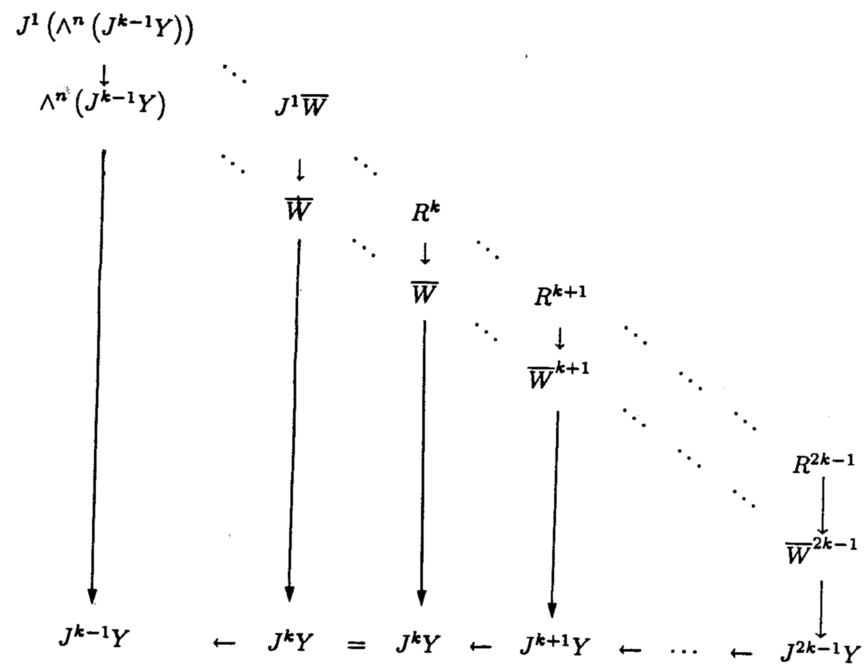
$$z = p\omega + \sum_{|\underline{\mu}|=0}^{k-1} p_A^{\underline{\mu}\nu}\, dy_{\underline{\mu}}^A \wedge \omega_\nu + \mathcal{O}\left(dy_{\underline{\mu}}^A \wedge dy_{\underline{\nu}}^B\right).$$

The reason why $\Lambda^n(J^{k-1}Y)$ and not $\Lambda^n(J^kY)$ appears here is explained by a comparison of (7.3) with the pullback of z to J^kY, which can be written

$$\left(\pi_{k-1}^k\right)^* z = \lambda\omega + \sum_{|\underline{\mu}|=0}^{k-1} \lambda_A^{\underline{\mu}\nu} \psi_{\underline{\mu}}^A \wedge \omega_\nu + \mathcal{O}\left(J_k \wedge J_k\right)$$

for some coefficients λ and $\lambda_A^{\underline{\mu}\nu}$.

From these expressions one sees that the "zeroth" step in the algorithm really should be to fix $\lambda = L$. As the Lagrangian is k^{th}-order, this requires prolonging the base once, from $J^{k-1}Y$ to J^kY. Thus $\overline{W}$ appears naturally as a subbundle of $(\pi_{k-1}^k)^{\#}(\Lambda^n(J^{k-1}Y))$, and the diagram (6.4) can be extended as follows:



Now the point of all this is that according to [17] the space $\Lambda^n(J^{k-1}Y)$ is exactly the covariant phase space corresponding to the Lagrangian system $(J^{2k-1}Y \xrightarrow{\pi^{2k-1}} X, \{0\}, \Theta)$ with Θ a strict classical Lepagean equivalent. In other words, $\Lambda^n(J^{k-1}Y)$ is the covariant analogue of the cotangent bundle in this context, and this is where the Hamiltonian formalism resides![11] Naturally it carries a canonical n-form – the exterior differential of which is the *multisymplectic form* – which, when pulled back to $\overline{W}$ in the manner of §6, is just the canonical Lepagean equivalent $\bar{\Theta}_{\mathcal{L}}$. Thus the left hand column of the above diagram is purely Hamiltonian, and by invoking the algorithm it is possible to "derive" (in some sense) the Lagrangian formalism from the Hamiltonian. To close the circle, one of course has the covariant Legendre transformation which maps $J^{2k-1}Y \to \Lambda^n(J^{k-1}Y)$. Thus the Legendre transformation is the "inverse" of the algorithm of §6. For further details on the covariant Hamiltonian formalism see [17].

