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Quantization and Bosonic BRST Theory

C. DUVAL*, J. ELHADAD*, M. J. GOTAY[†], J. ŚNIATYCKI[‡], AND G. M. TUYNMAN[§]

Mathematical Sciences Research Institute, Berkeley, California 94720

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We show that BRST symmetry has a natural bosonic analogue in symplectic geometry. In fact, bosonic BRST theory arises as a purely symplectic construction, which can naturally be viewed as a specific instance of symplectic induction. In this context, both the BRST charge and the total ghost number appear as the "components" of a momentum map on an extended symplectic phase space. Our approach to bosonic BRST theory is motivated by certain problems which arise in the quantization of constrained classical systems. We show that the usual Dirac quantization prescription is incorrect when the system has nonunimodular symmetries and demonstrate how bosonic BRST theory may be used to rectify this. As a byproduct we also prove, under certain circumstances, that both the processes of induction and reduction commute with quantization.

I. INTRODUCTION

We study the quantization of a system which is described classically by the vanishing of first class constraints $\phi_\mathbf{a} = 0$, $\mathbf{a} = 1, \ldots, k$. According to the Dirac prescription [D], the physically admissible quantum states of the system are those Ψ which satisfy

$$\mathcal{Q}\phi_\mathbf{a}[\Psi] = 0 \tag{1.1}$$

for all $\mathbf{a}$, where $\mathcal{Q}\phi_\mathbf{a}$ is the quantum operator corresponding to $\phi_\mathbf{a}$.

Let $\mathcal{H}$ be the quantum Hilbert space and denote by $\mathcal{H}_0$ the space of all wave functions satisfying (1.1). If zero is in the *discrete* spectra of the operators $\mathcal{Q}\phi_\mathbf{a}$, then $\mathcal{H}_0 \subset \mathcal{H}$ and $\mathcal{H}_0$ inherits an inner product from $\mathcal{H}$. But if zero is in the *continuous*

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* Permanent address: CPT-CNRS, Luminy Case 907, F-13288, Marseille Cedex 9, France; E-mail: Duval@FRCPTM51.bitnet and Elhadad@FRCPTM51.bitnet.

[†] Permanent address: Mathematics Department, U.S. Naval Academy, Annapolis, MD 21402-5002; E-mail: Gotay@n1.usna.navy.mil. Supported in part as a Ford Foundation Fellow and by NSF Grant DMS-8805699.

[‡] Permanent address: Department of Mathematics and Statistics, University of Calgary, Calgary, AB, T2N 1N4 Canada; E-mail: Sniatycki@uncamult.bitnet. Supported in part by NSERC Grant A8091.

[§] Permanent address: Département de Mathématiques, Université de Lille 1, F-59655 Villeneuve d'Ascq cedex, France; E-mail: GMT@FRCITL81.bitnet. Supported in part by the Netherlands Organization for Scientific Research (NWO).

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spectra of the $\mathcal{Q}\phi_a$, as is often the case in applications, then wave functions in $\mathcal{H}_0$ are not square-integrable. Then $\mathcal{H}_0 \cap \mathcal{H} = \{0\}$ and consequently the inner product on $\mathcal{H}$ no longer induces one on $\mathcal{H}_0$.

One way to recover the inner product on $\mathcal{H}_0$ is to quantize the reduced system obtained by imposing the constraints $\phi_a = 0$ classically and then eliminating gauge ambiguities. *Provided* this quantization can be carried out in a manner compatible with that of the original system, the inner product so obtained can be used to induce one on $\mathcal{H}_0$. This is a standard means of obtaining (positive definite) inner products for relativistic particles (e.g., positive frequency solutions of the Klein–Gordon equation) [Wo].

The above is an ambitious program and has been carried out in detail only in certain special situations [G, GoS, GuS1, Sn2, T1]. The simplest case to consider is that of a "constrained classical system with symmetry." In this context, let (M, ω) be a symplectic manifold and Φ a free and proper (left) Hamiltonian action of a connected Lie group G on (M, ω). We suppose that the constraint set is given by $J^{-1}(0)$, where $J: M \to \mathfrak{g}^*$ is the Ad*-equivariant momentum mapping for Φ and $\mathfrak{g}$ is the Lie algebra of G.[1] Upon quantizing (M, ω) we get a Hilbert space $\mathcal{H}$ and (1.1) becomes

$$\mathcal{Q}J_\xi[\Psi] = 0 \tag{1.2}$$

for all $\xi \in \mathfrak{g}$, where $J_\xi(m) := \langle J(m), \xi \rangle$. In general, $\mathcal{H}_0 \subset \mathcal{H}$ when G is compact, but not otherwise. It is the latter case in which our interest lies.

Since Φ is free and proper, $J^{-1}(0)$ is smooth and the symplectic reduced space $\bar{M} = G \backslash J^{-1}(0)$ is well defined [MW]. Consider now a quantization of $\bar{M}$ with corresponding Hilbert space $\bar{\mathcal{H}}$; we need to examine the conditions under which the quantizations of M and $\bar{M}$ are "compatible." Roughly speaking, this means that there exists a linear isomorphism $\mathcal{H}_0 \to \bar{\mathcal{H}}$ which intertwines the quantizations of (a certain class of) G-invariant observables on M and their projections to $\bar{M}$.

To be definite, we carry out these quantizations using the Kostant–Souriau geometric quantization theory [Sn1, So, Wo], in which the obstructions to compatibility are fairly well understood [G, GoS, GuS1, Sn2]. For the most part they comprise the conditions under which the various quantization structures (*viz.*, the prequantization line bundle, the polarization, and the metaplectic structure) are G-invariant and so pass to corresponding quantization structures on the quotient $\bar{M}$.

However, one obstruction is somewhat more subtle and lies at the core of the problem we wish to consider: it is the *unimodularity* of G. (Recall that a group G is "unimodular" if it carries a bi-invariant volume element.) It has been recently discovered [DET, T1] that the correlation between the original and reduced phase space quantizations breaks down when G is not unimodular, even when all other

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[1] For simplicity we consider only the case when the constraints are given by $J = 0$. Also, it technically suffices to assume only that G acts freely and properly along $J^{-1}(0)$. These matters will be discussed further in Section IX.

obstructions vanish. This is interesting for several reasons. One is that although unimodularity was implicitly realized to be an obstruction to compatibility,[2] no examples were known prior to [DET]. Another is that this obstruction appears to be purely mathematical; there is no obvious physical content in it. More importantly, in order to regain the correspondence between the original and reduced phase space quantizations, it was found necessary in [T1] to modify (1.2) as follows:

$$\mathcal{Q}J_\xi[\Psi] = -\frac{i}{2}\operatorname{tr}(ad_\xi)\,\Psi. \tag{1.3}$$

Such a modification to the Dirac quantization prescription was effectively proposed by Kostant and Sternberg in their paper [KS] on the quantization of reduced Poisson algebras using BRST techniques.[3] Although the reason for this "correction" is not evident from a physical standpoint, such shifts are well known in the context of the theory of induced representations [M, V]. (This observation will prove to be very useful later.)

In this paper we show how the unimodularity obstruction may be avoided using a purely bosonic version of classical BRST theory. This is based on the observations, to be made precise later, that *BRST symmetry has a natural bosonic analogue in symplectic geometry* and that in this context *the BRST charge* (*together with the total ghost number*) *appear as components of an ordinary momentum mapping on an extended symplectic phase space.*

When G is not unimodular, we extend the original system by adjoining ghost-antighost canonical pairs in a natural way. We also show how to extend the group action, by effectively replacing G by T^*G. The key points here are that (i) *T^*G is always unimodular* and (ii) the reductions of the extended and original systems coincide. We then quantize this extended system along with its associated bosonic BRST symmetry. Since by (i) we are now back in the unimodular case, we can apply the Smooth Equivalence Theorem (Section V) which guarantees the equivalence of compatible quantizations of a unimodular system and its reduction. This equivalence enables us to induce an inner product on the space of T^*G-invariant states of the extended system. We then invoke (ii) to assert, roughly speaking, that the quantization of the (constrained) extended system gives rise to a quantization of the (constrained) original system. This happens in such a way that the quantization condition (1.2), now written in terms of the momentum map for the action of T^*G on the extended phase space, is equivalent to (1.3), written in terms of the original momentum map for the G-action—with the correction $-(i/2)\operatorname{tr}(ad_\xi)\Psi$ *automatically included.* We thereby obtain an inner product, not on the space of

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[2] In [G, Sn2] G was actually required to carry a bi-invariant metric. But this is an overly restrictive condition; an examination of these papers shows that unimodularity will suffice.

[3] In Kostant and Sternberg's approach [KS], the correction in (1.3) is subsumed as an "extra" term in the quantum BRST operator. This term does not appear in the standard BRST literature [GSW], presumably because of factor ordering ambiguities, cf. Remark 9 in Section VI.

G-invariant states of the original system, but on the space of *quasi-invariant* states (i.e., those satisfying the modified Dirac prescription (1.3)). These results indicate that the Dirac prescription (1.2) is appropriate only for unimodular systems, and in the nonunimodular case should be superseded by the modified prescription (1.3).

Thus not only do we present a general scheme for quantizing systems with nonunimodular groups, we also provide a natural geometric explanation for the mysterious correction in (1.3) in terms of (bosonic) BRST theory. Our analysis demonstrates moreover the significance of BRST techniques for analyzing the structure, and in particular the quantization of constrained classical systems.

Although by now standard in physics, there remains the problem of really understanding the BRST construction at its most fundamental level. To this end we exploit the connection between the Dirac quantization problem and the theory of induced representations hinted at above to show that the quantizations of the extended and original systems are related by the process of unitary induction [M, V]. The crucial observation is that an analogous result is true classically: we prove that the extended system can be obtained from the original one by applying symplectic induction [GuS2, KKS] to the subgroup G of T^*G. Thus in this sense *bosonic BRST theory arises as a purely symplectic construction, which can be naturally viewed as a specific instance of symplectic induction.* The results on quantization in this paper may then to some extent be interpreted as establishing, under certain circumstances, that *both the processes of induction and reduction commute with quantization.*

The plan of this paper is as follows. In Section II we give the bosonic version of BRST theory; our discussion here is mostly heuristic. Then in Section III we derive the bosonic BRST formalism by means of symplectic induction. The relevant aspects of the geometric quantization procedure are briefly reviewed in Section IV, and in Section V we study the quantization of unimodular systems. The main result here states sufficient conditions which ensure, for such systems, that reduction and quantization commute. Next, in Section VI, we apply bosonic BRST theory to the quantization of nonunimodular systems; this section forms the core of the paper. In Section VII we examine the relationship between symplectic and unitary induction in this context, proving that induction commutes with quantization as well. We work out the details for a simple example in Section VIII, and then conclude with some remarks and open problems.

Our conventions for notation and terminology are those of [AM] for symplectic geometry and group theory, [Sn1] for geometric quantization theory, and [BM] for the BRST formalism.

II. A BOSONIC FORMULATION OF BRST SYMMETRY

Fadeev and Popov introduced ghosts as a byproduct of the path integral quantization of gauge theories. The effective action they constructed was no longer gauge invariant, but was observed to possess a new global symmetry, now called

Becchi–Rouet–Stora–Tyutin invariance. This symmetry mixes the (fermionic) ghosts with the other fields of the theory and is most naturally expressed in (co)homological terms.

Since then it has been realized that this symmetry (or its associated generator, the *BRST charge*) has a classical counterpart: indeed, there is a BRST charge associated to every Hamiltonian system with first class constraints. The classical version of BRST theory grew out of the work of Batalin, Fradkin, and Vilkovisky, as generalized and reinterpreted by Henneaux, McMullan, and others. Overall references include the papers by Henneaux and Teitelboim [HT], Kostant and Sternberg [KS], and Loll [L].

On the classical level the main use of the BRST/BFV formalism is to give a cohomological description of the symplectic reduction process, as emphasized by Kostant and Sternberg [KS] and Stasheff [St]. In the specific context of constrained systems with symmetry, these techniques enable one to compute the reduced Poisson algebra

$$C^\infty(\bar{M}) \approx [C^\infty(M)/\mathscr{J}]^G$$

of physical observables without having to pass through the quotient $C^\infty(M)/\mathscr{J}$, where $\mathscr{J}$ denotes the ideal in $C^\infty(M)$ generated by the components of the momentum map J. The procedure may be nicely summarized in terms of supergeometry [H2] as follows. (See [KS, L] for detailed discussions.)

One considers the space

$$\mathscr{P} = C^\infty(M) \otimes \Lambda(\mathfrak{g}^* \oplus \mathfrak{g})$$

and equips it with a super Poisson bracket $\{\cdot,\cdot\}$, so that $\mathscr{P}$ becomes a super Poisson algebra. Let $\{\rho_a\}$ be a basis for $\mathfrak{g}$ and $\{\eta^a\}$ the corresponding dual basis for $\mathfrak{g}^*$. Then one defines the BRST charge to be

$$\Omega = J_a \eta^a - \tfrac{1}{2} C^c_{ab} \eta^a \wedge \eta^b \wedge \rho_c. \tag{2.1}$$

Similarly, the *total ghost number* is

$$\Theta = \eta^a \wedge \rho_a. \tag{2.2}$$

Then one can prove that

$$C^\infty(\bar{M}) \approx \{f \in \mathscr{P} \mid \{\Omega, f\} = 0 = \{\Theta, f\}\}/\{\Omega, \mathscr{P}\}.$$

It is possible to realize this super Poisson algebra as the Poisson algebra of a super symplectic manifold and hence Ω and Θ become super functions on this space [L, T2]. Thus the *super* symplectic geometry of the BRST/BFV formalism is fairly well understood, but the underlying *bosonic* symplectic geometry is not. What we will do now is recover bosonic analogues of the BRST charge and the total ghost number as components of a momentum map, purely in terms of standard symplectic geometry, without any odd degrees of freedom in the formalism.

The basic idea is to extend both the phase space *and* the group by adding new bosonic degrees of freedom. Following the BRST/BFV philosophy, to every constraint $J_a = \langle J, \rho_a \rangle$ we will adjoin a *ghost* $\eta^a \in \mathfrak{g}^*$ and its canonically conjugate *antighost* $\rho_a \in \mathfrak{g}^{**} \approx \mathfrak{g}$ to the set of variables describing the system classically.[4] Geometrically, this amounts to extending the phase space M to

$$\tilde{M} = M \times (\mathfrak{g}^* \oplus \mathfrak{g}) \tag{2.3}$$

with symplectic form

$$\tilde{\omega} = \omega + \omega_{\mathfrak{g}}, \tag{2.4}$$

where $\omega_{\mathfrak{g}}$ is the canonical symplectic structure on $\mathfrak{g}^* \oplus \mathfrak{g} \approx T^*\mathfrak{g}^*$. The manifold $\tilde{M}$ is the ordinary symplectic manifold corresponding to the super symplectic manifold which gives rise to the super Poisson algebra $\mathscr{P}$.

Again appealing to the dictum that "for every constraint there is an associated ghost," we extend the group G to T^*G.[4] We use the left trivialization to identify T^*G with $G \times \mathfrak{g}^*$ (which makes the presence of the "group ghosts" manifest) and then endow the latter with a semi-direct product group structure $G \circledS \mathfrak{g}^*$ according to

$$(g, \alpha) \cdot (h, \beta) = (gh, \mathrm{Ad}^*_{g^{-1}}(\beta) + \alpha).$$

It is readily verified that $\tilde{\Phi}: (G \circledS \mathfrak{g}^*) \times \tilde{M} \to \tilde{M}$ defined by

$$\tilde{\Phi}_{(g,\alpha)}(m, \eta, \rho) = (\Phi_g(m), \mathrm{Ad}^*_{g^{-1}}(\eta) + \alpha, \mathrm{Ad}_g(\rho)) \tag{2.5}$$

is a free and proper left action of the extended group $G \circledS \mathfrak{g}^*$ on the extended phase space $\tilde{M}$. It consists of three pieces: the original action Φ of G on M, the cotangent lift of the coadjoint action of G on $\mathfrak{g}^*$ to $\mathfrak{g}^* \oplus \mathfrak{g}$ and translations along $\mathfrak{g}^*$.

In fact, this action is Hamiltonian with momentum map $\tilde{J}: \tilde{M} \to [\mathrm{Lie}(G \circledS \mathfrak{g}^*)]^* \approx \mathfrak{g}^* \oplus \mathfrak{g}$ given by

$$\langle \tilde{J}(m, \eta, \rho), (\xi, \alpha) \rangle = \langle J(m), \xi \rangle - \langle ad^*_\xi(\eta), \rho \rangle + \langle \alpha, \rho \rangle \tag{2.6}$$

for all $(\xi, \alpha) \in \mathfrak{g} \circledS \mathfrak{g}^*$. This momentum map is Ad^*-equivariant with respect to the semi-direct product structure on $G \circledS \mathfrak{g}^*$. Writing $\tilde{J}$ out with respect to the above bases, we obtain

$$\tilde{J}_{(\xi,\alpha)} = J_a \xi^a - C^c_{ab} \xi^a \eta^b \rho_c + \alpha_a \eta^a, \tag{2.7}$$

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[4] Our convention is that the ghosts η^a are elements of $\mathfrak{g}^*$ and that the antighosts ρ_a are the dual variables in $\mathfrak{g}^{**} \approx \mathfrak{g}$. (Thus, if $M = T^*C$, the extended configuration space would be $\tilde{C} = C \times \mathfrak{g}^*$ and $\tilde{M} = T^*\tilde{C}$, etc.) The distinction is important, however. For if we took the ghosts to be elements of $\mathfrak{g}$ rather than $\mathfrak{g}^*$, we would naturally be led to consider the group TG as opposed to T^*G. But TG is *not* automatically unimodular, and this would render our constructions pointless. (In fact, a computation similar to that given in the proof of Proposition 6 shows that TG is unimodular iff G is.)

where in this expression we regard η^{a} and ρ_{a} as being the coordinate functions on $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively.

A comparison of (2.7) with (2.1) and (2.2) is provocative. To be more precise, viewing $\xi = \xi^{\mathrm{a}}\rho_{\mathrm{a}}$, $\alpha = \alpha_{\mathrm{a}}\eta^{\mathrm{a}}$, and $\tilde{J}_{(\xi,\alpha)}$ as superfunctions in the obvious way, a short calculation formally establishes

$$\tilde{J}_{(\xi,\alpha)} = \{\Omega, \xi\} + \{\Theta, \alpha\}. \tag{2.8}$$

We may conclude that $\tilde{J}$ constitutes the bosonic analogue of the BRST charge and the total ghost number together and that the action $\tilde{\Phi}$ is the natural symplectic counterpart of BRST symmetry.

The observation that *both* Ω and Θ are just "components" of $\tilde{J}$ is intriguing, as it suggests that they are not of primary importance in themselves, but rather only together in the combination $\tilde{J}$. This contention is supported by the work of Cariñena and Ibort on Yang–Mills theory [CI]. There they showed, under certain circumstances, that Θ actually plays the role of the BRST charge as the generator of BRST symmetry. Their construction also yields a natural symplectic interpretation of BRST symmetry and ghosts in the Yang–Mills case; in a sense it is a "reduced" version of that presented here.