All this might be viewed as the "Lepagean approach" to the classical calculus of variations. Turn next to the more familiar "De Donder-Weyl

[11] Some authors view the Hamiltonian formalism as occurring elsewhere, *e.g.*, on $J^{2k-1}Y$ [27] or R^k [19], but this conflicts with common practice.

approach," in which Θ is taken to be a classical Cartan form. The situation is then essentially similar to that above, except that $\overline{W}$ is replaced by

$$\overline{V} = \left\{ \overline{w} \in \overline{W} \mid i_\xi i_\eta \overline{w} = 0 \;\forall\, \xi, \eta \in V\pi^{k-1} \right\},$$

etc., and so $\Lambda^n(J^{k-1}Y)$ must be replaced by its subbundle

$$Z^{k-1} = \left\{ z \in \Lambda^n\left(J^{k-1}Y\right) \mid i_\xi i_\eta z = 0 \;\forall\, \xi, \eta \in V\pi^{k-1} \right\}.$$

The resulting Hamiltonian formalism now lives on Z^{k-1} which is the multisymplectic manifold appropriate in this context [17].

An interesting corollary of this discussion is that the covariant Hamiltonian formalism depends crucially upon the specific type of classical Lepagean equivalent under consideration, and not just the original variational problem $(J^kY \xrightarrow{\pi^k} X, \mathcal{J}_k, \mathcal{L})$. Moreover, as illustrated above, the notion of regularity shares this property.

These observations are tantalizing, and deserve further investigation. They may also have analogues in the general nonclassical case, but it is not so clear how this might go. When I is Pfaffian, the analogue of the covariant Hamiltonian formalism should in some sense live on a subbundle of $\Lambda^n M$, whereas the rest of the algorithm should live on bundles over prolongations of $(M \xrightarrow{\pi} X, \mathcal{I}, \mathcal{L})$.

Finally, note that besides the Hamiltonian formalism on the far left of the diagram and the classical Lagrangian formalism on the far right, there are "intermediate" formalisms based on the bundles $R^q \to J^qY$, where $q = k, \ldots, 2k-1$. Each of these bundles carries a canonical form induced from $\overline{\Theta}_{\mathcal{L}}$ on $\overline{W}$ (or, equivalently, from the canonical form on $\Lambda^n(J^{k-1}Y)$) which makes it into a Lepagean equivalent of the original problem. These Lepagean equivalents are distinctly "nonclassical"; nonetheless, people have looked at variational problems on (what amounts to) R^k. Dedecker [11] has pursued this for first order field theories, and his work was extended to the higher order case by Szapiro [39]. Particularly interesting is the fact that they obtained the multisymplectic manifold $\Lambda^n(J^{k-1}Y)$ in the Hamiltonian versions of their theories. Griffiths [19] has studied both classical and general single integral problems in an analogous context.

Appendix: Notation and Terminology

If X is an n-dimensional manifold with coordinates x^μ, define $\omega = dx^1 \wedge \ldots \wedge dx^n$, $\omega_\nu = i_{\partial_\nu}\omega$, and $\omega_{\mu\nu} = i_{\partial_\nu}\omega_\mu$ etc. On occasion I will employ multi-index notation. If $\underline{\mu} = (\mu_1, \ldots, \mu_r)$ is a multi-index, set $|\underline{\mu}| = r$. By convention $|\underline{\mu}| = 0$ means there is no multi-index. When ν is an ordinary index, the combination $\underline{\mu}\nu$ stands for the "augmented" multi-index $(\mu_1, \ldots, \mu_r, \nu)$. Parentheses about a multi-index denotes symmetrization in its last two slots, *e.g.*, $\lambda_A^{(\underline{\mu})} = \lambda_A^{\mu_1 \ldots (\mu_{r-1}\mu_r)}$, while brackets denote antisymmetrization. Set $\partial_{\underline{\mu}} = \frac{\partial^r}{\partial x^{\mu_1} \ldots \partial x^{\mu_r}}$. Throughout I use the summation convention, except with multi-indices where the summation is made explicit.

Let $\pi : Y \to X$ be a fibration. Adapted coordinates on Y are y^A along the fibers and x^μ along the base. The space of sections of π will be denoted $\Gamma(\pi)$. If Y is a tensor bundle then I will often write $C^\infty(Y)$ for $\Gamma(\pi)$. The vertical bundle $V\pi$ is the subbundle $\ker T\pi$ of TY. A form α on Y is *semi-basic* or *π-horizontal* if $i(\ker T\pi)\alpha = 0$.