Finally, we observe from (2.6) that the extended constraint set

$$\tilde{J}^{-1}(0,0) = J^{-1}(0) \times (\mathfrak{g}^* \oplus \{0\})$$

is strictly larger than the original one. (The antighosts are constrained to vanish by (2.6), but the ghosts remain freely specifiable.) Nonetheless, (2.5) shows that the reduced spaces coincide:

$$(G \circledS \mathfrak{g}^*)\backslash \tilde{J}^{-1}(0,0) \approx G\backslash J^{-1}(0). \tag{2.9}$$

Remarks. 1. In the bosonic formulation (2.5) of BRST symmetry, the ghosts and antighosts are completely decoupled from the original variables. The coupling appears only in the superized theory and is a characteristic feature thereof. (See also Footnote 5 in this regard).

2. Within this framework we are automatically able to incorporate—without introducing odd variables—the kinematical invariance of the system corresponding to changes

$$J_{\mathrm{a}} \to A_{\mathrm{a}}^{\mathrm{b}} J_{\mathrm{b}}$$

in the defining set of first class constraints, where $A \in GL(\mathfrak{g})$ [BM, HT]. Indeed, transformations of this type arise from the action of $GL(\mathfrak{g})$ on $\mathfrak{g}^* \oplus \mathfrak{g}$ given by $(\eta, \rho) \to ({}^{t}A^{-1}\eta, A\rho)$, with momentum map $K\colon \mathfrak{g}^* \oplus \mathfrak{g} \to gl(\mathfrak{g})^*$ of the form $K_\xi = \eta^{\mathrm{a}}\xi_{\mathrm{a}}^{\mathrm{b}}\rho_{\mathrm{b}}$, where $\xi \in gl(\mathfrak{g})$.

3. It seems clear that the formalism we have presented here can be straightforwardly superized by allowing all ghosts and antighosts to become

fermionic. Then, viewing $T^*G \approx G \circledS \mathscr{g}^*$ as a supergroup acting on $\tilde{M}$, now regarded as a super symplectic manifold, we obtain a super momentum mapping $\tilde{J}: \tilde{M} \rightarrow \mathscr{g}^* \oplus \mathscr{g}$. If we view $\tilde{J}$ instead as a map

$$\tilde{J}: M \times (\mathscr{g}^* \oplus \mathscr{g}) \times (\mathscr{g} \oplus \mathscr{g}^*) \rightarrow R_0$$

(where R_0 is the even part of the graded commutative ring R which underlies the super structure) and then appropriately restrict to the "diagonal" in $(\mathscr{g}^* \oplus \mathscr{g}) \times (\mathscr{g} \oplus \mathscr{g}^*)$, one recovers BRST symmetry in its standard form.[5] However, we do not pursue this here as it is not needed for the rest of the paper.

III. Bosonic BRST Theory and Symplectic Induction

We now place the bosonic BRST theory just developed on a more solid mathematical footing, by showing that this construction is an entirely natural—and symplectic—one. In fact, we will show that it is but a specific instance of "symplectic induction."

Symplectic induction is a method whereby one can build a Hamiltonian K-space $(\tilde{M}, \tilde{\omega}, \tilde{\Phi}, \tilde{J})$, given a closed subgroup G of a Lie group K and a Hamiltonian G-space (M, ω, Φ, J). Thus one induces a symplectic realization of K from a symplectic realization of its subgroup G. We will see in Section VII that it is the classical counterpart of the unitary induction technique, where the link between the two constructions is given by geometric quantization. Symplectic induction and its generalizations are explained in [GuS2, We]; for our purposes the following version will suffice [KKS].

Let G be a closed Lie subgroup of K, and consider a Hamiltonian G-space (M, ω, Φ, J). The group G acts on K by $(g, k) \rightarrow kg^{-1}$, and on T^*K by cotangent lift, i.e., $(g, (k, \kappa)) \rightarrow (kg^{-1}, \mathrm{Ad}^*_{g^{-1}}(\kappa))$. (Again we use the canonical left trivialization to identify T^*K with $K \times \mathscr{k}^*$.) This action admits a canonically defined Ad*-equivariant momentum map $T^*K \rightarrow \mathscr{g}^*$ given by $(k, \kappa) \rightarrow -i^*\kappa$, where i^* is the projection $\mathscr{k}^* \rightarrow \mathscr{g}^*$ dual to the inclusion $i: \mathscr{g} \rightarrow \mathscr{k}$.

We now construct an action $\check{\Phi}$ of G on $\check{M} = M \times T^*K$ by combining its actions on M and T^*K:

$$\check{\Phi}_g(m, k, \kappa) = (\Phi_g(m), kg^{-1}, \mathrm{Ad}^*_{g^{-1}}(\kappa)).$$

This action is symplectic for the product symplectic form on $\check{M}$ and moreover admits an Ad*-equivariant momentum map $\check{J}: \check{M} \rightarrow \mathscr{g}^*$ given by $\check{J}(m, k, \kappa) = J(m) - i^*\kappa$. If we suppose that G acts properly and freely on M, then G will act properly (as G is closed in K) and freely on $\check{M}$. We can thus construct the

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[5] By restricting to the diagonal in this fashion we effectively introduce the coupling between the original variables and the ghosts and antighosts which is absent in the bosonic theory (cf. Remark 1). Note that the coupling is nontrivial only if the ghosts and antighosts are fermionic.

Marsden–Weinstein reduced symplectic manifold $(\tilde{M}, \tilde{\omega}) = G \backslash \check{J}^{-1}(0)$, which is the sought after induced symplectic manifold.

To obtain a Hamiltonian action of K on $(\tilde{M}, \tilde{\omega})$ is a straightforward matter. The group K acts on itself by left translations, and this action lifts to T^*K as $(k', (k, \kappa)) \to (k'k, \kappa)$. Let K act trivially on M, thereby giving rise to a Hamiltonian action of K on $(\check{M}, \check{\omega})$ with canonical Ad*-equivariant momentum map $\check{L}: \check{M} \to \mathfrak{k}^*$ given by $\check{L}(m, k, \kappa) = \mathrm{Ad}^*_{k^{-1}}(\kappa)$. This action commutes with the G-action on $\check{M}$ and leaves $\check{J}$ invariant; hence it induces a symplectic action $\tilde{\Phi}$ of K on $\tilde{M}$. Since $\check{L}$ is invariant under the G-action, it descends as an Ad*-equivariant momentum map for the K-action on $\tilde{M}$ which we denote by $\tilde{J}$. Then $(\tilde{M}, \tilde{\omega}, \tilde{\Phi}, \tilde{J})$ is the required Hamiltonian K-space.[6]

Now we return to bosonic BRST theory. The only "problem" with applying symplectic induction here is that it is not immediately apparent what K should be. But a glance back at the previous section tells us that we should take $K = T^*G$, and indeed this is the simplest most natural choice under the given circumstances. Turning the symplectic induction crank as applied to $G \subset T^*G$, a series of routine verifications establishes:

BOSONIC BRST THEOREM. *$(\tilde{M}, \tilde{\omega}, \tilde{\Phi}, \tilde{J})$, with these objects defined as in Section II, is the Hamiltonian T^*G-space symplectically induced from the Hamiltonian G-space (M, ω, Φ, J).*

Thus the bosonic BRST theory of Section II, which admittedly looks rather ad hoc in that context, is actually a canonical symplectic construction. In particular, our definition (2.3) of $\tilde{M}$ as a *trivial* vector bundle is by no means arbitrary; this answers a question of Loll [L, p. 515].

IV. ELEMENTS OF GEOMETRIC QUANTIZATION THEORY

In this section we give a rapid review of the relevant aspects of the geometric quantization procedure. For further information the reader is referred to [Sn1, So, Wo].

Let (M, ω) be a $2n$-dimensional symplectic manifold. For our purposes, the supplementary structures needed for the geometric quantization of (M, ω) are a prequantization line bundle and a polarization.

A *prequantization* of (M, ω) consists of a complex line bundle $l: L \to M$ with a connection ∇ and a compatible Hermitian structure $(\cdot, \cdot)$ such that

$$\text{curvature } \nabla = -(1/2\pi)\, l^*\omega,$$

where we take units such that Planck's reduced constant $h/2\pi = 1$.

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[6] Note that symplectic reduction plays an essential role in the symplectic induction process. This fact will be important for our considerations in Section VII.

A (real) *polarization* of (M, ω) is an involutive n-dimensional Lagrangian distribution P on M. We assume that P is "strongly admissible," i.e., the leaf space M/P is a manifold and the projection $p\colon M \to M/P$ is a submersion. We furthermore suppose that the leaves of P are simply connected so that there are no Bohr–Sommerfeld conditions.

Remarks. 4. We consider only real polarizations in this paper. Analogous results for Kähler polarizations are given in [GuS1].

5. For the sake of simplicity, and since we will not be moving polarizations, we do not utilize metaplectic structures or half-forms. Thus we shall focus on the half-density quantization. This captures the essence of our ideas and constructions without introducing undue complications; in any case, the extension to the half-form quantization should be straightforward.

Fix a polarization P of (M, ω) and let FP be the linear frame bundle of P; it is a right principal $GL(n, \mathbf{R})$-bundle over M. Let $\Delta\colon GL(n, \mathbf{R}) \to \mathbf{R}_+$ be defined by $\Delta(A) = |\det A|^{1/2}$. The bundle $|\Lambda^n P|^{1/2}$ of *half-densities* relative to P is the complex line bundle associated to FP on which $GL(n, \mathbf{R})$ acts by multiplication by Δ. This bundle has a canonically defined partial flat connection covering P. Each section ν of $|\Lambda^n P|^{1/2}$ can be identified with a function $\nu^{\#}\colon FP \to \mathbf{C}$ satisfying

$$\nu^{\#}(\mathbf{f}A) = \Delta(A)^{-1} \cdot \nu^{\#}(\mathbf{f}) \tag{4.1}$$

for all frames $\mathbf{f} \in FP$ and all $A \in GL(n, \mathbf{R})$.

Consider the bundle $L \otimes |\Lambda^n P|^{1/2}$. It carries a partial flat connection covering P induced from those on L and $|\Lambda^n P|^{1/2}$. Denote by $\Gamma(L \otimes |\Lambda^n P|^{1/2})$ the space of all smooth sections of $L \otimes |\Lambda^n P|^{1/2}$. A section $\Psi \in \Gamma(L \otimes |\Lambda^n P|^{1/2})$ is said to be *polarized* if it is covariantly constant along P. Let $\mathscr{h}$ be the subspace of all polarized sections. Elements of $\mathscr{h}$ are interpreted as smooth wave functions, i.e., $\mathscr{h}$ is the *smooth* quantum state space associated to (M, ω) by the geometric quantization procedure in the representation defined by the polarization P.

We construct an inner product on $\mathscr{h}$ as follows. Let $(\mathbf{e}, \mathbf{f})$ be a symplectic basis for $T_m M$ such that $\mathbf{f} \in F_m P$; then $Tp(\mathbf{e})$ is a basis for $T_{p(m)}(M/P)$. If $\Psi = \lambda \otimes \nu$ and $\Upsilon = \chi \otimes \mu$ are any two elements of $\mathscr{h}$, we pair them to obtain a density $\langle \Psi, \Upsilon \rangle$ on the leaf space M/P by setting

$$\langle \Psi, \Upsilon \rangle (Tp(\mathbf{e})) = (\lambda(m), \chi(m))\, \nu^{\#}(\mathbf{f})\, \overline{\mu^{\#}(\mathbf{f})},$$

where the bar denotes complex conjugation. Then we define

$$\langle\!\langle \Psi, \Upsilon \rangle\!\rangle = \int_{M/P} \langle \Psi, \Upsilon \rangle. \tag{4.2}$$

The completion of the space of all compactly supported (modulo P) elements of $\mathscr{h}$ with respect to this inner product is the quantum Hilbert space $\mathscr{H}$.

To complete the geometric quantization program, we must indicate how to quantize (certain) classical observables. Let $k \in C^\infty(M)$ preserve P, i.e., $[X_k, P] \subset P$, where X_k is the Hamiltonian vector field of k. Then the corresponding (formally) self-adjoint quantum operator $\mathcal{Q}k$ on $\mathcal{H}$ is given by

$$\mathcal{Q}k[\Psi] = (-i\nabla_{X_k} + k)\,\lambda \otimes \nu - i\lambda \otimes \mathcal{L}_{X_k}\nu, \tag{4.3}$$

where $\Psi = \lambda \otimes \nu$ and $\mathcal{L}$ denotes the Lie derivative.

For future reference we give local expressions for the inner product (4.2) and the quantum operators (4.3). Choose Darboux coordinates (q^i, p_i), $i = 1, ..., n$, on an open set $U \subset M$ which are adapted to P in the sense that

$$\mathbf{f} = \left(\frac{\partial}{\partial p_1}, ..., \frac{\partial}{\partial p_n}\right)$$

is a local frame field for P. Define a half-density ν for P according to $\nu^\# \circ \mathbf{f} = 1$. Fix a 1-form θ with $\omega \mid U = -d\theta$ and let λ be a trivializing section of $L \mid U$ which is normalized to unity and is such that $\nabla\lambda = -i\theta \otimes \lambda$. Then, relative to these choices, every polarized section Ψ of $L \otimes |\Lambda^n P|^{1/2}$ can be written locally as

$$\Psi \mid U = \psi(q)\lambda \otimes \nu. \tag{4.4}$$

The inner product of two such wave functions becomes

$$\langle\!\langle \Psi, \Upsilon \rangle\!\rangle = \int_{M/P} \psi(q)\,\overline{\upsilon(q)}\, d^n q \tag{4.5}$$

and, after a short computation, (4.3) reduces to

$$\mathcal{Q}k[\Psi] \mid U = \left\{\left[-iX_k - (X_k \,\lrcorner\, \theta) + k - \frac{i}{2}\operatorname{tr}(A_{\mathbf{f}}(X_k))\right]\psi\right\} \lambda \otimes \nu, \tag{4.6}$$

where the components a^i_j of the matrix $A_{\mathbf{f}}(X_k)$ are found from

$$\left[X_k, \frac{\partial}{\partial p_i}\right] = \sum_{j=1}^{n} a^i_j \frac{\partial}{\partial p_j}.$$

V. QUANTIZATION AND UNIMODULAR GROUPS

Now consider the case of a constrained classical system with symmetry. As before, we assume that the action of G on M is free and proper. In particular, then, 0 is a regular value of J. Let $\dim M = 2n$ and $\dim G = r$, so that $2\bar{n} := \dim \bar{M} = 2(n-r)$. Denote the projection $J^{-1}(0) \to \bar{M}$ by π.

We will show, when G is unimodular, how the reduced phase space quantization can be used to induce an inner product on the space of G-invariant wave functions defined by (1.2). Then, when G is not unimodular, we will "extend" the system to a unimodular one using the bosonic BRST theory developed in Section II, and apply the above strategy. This section (slightly) generalizes the results of [G], which were restricted to the cotangent bundle setting.

Our first task is to obtain compatible quantizations of (M, ω) and the reduced space $(\bar{M}, \bar{\omega})$. Let (L, ∇) be a prequantization line bundle for (M, ω) and P a polarization.

PROPOSITION 1. *Suppose that the action of G on M lifts to a connection-preserving action on $L \mid J^{-1}(0)$. Then the prequantization line bundle (L, ∇) for (M, ω) induces a prequantization line bundle $(\bar{L}, \bar{\nabla})$ for $(\bar{M}, \bar{\omega})$.*

Proof. Since the action of G on M is free and proper, the lifted action on $L \mid J^{-1}(0)$ is also. Thus $\bar{L} = G\backslash(L \mid J^{-1}(0))$ is well defined and has the structure of a complex line bundle over $\bar{M}$. Moreover, since G acts by connection-preserving automorphisms, the connection ∇ on L induces a connection $\bar{\nabla}$ on $\bar{L}$. That $(\bar{L}, \bar{\nabla})$ actually provides a prequantization of $(\bar{M}, \bar{\omega})$ now follows from Theorem 3.5 of [G]. ▮

The hypothesis of this proposition will be satisfied if, for example, G is simply connected or M is a cotangent bundle and the action is lifted from the base [G].

PROPOSITION 2. *Suppose that the polarization P of (M, ω) is G-invariant and satisfies*

$$P \cap TJ^{-1}(0)^{\perp} = \{0\}. \tag{5.1}$$

Then P induces a polarization $\bar{P}$ of $(\bar{M}, \bar{\omega})$.

Here, $\perp$ denotes the symplectic polar with respect to ω.

Proof. Taking polars in (5.1) yields

$$P + TJ^{-1}(0) = TM \mid J^{-1}(0).$$

Since 0 is a regular value of J, it follows that $P \cap TJ^{-1}(0)$ is an $(n-r)$-dimensional distribution along $J^{-1}(0)$ transverse to the orbits of G. As P is G-invariant,

$$\bar{P} = T\pi(P \cap TJ^{-1}(0))$$

is an $\bar{n}$-dimensional distribution on $\bar{M}$. It is clearly involutive and Lagrangian, and so defines a polarization of the reduced space. ▮

With these assumptions the quantization structures on (M, ω) project to quantization structures on $(\bar{M}, \bar{\omega})$. We suppose that $\bar{P}$ is strongly admissible. Let $\bar{\mathscr{H}}$ be the smooth quantum state space corresponding to the data $\bar{P}$ and $\bar{L}$ on $(\bar{M}, \bar{\omega})$.

To ensure the equivalence of thse two quantizations, we need one further restriction on the polarization P. We say that P is *compatible with the constraints* provided that, in addition to the assumptions of Proposition 2, every leaf of P

intersects $J^{-1}(0)$.[7] We always assume this intersection is connected. Since the quantum wave functions are covariantly constant along P, compatibility implies that each wave function is uniquely determined by its restriction to the constraint set. In essence, this means that the quantization of the constrained system (M, ω) is insensitive to what happens "off shell." This requirement is crucial, since the reduced phase space quantization is totally "on shell." For a more detailed discussion of this point, see [G, GoS].

Our main result is the

SMOOTH EQUIVALENCE THEOREM. *Let P be compatible with the constraints and assume that G is unimodular. Then there exists a canonical isomorphism between the space of G-invariant smooth sections of $L\otimes|\Lambda^n P|^{1/2}$ which are covariantly constant along P and the space of smooth sections of $\bar{L}\otimes|\Lambda^{\bar{n}}\bar{P}|^{1/2}$ which are covariantly constant along $\bar{P}$.*

Proof. We will prove the theorem in several steps. First we establish some technical results.