The k^{th} jet bundle of Y is J^kY with coordinates $x^\mu, y^A, y^A_{\mu_1}, \ldots, y^A_{\mu_1 \ldots \mu_k}$. Set $J^0Y = Y$, $\pi^0 = \pi$ and $y^A_{\mu_0} = y^A$. The various projections are:

$$\begin{array}{ccc} J^rY & \xrightarrow{\pi^r_s} & J^sY \\ {\scriptstyle \pi^r_0}\downarrow & & \downarrow{\scriptstyle \pi^s} \\ Y & \xrightarrow[\pi]{} & X \end{array}$$

for $r \geq s$. The k^{th} jet prolongation is denoted j^k. A section ϕ of $J^rY \to X$ is said to be *s-holonomic* if $\pi^r_s \circ \phi = j^s(\pi^r_0 \circ \phi)$.

The contact system on J^kY is denoted J_k and is locally spanned by the 1-forms

$$\psi^A_{\underline{\mu}} = dy^A_{\underline{\mu}} - y^A_{\underline{\mu}\nu}dx^\nu,$$

where $0 \leq |\underline{\mu}| \leq k-1$. The differential ideal generated by J_k is denoted $\mathcal{J}_k$. A form α on J^kY is *contact* provided $(j^k\varphi)^*\alpha = 0$ for all $\varphi \in \Gamma(\pi)$. Equivalently $(\pi^{k+1}_k)^*\alpha \equiv 0 \bmod J_{k+1}$.

For a k^{th}-order function $f \in C^\infty(J^kY, \mathbb{R})$, the *formal* (or *total*) partial derivative $D_\mu f \in C^\infty(J^{k+1}Y, \mathbb{R})$ of f in the direction x^μ is defined by $(j^{k+1}\varphi)^*(D_\mu f) = \partial_\mu(f \circ j^k\varphi)$ for all $\varphi \in \Gamma(\pi)$. In jet charts

$$D_\mu f = \partial_\mu f + \sum_{|\underline{\nu}|=0}^{k} \frac{\partial f}{\partial y^A_{\underline{\nu}}} y^A_{\underline{\nu}\mu}.$$

The *variational derivative* of f is the function on $J^{2k}Y$ given by

$$\frac{\delta f}{\delta y^A} = \sum_{s=0}^{k} (-1)^s D_{\mu_1} \ldots D_{\mu_s} \left(\frac{\partial f}{\partial y^A_{\mu_1 \ldots \mu_s}} \right).$$

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The Rotor and the Pendulum

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In Honor of J.-M. Souriau

Abstract

We show that Euler's equations for a free rigid body, and for a rigid body with a controlled feedback torque each reduce to the classical simple pendulum equation under an explicit cylindrical coordinate change of variables. These examples illustrate several ideas in Hamiltonian mechanics: Lie-Poisson reduction, cotangent bundle reduction, singular Lie-Poisson maps, deformations of Lie algebras, brackets on $\mathbb{R}^3$, simplifications obtained by utilizing the representation-dependence of Lie-Poisson reduction, and controlling instability by inducing global bifurcations among a set of equilibria using a control parameter.

1. Introduction

Even though the free rigid body is a classical and well understood system, some new and interesting features are still being uncovered. Notable amongst these is Montgomery's [1990] formula for the change in the geometric phase for the attitude of the body when the body angular momentum vector executes one period of its motion. In this paper we present a number of other results that also seem to be new. Perhaps the most interesting of these is the fact that the rigid body system in body angular momentum space (identified with $\mathbb{R}^3$) is filled with invariant elliptical cylinders on each of which the dynamics is, in elliptical cylindrical coordinates, *exactly* the dynamics of a standard simple pendulum.

Related to this is the variety of ways the rigid body equations can be written in Hamiltonian form as a Lie-Poisson system associated to a Lie algebra structure on $\mathbb{R}^3$. The standard choice is to use the Lie algebra $SO(3)$, but one can also use the Euclidean Lie algebra $SE(2)$ or the Lie algebra $SO(2,1)$. The deformation through these algebras, discussed abstractly in Weinstein [1983], is achieved explicitly in the rigid body simply by defining new Hamiltonians and Casimirs using linear combinations of the standard ones.

We make a similar analysis of the rigid body with the stabilizing torque feedback law introduced by Bloch and Marsden [1990] (see also Bloch, Krishnaprasad, Marsden, and Sanchez de Alvarez [1990]). In particular, this sort of analysis enables one to see how the stabilization is achieved from