Let E be the characteristic bundle of $J^{-1}(0)$, i.e., E is the vector subbundle of $TJ^{-1}(0)$ whose fiber at $m\in J^{-1}(0)$ is $E_m = T_m J^{-1}(0)^{\perp}$. Since G acts freely on $J^{-1}(0)$, there is a canonical G-equivariant identification of $J^{-1}(0)\times \mathfrak{g}$ with E given by $(m, \xi)\to \xi_M(m)$, where ξ_M is the fundamental vector field on M corresponding to $\xi\in\mathfrak{g}$ [MW]. By assumption G is unimodular, so $\mathfrak{g}$ carries an Ad-invariant volume. By transference, E carries a G-invariant volume. In other words, the structure group of E can be invariantly reduced to $SL(r, \mathbf{R})$. This will enable us to relate half-densities for P on M with those for $\bar{P}$ on $\bar{M}$.

To this end fix a global frame $\mathbf{e} = (e_1, ..., e_r)$ for E which has volume one. Consider the subbundle B of $FP \mid J^{-1}(0)$ consisting of frames of the form $\mathbf{f} = (\mathbf{u}, \mathbf{v})$, where $\mathbf{u}$ is a frame for $P\cap TJ^{-1}(0)$ and $\mathbf{v}$ is canonically conjugate to $\mathbf{e}$; that is, $\omega(v_i, e_j) = \delta_{ij}$ for $i, j = 1, ..., r$. Then B is a right principal H-bundle, where

$$H = \left\{ \begin{pmatrix} U & W \\ 0 & I \end{pmatrix} \,\middle|\, U\in GL(\bar{n}, \mathbf{R}) \right\}$$

is the subgroup of $GL(n, \mathbf{R})$ that stabilizes B. Let K be the subgroup of H consisting of those matrices which leave invariant the projection of $\mathbf{u}$ to $T(M/P)$; explicitly,

$$K = \left\{ \begin{pmatrix} I & W \\ 0 & I \end{pmatrix} \right\}.$$

It is a normal subgroup of H and $H/K \approx GL(\bar{n}, \mathbf{R})$. We may therefore identify

$$B/K \approx \pi^* F\bar{P}. \tag{5.2}$$

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[7] If M is connected and $J^{-1}(0)$ is compact, then any polarization satisfying (5.1) is necessarily compatible with the constraints, cf. [GoS].

Proposition 3. *There exists a canonical isomorphism* $|\Lambda^n P|^{1/2} \mid J^{-1}(0) \approx \pi^* |\Lambda^{\bar n} \bar P|^{1/2}$.

Proof. Fix $m \in J^{-1}(0)$ and consider $\nu \in |\Lambda^n P|^{1/2}_m$. Then (4.1) implies that $\nu^{\#}(\mathbf{f}A) = \nu^{\#}(\mathbf{f})$ for all $\mathbf{f} \in B_m$ and all $A \in K$. It follows from (5.2) that for $\mathbf{f} \in B_m$, the equation

$$\bar\nu^{\#}([\mathbf{f}]) = \nu^{\#}(\mathbf{f}) \tag{5.3}$$

defines an element $\bar\nu$ of $(\pi^* |\Lambda^{\bar n} \bar P|^{1/2})_m$, where the brackets denote K-equivalence classes. Conversely, given $\bar\nu \in (\pi^* |\Lambda^{\bar n} \bar P|^{1/2})_m$, (5.3) defines an element $\nu \in |\Lambda^n P|^{1/2}_m$ since, according to (4.1), any half-density is completely determined by its restriction to B_m.

The association (5.3) is the desired isomorphism. To prove that it is canonical, it is only necessary to show that it does not depend upon the choice of the frame $\mathbf{e}$ for E. But if in the above constructions we replace $\mathbf{e}$ by $\mathbf{e}' = \mathbf{e}V$ for some $V \in SL(r, \mathbf{R})$, then this has the effect of replacing $\mathbf{f} = (\mathbf{u}, \mathbf{v})$ by $\mathbf{f}' = (\mathbf{u}', \mathbf{v}')$, where

$$(\mathbf{u}', \mathbf{v}') = (\mathbf{u}, \mathbf{v}) \begin{pmatrix} I & 0 \\ 0 & {}^tV^{-1} \end{pmatrix}.$$

Since V is unimodular, it follows from (4.1) that $\nu^{\#}(\mathbf{f}') = \nu^{\#}(\mathbf{f})$. Thus the choice of frame $\mathbf{e}$ is immaterial. ▌

The action of G on FP gives rise to a left action of G on $|\Lambda^n P|^{1/2}$ by pull back. Since the volume on E is G-invariant, the argument immediately above shows that this induced action is compatible with the isomorphism (5.3). Thus Proposition 3 yields $G \backslash (|\Lambda^n P|^{1/2} \mid J^{-1}(0)) \approx |\Lambda^{\bar n} \bar P|^{1/2}$. Combining this result with Proposition 1, we have

Corollary 4. $G \backslash [(L \otimes |\Lambda^n P|^{1/2}) \mid J^{-1}(0)] \approx \bar L \otimes |\Lambda^{\bar n} \bar P|^{1/2}$.

We are at last ready to prove the Smooth Equivalence Theorem. The polarization P is G-invariant, so the components J_ξ of the momentum map are directly quantizable according to formula (4.2). Thus the assignment $\xi \to \mathcal{Q}J_\xi$ provides a representation of $\mathfrak{g}$ on $\mathfrak{h}$. Let $\mathfrak{h}_0 \subset \mathfrak{h}$ be the subspace consisting of G-invariant states; $\mathfrak{h}_0$ is thus given by (1.2). We must establish the existence of a canonical isomorphism $\mathfrak{h}_0 \approx \bar{\mathfrak{h}}$.

Let $\Psi \in \mathfrak{h}_0$. By Corollary 4, $\Psi \mid J^{-1}(0)$ projects to a smooth section $\bar\Psi$ of $\bar L \otimes |\Lambda^{\bar n} \bar P|^{1/2}$. Since Ψ is polarized, Proposition 2 shows that $\bar\Psi$ is also. Thus $\bar\Psi \in \bar{\mathfrak{h}}$. The association $\Psi \to \bar\Psi$ is injective: for if $\bar\Psi = \bar\Upsilon$, then $(\Psi - \Upsilon) \mid J^{-1}(0) = 0$. But then $\psi = \Upsilon$ as P is compatible with the constraints.

To show surjectivity, let $\bar\Psi \in \bar{\mathfrak{h}}$. Corollary 4 and Proposition 2 show that $\bar\Psi$ pulls back to a unique G-invariant section Ψ_0 of $(L \otimes |\Lambda^n P|^{1/2}) \mid J^{-1}(0)$ which is covariantly constant along $P \cap TJ^{-1}(0)$. Since every leaf of P is simply connected and intersects $J^{-1}(0)$ in a connected set, parallel transport along P produces a globally defined polarized section Ψ of $L\otimes|\Lambda^n P|^{1/2}$ which agrees with Ψ_0 on $J^{-1}(0)$. Now consider the polarized sections $\Psi_\xi := \mathscr{Q}J_\xi[\Psi]$ for each $\xi \in \mathfrak{g}$. Every Ψ_ξ is uniquely determined by its restriction to $J^{-1}(0)$. But $\Psi_\xi \mid J^{-1}(0) = 0$ by construction, so $\Psi_\xi \equiv 0$ for all ξ and hence $\Psi \in \mathscr{h}_0$.

Finally, the isomorphism $\mathscr{h}_0 \to \bar{\mathscr{h}}$ so obtained is canonical since all our constructions are.[8] ▌

Compatible quantizations thus have canonically isomorphic spaces of physically admissible wave functions. But compatibility must ensure more than this: it should also intertwine the quantizations of G-invariant observables. More precisely, let $k \in C^\infty(M)$ be G-invariant in which case it reduces to $\bar{k} \in C^\infty(\bar{M})$, and let $\mathscr{Q}k$ and $\bar{\mathscr{Q}}\bar{k}$ be the corresponding operators on $\mathscr{h}_0$ and $\bar{\mathscr{h}}$, respectively.

THEOREM 5. *Let k be a polarization-preserving G-invariant observable. Then $\bar{k}$ is polarization-preserving, both $\mathscr{Q}k$ and $\bar{\mathscr{Q}}\bar{k}$ exist and the diagram*

$$\begin{tikzcd}
\mathscr{h}_0 \arrow[r, "\mathscr{Q}k"] \arrow[d] & \mathscr{h}_0 \arrow[d] \\
\bar{\mathscr{h}} \arrow[r, "\bar{\mathscr{Q}}\bar{k}"'] & \bar{\mathscr{h}}
\end{tikzcd}$$

commutes, where the vertical arrows are the isomorphisms provided by the Smooth Equivalence Theorem.

The proof is given in [G].

We now turn to the question of the inner product on $\mathscr{h}_0$. When the G-invariant states in $\mathscr{h}$ are not normalizable in $\mathscr{H}$, but G is unimodular, we may use the isomorphism $\mathscr{h}_0 \to \bar{\mathscr{h}}$ provided by the Smooth Equivalence Theorem to induce an inner product on $\mathscr{h}_0$ from that on $\bar{\mathscr{h}}$. Upon taking completions we obtain a unitary equivalence of the corresponding Hilbert spaces $\mathscr{H}_0$ and $\bar{\mathscr{H}}$. If G is compact, whence $\mathscr{H}_0 \subset \mathscr{H}$, it is straightforward to show that the two induced inner products on $\mathscr{h}_0$ coincide up to a scale factor [G].

VI. QUANTIZATION OF NONUNIMODULAR SYSTEMS

But what if G is not unimodular, so that the Smooth Equivalence Theorem, as stated, is no longer applicable? To circumvent this problem we "unimodularize" the system by replacing G by T^*G according to the bosonic BRST construction. Our approach rests on the following fundamental observation.

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[8] Strictly speaking, the isomorphism $\mathscr{h}_0 \to \bar{\mathscr{h}}$ is not quite canonical, as it depends upon the choice of an Ad-invariant volume on $\mathfrak{g}$. But any two such volumes differ by a nonzero multiple, and so the isomorphism *is* canonical when viewed as an equivalence of the corresponding projectivized (pre-) Hilbert spaces (which are, after all, the actual quantum state spaces).

PROPOSITION 6. *$T^*G \approx G \circledS \mathscr{g}^*$ is unimodular.*

Proof.[9] Infinitesimally, a group G is unimodular iff the trace of $ad_\xi\colon \mathscr{g} \to \mathscr{g}$ vanishes for all $\xi \in \mathscr{g}$.

For the group $G \circledS \mathscr{g}^*$, we have $\mathrm{Lie}(G \circledS \mathscr{g}^*) = \mathscr{g} \circledS \mathscr{g}^*$ with

$$ad_{(\xi,\alpha)}(\zeta, \beta) = (ad_\xi(\zeta), ad_\zeta^*(\alpha) - ad_\xi^*(\beta)). \tag{6.1}$$

With respect to the dual bases $\{\rho_\mathrm{a}\}$ of $\mathscr{g}$ and $\{\eta^\mathrm{a}\}$ of $\mathscr{g}^*$, (6.1) takes the matrix form

$$ad_{(\xi,\alpha)}(\rho_\mathrm{c}\ \eta^\mathrm{d}) = (\rho_\mathrm{e}\ \eta^\mathrm{f}) \begin{pmatrix} \xi^\mathrm{a} C^\mathrm{e}_\mathrm{ac} & 0 \\ \alpha_\mathrm{b} C^\mathrm{b}_\mathrm{cf} & -\xi^\mathrm{a} C^\mathrm{d}_\mathrm{af} \end{pmatrix}.$$

Then

$$\mathrm{tr}(ad_{(\xi,\alpha)}) = \xi^\mathrm{a}(C^\mathrm{e}_\mathrm{ae} - C^\mathrm{f}_\mathrm{af}) = 0. \quad ∎$$

We quantize the extended system $(\tilde{M}, \tilde{\omega})$ with momentum map $\tilde{J}$. Let P be a strongly admissible polarization of (M, ω) which is compatible with the constraints $J = 0$ and let $V_{\mathscr{g}^*}$ be the vertical polarization on $\mathscr{g}^* \oplus \mathscr{g} \approx T^*\mathscr{g}^*$. Then $\tilde{P} = P \oplus V_{\mathscr{g}^*}$ is a strongly admissible polarization of $(\tilde{M}, \tilde{\omega})$ which is compatible with the constraints $\tilde{J} = 0$. The leaves of $\tilde{P}$ are simply connected if those of P are. Similarly, $\tilde{L} = L \times (\mathscr{g}^* \oplus \mathscr{g})$ is a prequantization line bundle for $(\tilde{M}, \tilde{\omega})$, where L is one for (M, ω). Moreover, if the G-action lifts to $L \mid J^{-1}(0)$, then the T^*G-action lifts to $\tilde{L} \mid \tilde{J}^{-1}(0, 0)$. Thus if the quantization structures on (M, ω) satisfy the conditions set forth in the Smooth Equivalence Theorem, then these induced quantization structures on $(\tilde{M}, \tilde{\omega})$ do also.

Note that the reductions of $\tilde{J}^{-1}(0, 0)$ and $J^{-1}(0)$ are the same by (2.9), so the reduced phase space quantization is fixed.

Since $G \circledS \mathscr{g}^*$ is unimodular we may apply the Smooth Equivalence Theorem to the extended phase space quantization, obtaining a canonical isomorphism $\tilde{\mathscr{h}}_0 \to \bar{\mathscr{h}}$, where

$$\tilde{\mathscr{h}}_0 = \{\tilde{\Psi} \in \tilde{\mathscr{h}} \mid \tilde{\mathscr{Q}}\tilde{J}_{(\xi,\alpha)}[\tilde{\Psi}] = 0 \text{ for all } (\xi, \alpha) \in \mathscr{g} \circledS \mathscr{g}^*\}$$

is the subspace of $(G \circledS \mathscr{g}^*)$-invariant states in the extended quantum representation space $\tilde{\mathscr{h}}$.

Remarks. 6. The quantization structures we use on $(\tilde{M}, \tilde{\omega})$ are very natural, but are by no means the only possible ones. In particular, any real polarization on $(\tilde{M}, \tilde{\omega})$ will do, as long as it is compatible with the constraints $\tilde{J} = 0$, etc. However, other choices of $\tilde{P}$ may yield different quantizations and do not necessarily give rise to polarizations of the original phase space (M, ω). On the other hand, since

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[9] Alternately, one verifies that $\mathrm{vol}(g, \alpha) = [\det(\mathrm{Ad}_g)]\, \varepsilon(g, \alpha)$ is a bi-invariant volume element on $G \circledS \mathscr{g}^*$, where ε is the Liouville volume.

$\mathfrak{g}^* \oplus \mathfrak{g}$ is contractible, every prequantization $\tilde{L}$ of $\tilde{M}$ must be of the form $\tilde{L} = L \times (\mathfrak{g}^* \oplus \mathfrak{g})$ for some prequantization L of M.

7. In contrast to the fermionic case, the inner product we obtain on the extended Hilbert space $\tilde{\mathscr{H}}$ is positive-definite.

We now turn to a discussion of the quantum constraint conditions

$$\tilde{\mathscr{Q}}\tilde{J}_{(\xi,\alpha)}[\tilde{\Psi}] = 0. \tag{6.2}$$

First observe that these conditions are more restrictive than the original quantum constraints (1.2), since here we require the wave functions to be $(G \circledS \mathfrak{g}^*)$-invariant as opposed to merely being G-invariant. This is partly a reflection of the fact that there are more variables in the extended system than originally. Notice, however, that the G-actions on $\tilde{M}$ and M are *not* the same—$\tilde{\Phi}$ incorporates, in addition, the cotangent lift of the coadjoint action of G on $\mathfrak{g}^*$ to $\mathfrak{g}^* \oplus \mathfrak{g}$. We shall exploit this circumstance momentarily.

Now suppose in (6.2) that we set $\alpha = 0$. Then, by comparison with (2.8), we see that the resulting quantum constraints form the bosonic analogue of the *quantum BRST condition* $\mathscr{Q}\Omega[\tilde{\Psi}] = 0$ [KS, T2]. This indicates that our formalism is physically correct. Similarly, setting $\xi = 0$ in (6.2) and recalling that we are quantizing in the vertical polarization on the $\mathfrak{g}^* \oplus \mathfrak{g}$ factor, we obtain the bosonic counterpart of the requirement that the physically admissible quantum states have total ghost number zero: $\mathscr{Q}\Theta[\tilde{\Psi}] = 0$.

To gain further insight into the quantum constraints (6.2) we work out their local expressions. In Darboux coordinates $(q^i, p_i, \rho_a, \eta^a)$ adapted to $\tilde{M} = M \times (\mathfrak{g}^* \oplus \mathfrak{g})$ and $\tilde{P} = P \oplus V_{\mathfrak{g}^*}$, a local frame field for $\tilde{P}$ is $\tilde{\mathbf{f}} = (\mathbf{f}, \zeta)$, where

$$\mathbf{f} = \left(\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}\right) \quad \text{and} \quad \zeta = \left(\frac{\partial}{\partial \eta^1}, \dots, \frac{\partial}{\partial \eta^r}\right)$$

are local frame fields for P and $V_{\mathfrak{g}^*}$, respectively. Then, as in Section IV, every element $\tilde{\Psi} \in \tilde{\mathscr{h}}$ can be written

$$\tilde{\Psi} = \tilde{\psi}(q, \rho)\tilde{\lambda} \otimes \tilde{\nu},$$

where $\tilde{\nu} \circ \tilde{\mathbf{f}} = 1$ and $\tilde{\lambda}$ is a normalized locally trivializing section of $\tilde{L}$.

Taking $\xi = 0$ in (2.7), the extended version of (4.6) yields

$$\tilde{\mathscr{Q}}\tilde{J}_{(0,\alpha)}[\tilde{\Psi}] = \left(-i\alpha_a \frac{\partial \tilde{\psi}}{\partial \rho_a}\right)\tilde{\lambda} \otimes \tilde{\nu},$$

and (6.2) then implies that $\tilde{\psi} = \tilde{\psi}(q)$ only. Thus we may factor $\tilde{\Psi} = \Psi \otimes \delta$, where $\Psi \in \mathscr{h}$ and δ is the covariantly constant half-density on $\mathfrak{g}^* \oplus \mathfrak{g}$ defined by $\delta^{\#}(\zeta) = 1$. Setting $\alpha = 0$ in (2.7), this enables us to reduce the expression (4.6) for $\tilde{\mathscr{Q}}\tilde{J}_{(\xi,0)}$ to

$$\tilde{\mathscr{Q}}\tilde{J}_{(\xi,0)}[\Psi \otimes \delta] = \left\{\left(\mathscr{Q}J_\xi + \frac{i}{2}\operatorname{tr}(ad_\xi)\right)[\Psi]\right\} \otimes \delta. \tag{6.3}$$

Thus the effective content of (6.2) is[10]

$$\mathcal{Q}J_\xi[\Psi] = -\frac{i}{2}\operatorname{tr}(ad_\xi)\,\Psi. \tag{6.4}$$

We have therefore recovered the modified Dirac quantization condition (1.3). When the group G is not unimodular, we conclude that the physically admissible quantum states are *not* those which are G-invariant (with respect to the action Φ), but rather those which are quasi-invariant or, equivalently, those which are $(G \circledS \mathfrak{g}^*)$-invariant (with respect to the action $\tilde{\Phi}$). Put somewhat differently, in the nonunimodular case it is essential to take into account the (bosonic) BRST symmetry of the theory, even though this symmetry appears only when the system has been appropriately extended.

Remarks. 8. In this connection we emphasize that the original Dirac prescription (1.2) can lead to demonstrably wrong results in the presence of nonunimodular symmetries [DEGT, T1], despite claims to the contrary [1, Section VIII.3]. See also the example in the next section.

9. Referring back to Footnote 3, Eq. (6.3) shows that our quantum BRST operators $\mathcal{Q}\tilde{J}_{(\xi,0)}$ automatically incorporate the required modification to the Dirac quantization prescription. Since they are formally self-adjoint in addition, this indicates that the Kostant–Sternberg factor ordering is the correct one (as their quantum BRST charge shares these features, cf. [KS]).

10. There is no shift here in the quantum ghost number relative to the classical ghost number as there is in the fermionic BRST theory [HT, KS]. This is because the operators $\mathcal{Q}\tilde{J}_{(0,\alpha)}$ are already formally self-adjoint, and so no factor reordering (which is responsible for the shift) is necessary. Ultimately, this is a consequence of the decoupling of the ghosts/antighosts and the original variables in the extended momentum map $\tilde{J}$, cf. Remark 1.

To summarize: the Smooth Equivalence Theorem states, for unimodular systems, that reduction commutes with quantization. But the results of this section show that we can remove the unimodularity restriction, provided we either (i) perform the bosonic BRST construction classically and then quantize the extended system so obtained using the Dirac prescription (6.2), or (ii) quantize the original system using instead the modified Dirac prescription (6.4).

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[10] One must be somewhat careful in interpreting (6.4). The "correction term" on the right-hand side causes a shift in the eigenspaces of the operators $\mathcal{Q}J_\xi$ consisting of physically admissible states from eigenvalue 0 to eigenvalue $-(i/2)\operatorname{tr}(ad_\xi)$. It may seem odd that the latter eigenvalue is imaginary, since $\mathcal{Q}J_\xi$ is formally self-adjoint. However, this is not a problem since states satisfying (6.4) cannot be normalizable, and on domains including non-normalizable states the notion of self-adjointness is not defined.

VII. INDUCTION AND QUANTIZATION

We have shown that the bosonic BRST construction yields a Hamiltonian action of T^*G on the extended phase space $(\tilde{M}, \tilde{\omega})$ which, when quantized, gives a unitary representation of T^*G on the extended quantum state space $\tilde{\mathscr{h}}$. But there is another way to generate a unitary representation of T^*G: first quantize the original phase space (M, ω) and then apply unitary induction to $G \subset T^*G$. In this section we show that these two procedures give identical results. That this is so should not be surprising in view of the Bosonic BRST Theorem, since symplectic induction is the classical analogue of unitary induction. Our goal is to make this correspondence very precise; in doing so, we prove more generally that "induction commutes with quantization."

We first recall the basic facts concerning induced representations. By analogy with its classical symplectic counterpart, unitary induction manufactures a unitary representation of a Lie group K, given a unitary representation of a closed subgroup G.

So let $U: G \to \mathscr{U}(\mathscr{h})$ be a unitary representation of G on an inner product space $\mathscr{h}$, and let $\mathscr{D}^{1/2}(K)$ denote the space of all smooth half-densities on K. We then have the product representation $\check{U}$ of G on $\check{\mathscr{h}} = \mathscr{h} \otimes \mathscr{D}^{1/2}(K)$ given by

$$\check{U}_g(\Psi \otimes \delta) = U_g \Psi \otimes R_g^* \delta.$$

The inner product on $\check{\mathscr{h}}$ is

$$\langle\!\langle \check{\Psi}, \check{\Upsilon} \rangle\!\rangle^{\vee} = \int_K \langle\!\langle \Psi, \Upsilon \rangle\!\rangle (\delta \otimes \bar{\rho}), \tag{7.1}$$

where $\check{\Psi} = \Psi \otimes \delta$, $\check{\Upsilon} = \Upsilon \otimes \rho$, and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the given inner product on $\mathscr{h}$.

The induced representation space will be the subspace $\tilde{\mathscr{h}}$ of quasi-invariant states, i.e.,

$$\tilde{\mathscr{h}} = \{ \check{\Psi} \in \check{\mathscr{h}} \mid \check{U}_g \check{\Psi} = [\det(\mathrm{Ad}_g)]^{-1/2} \check{\Psi} \text{ for all } g \in G \}, \tag{7.2}$$

where the adjoint representation is taken with respect to the reducing group G. The inner product on $\tilde{\mathscr{h}}$ can be described intrinsically as follows: Fix a frame ζ for $\mathscr{g}$, and set $\zeta_K(k) = TL_k \cdot \zeta$. For $\tilde{\Psi}, \tilde{\Upsilon} \in \tilde{\mathscr{h}}$, consider the density $\sigma = \langle\!\langle \Psi, \Upsilon \rangle\!\rangle (\delta \otimes \bar{\rho})$ on K appearing in (7.1) and observe that, by virtue of (7.2), $\sigma(\zeta_K, \cdot)$ is the pullback of a density on $G \backslash K$. Integrating this latter density over $G \backslash K$ then gives the inner product on $\tilde{\mathscr{h}}$. Note that a different choice for ζ changes the inner product by a constant factor.

Now consider the representation of K on $\check{\mathscr{h}}$ defined by

$$(k, \Psi \otimes \delta) \to \Psi \otimes L_{k^{-1}}^* \delta.$$

This representation clearly commutes with $\check{U}$ and so restricts to a representation $\tilde{U}$ of K on $\tilde{\mathscr{h}}$. Finally, it is easily verified that this induced representation is unitary.

To forge the link between symplectic and unitary induction in general, we now suppose that geometric quantization applied to (M, ω, Φ, J) yields a unitary representation U of G on the quantum state space $\mathscr{h}$. Assume that the polarization P and the prequantization line bundle L on M satisfy all the conditions set forth in the Smooth Equivalence Theorem. The aim is to prove that the representation of K on the extended quantum state space obtained via the geometric quantization of the induced symplectic manifold $(\tilde{M}, \tilde{\omega}, \tilde{\Phi}, \tilde{J})$ exactly coincides with the induced representation of K on $\tilde{\mathscr{h}}$ just described.

For the most part the proof simply consists of a comparison of the two approaches. We start by applying geometric quantization to the symplectic manifold $\check{M} = M \times T^*K$. Equip T^*K with the vertical polarization V_K and define on $\check{M}$ the composite polarization $\check{P} = P \oplus V_K$. We shall use the natural identification of the space of covariantly constant half-densities relative to V_K with $\mathscr{D}^{1/2}(K)$. Similarly, take the prequantization line bundle on $\check{M}$ to be $\check{L} = L \times T^*K$. Then this quantization data $(\check{P}, \check{L})$ on $(\check{M}, \check{\omega})$ will satisfy all the hypotheses of the Smooth Equivalence Theorem as well.

The quantization of $(\check{M}, \check{\omega})$ is then clearly $\check{\mathscr{h}}$ with the inner product (7.1). Moreover, it is obvious that quantization intertwines the actions of G and K on $\check{M}$ defined in Section III with the representations of G and K on $\check{\mathscr{h}}$ defined above. Thus the first steps of the two induction procedures are identical.

We must now consider the classical reduction from $(\check{M}, \check{\omega})$ to $(\tilde{M}, \tilde{\omega})$. We first remark that, by construction, the quantization data on $(\check{M}, \check{\omega})$ induces compatible quantization data on $(\tilde{M}, \tilde{\omega})$. (When $K = T^*G$, this data is just that given in Section VI.) But the Smooth Equivalence Theorem, in conjunction with the observations at the end of the last section, shows that the quantization of $(\tilde{M}, \tilde{\omega})$ is equivalent to the quantization of $(\check{M}, \check{\omega})$ provided the *modified* Dirac conditions

$$\check{\mathscr{Q}}\check{J}_\xi[\check{\Psi}] = -\frac{i}{2}\operatorname{tr}(ad_\xi)\check{\Psi} \tag{7.3}$$

hold for all $\xi \in \mathfrak{g}$. But this is simply the infinitesimal version of the defining property (7.2) of the induced representation space $\tilde{\mathscr{h}}$! Thus (modulo several routine verifications) we have proved:

INDUCTION EQUIVALENCE THEOREM. *Suppose that the quantization data on (M, ω) satisfies all the assumptions of the Smooth Equivalence Theorem. Then there exists a canonical unitary equivalence of the quantization of the symplectically induced Hamiltonian K-space $(\tilde{M}, \tilde{\omega}, \tilde{\Phi}, \tilde{J})$ with the representation of K unitarily induced from the quantization of the original Hamiltonian G-space (M, ω, Φ, J).*

Succinctly, the diagram

$$\begin{tikzcd}
(\tilde{M}, \tilde{\omega}, \tilde{\Phi}, \tilde{J}) \arrow[r, "\text{Geometric quantization}"'] & (\tilde{\mathscr{h}}, \tilde{U}) \\
(M, \omega, \Phi, J) \arrow[u, "\text{Symplectic}", "\text{Induction}"'] \arrow[r, "\text{Geometric quantization}"'] & (\mathscr{h}, U) \arrow[u, "\text{Unitary}", "\text{Induction}"']
\end{tikzcd}$$

commutes.

Remarks. 11. Note that G need not be unimodular; the effects of nonunimodularity are automatically taken into account by means of the modified Dirac conditions (7.3).

12. We caution here against one possible source of confusion. In this section we are studying the BRST extended phase space $(\tilde{M}, \tilde{\omega}, \tilde{\Phi}, \tilde{J})$—viewed as an *unconstrained* system—and its quantization; we have shown that the former can be obtained by the reduction of an even "grander" constrained system $(\breve{M}, \breve{\omega}, \breve{\Phi}, \breve{J})$. We have also demonstrated how this quantization can be unitarily induced from the *unconstrained* quantization of (M, ω, Φ, J). But none of this has anything to do with our original problem, viz., the *constrained* quantizations of $(\tilde{M}, \tilde{\omega}, \tilde{\Phi}, \tilde{J})$ and (M, ω, Φ, J) and their relation to the quantization of the reduced phase space $(\bar{M}, \bar{\omega})$. These are separate issues altogether. In particular, the modified Dirac conditions (7.3) and (6.4) occur in entirely different settings; the former arise via unitary induction as we have seen, whereas the latter (apparently) have no connection with induced representations.

VIII. AN EXAMPLE

We illustrate here our results on BRST quantization in the specific case of a nonunimodular reducing group.

To construct the example we consider the simplest nonunimodular group: the Borel subgroup G of $SL(2, \mathbf{R})$. This is the set of all 2×2 matrices of the form

$$(a, b) := \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

with $a \in \mathbf{R}_+$ and $b \in \mathbf{R}$. Let

$$\xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

be the standard basis for $\mathscr{g}$; note that

$$\operatorname{tr}(ad_{\xi_1}) = 2 \quad \text{and} \quad \operatorname{tr}(ad_{\xi_2}) = 0.$$

We let this group act on $\dot{\mathbf{R}}^3 \times \mathbf{R}$ by

$$(a, b) \cdot (q, t) = (aq, a^2 t + ab).$$

The cotangent lift of this action to $M = T^*(\dot{\mathbf{R}}^3 \times \mathbf{R})$ is given by

$$(a, b) \cdot (q, p, t, e) = (aq, a^{-1}p, a^2 t + ab, a^{-2} e).$$

With respect to the dual basis for $\mathscr{g}^*$, the momentum map is

$$J(q, p, t, e) = (p \cdot q + 2et, e).$$

An elementary calculation shows that the reduced space $G\backslash J^{-1}(0)$ is symplectomorphic to T^*S^2 with its canonical symplectic structure.

We now quantize M in the vertical polarization and with the trivial prequantization line bundle. This yields in the usual way the quantum Hilbert space $\mathscr{H} = L^2(\dot{\mathbf{R}}^3 \times \mathbf{R})$, elements of which we identify with functions $\psi(q, t)$ by the choice of the Lebesgue measure. In this representation the quantized operators read:

$$\mathscr{Q}J_{\xi_1} = -i\left(q\cdot\nabla + 2t\frac{\partial}{\partial t} + \frac{5}{2}\right) \quad \text{and} \quad \mathscr{Q}J_{\xi_2} = -i\frac{\partial}{\partial t}.$$

Imposing the *modified* Dirac quantization conditions (1.3) gives $\psi = \psi(q)$ along with

$$q\cdot\nabla\psi = -\tfrac{3}{2}\psi. \tag{8.1}$$

Going over to polar coordinates on $\dot{\mathbf{R}}^3$, this implies that

$$\psi(q) = r^{-3/2}\phi(\theta, \varphi). \tag{8.2}$$

Thus the space $\mathscr{H}_*$ of quasi-invariant states may be identified with $L^2(S^2)$; the Smooth Equivalence Theorem tells us that the induced inner product is actually given by integration over S^2 with respect to the standard volume.

Let us now investigate what would happen if we quantize the system using the *uncorrected* Dirac conditions (1.2). We find that (8.1) is replaced by

$$q\cdot\nabla\psi = -\tfrac{5}{2}\psi$$

and thus in this case

$$\psi(q) = r^{-5/2}\phi(\theta, \varphi). \tag{8.3}$$

Again we may identify the space $\mathscr{H}_0$ of G-invariant states with functions on S^2, but we no longer have a theorem that tells us how to put an inner product on this space.

We now show that the apparently innocuous difference between these two quantizations is crucial and that the unmodified Dirac prescription is not consistent with the quantization of the reduced phase space. To do this we consider the observable

$$L = \|q\| \, \langle v, p\rangle$$

on M, where v is any fixed nonzero vector in $\mathbf{R}^3$. It is clearly G-invariant and hence induces an observable $\bar{L}$ on the reduced space T^*S^2. As L is linear in the momenta, it is quantizable and we obtain

$$\mathscr{Q}L = -i\left(\|q\|\, v\cdot\nabla + \frac{1}{2}\frac{v\cdot q}{\|q\|}\right) \tag{8.4}$$

as an operator on $L^2(\dot{\mathbf{R}}^3 \times \mathbf{R})$. On the other hand, we may quantize the reduced space $\bar{M} \approx T^*S^2$ directly; as an operator on $\bar{\mathscr{H}} \approx L^2(S^2)$,

$$\bar{\mathscr{Q}}\bar{L} = -i\left(v_\theta \frac{\partial}{\partial\theta} + v_\varphi(\csc\theta)\frac{\partial}{\partial\varphi} - \frac{1}{2}v_r\right). \tag{8.5}$$

Then a calculation establishes that the isomorphism $\mathscr{H}_* \approx \bar{\mathscr{H}}$ given by (8.2) intertwines the operators (8.4) and (8.5) (as it must according to Theorem 5). But the correspondence $\mathscr{H}_0 \approx \bar{\mathscr{H}}$ defined by (8.3) does *not*, due to the different dependence on r.

Remarks. 13. We have been content here to merely illustrate the essential role played by the modified Dirac prescription. (It would be overwhelming to exhibit the full paraphernalia of the bosonic BRST machinery in this case.) To see this machinery in action, we refer the reader to the treatment of the pseudo-rigid body given in [DEGT].

14. If so desired, one can give this example a physical flavor by viewing the reduced phase space as that of a spherical pendulum. Although we will not pursue this, we point out that the setup here can be used to provide a nice interpretation of the "homogeneity trick" employed by Guillemin and Uribé in their paper [GU] on the quantum mechanical spherical pendulum.

IX. DISCUSSION

There has been much interest in the Dirac quantization program for constrained systems. The motivation is that the reduced phase space quantization is often hard to describe (the reduced space itself may not even exist in any reasonable sense), since it requires taking a quotient; whereas quantum reduction essentially consists of computing subspaces of (rigged) Hilbert space and so is in principle easier. But regardless of how one proceeds, one needs theorems to the effect that first quantizing and then reducing on the quantum level gives the same results as first reducing on the classical level and then quantizing. Our results here are of this nature.

While certainly an improvement over previous work in that (i) the unimodularity requirement in [G, Sn2] has been lifted, and (ii) one is no longer confined to either the cotangent or Kähler categories of [G, GuS1], our theorems are in some ways still far from optimal. To obtain hard results we found it necessary, for instance, to place severe restrictions on the group action and the choice of polarization. It is not at all apparent how one might relax these restrictions to any significant extent.

Since our approach to the quantization/reduction problem is closely tied to the bosonic BRST formalism, it is natural to try to ascertain the circumstances under which the latter will be well defined. There are two main avenues of generalization here. One is to consider non-free actions. In this case one has some idea as to how the (fermionic) BRST formalism goes [FHST]; one must introduce ghosts of ghosts of ... depending upon the degree of redundancy between the components of the momentum map. Is there a purely bosonic analogue of this procedure in symplectic geometry? Is symplectic induction the appropriate construction in this context as well? A second generalization is to consider the case when the constrained system is defined by the vanishing of globally defined functionally independent first class constraints which however do not arise from a group action. Here again the fermionic theory is reasonably well understood, although it is complicated by the fact that one now has structure functions instead of constants [BM, H1, H2, HT, St]. A possible first step would be to try to extend the bosonic BRST framework to the cotangent bundle setting and replace the group action on the configuration space by a foliation. Some results in this direction have recently been obtained in [T2, T3].

But the bosonic BRST formalism aside, one can ask: what is the counterpart of the modified Dirac prescription in these types of situations? A study of what happens when all isotropy groups are conjugated to one another can be found in [T1].

Another major problem is to obtain results valid for more general types of polarizations. One expects that there should be versions of the Smooth Equivalence Theorem for a variety of polarizations (viz., those not satisfying the compatibility condition (5.1)), but there are no results as yet other than [GuS1]. Other difficulties are easier to circumvent. For example, it should be straightforward to extend these results to the case when $J=\mu$, provided $\mu \in \mathfrak{g}^*$ is Ad*-invariant (so that $J^{-1}(\mu)$ is coisotropic in M or, equivalently, the constraints $J_\xi = \langle \mu, \xi \rangle$ are first class); see [G] in this regard. If $\mu \in \mathfrak{g}^*$ is not invariant, then our constructions may still be applied provided we make use of a standard trick, cf. Section 26 of [GuS2].

A further conundrum concerns one of the tenets of the BRST/BFV philosophy: that one should be able to dispense with the reduced phase space altogether. But we are unable to avoid having to implement the classical reduction—in fact, we are quite unable to do *without* the reduced phase space—because otherwise there is in general no way to construct the inner product on the reduced quantum state space.

So there remains much interesting and hard work to be done. But even in the absence of resolutions of these many problems, we hope it is apparent that, contrary to some popular belief (e.g., [L]), it *is* profitable to study constrained systems in purely symplectic terms, either super or not.

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