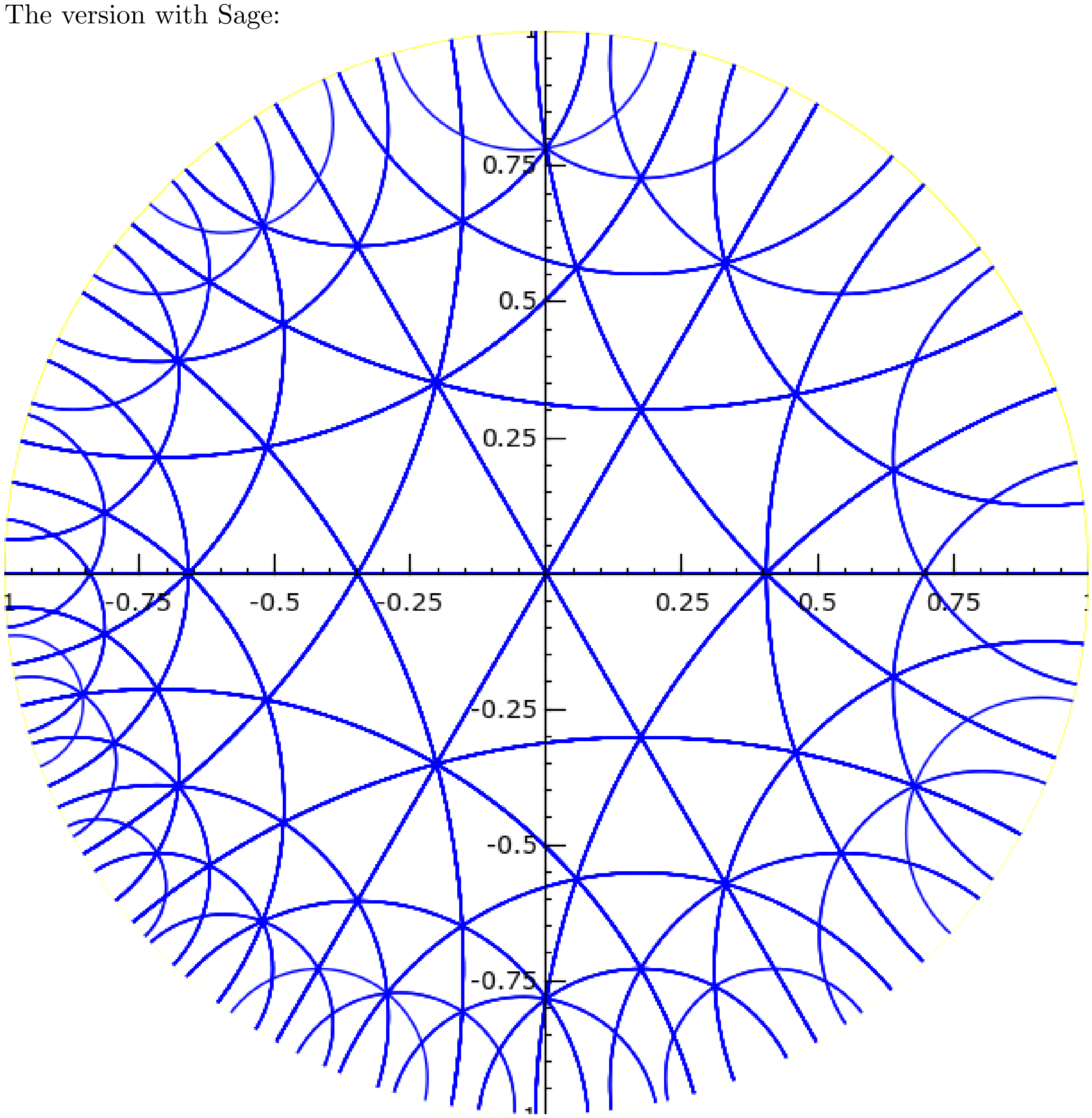
HYPERBOLIC TRIANGLE GROUPS

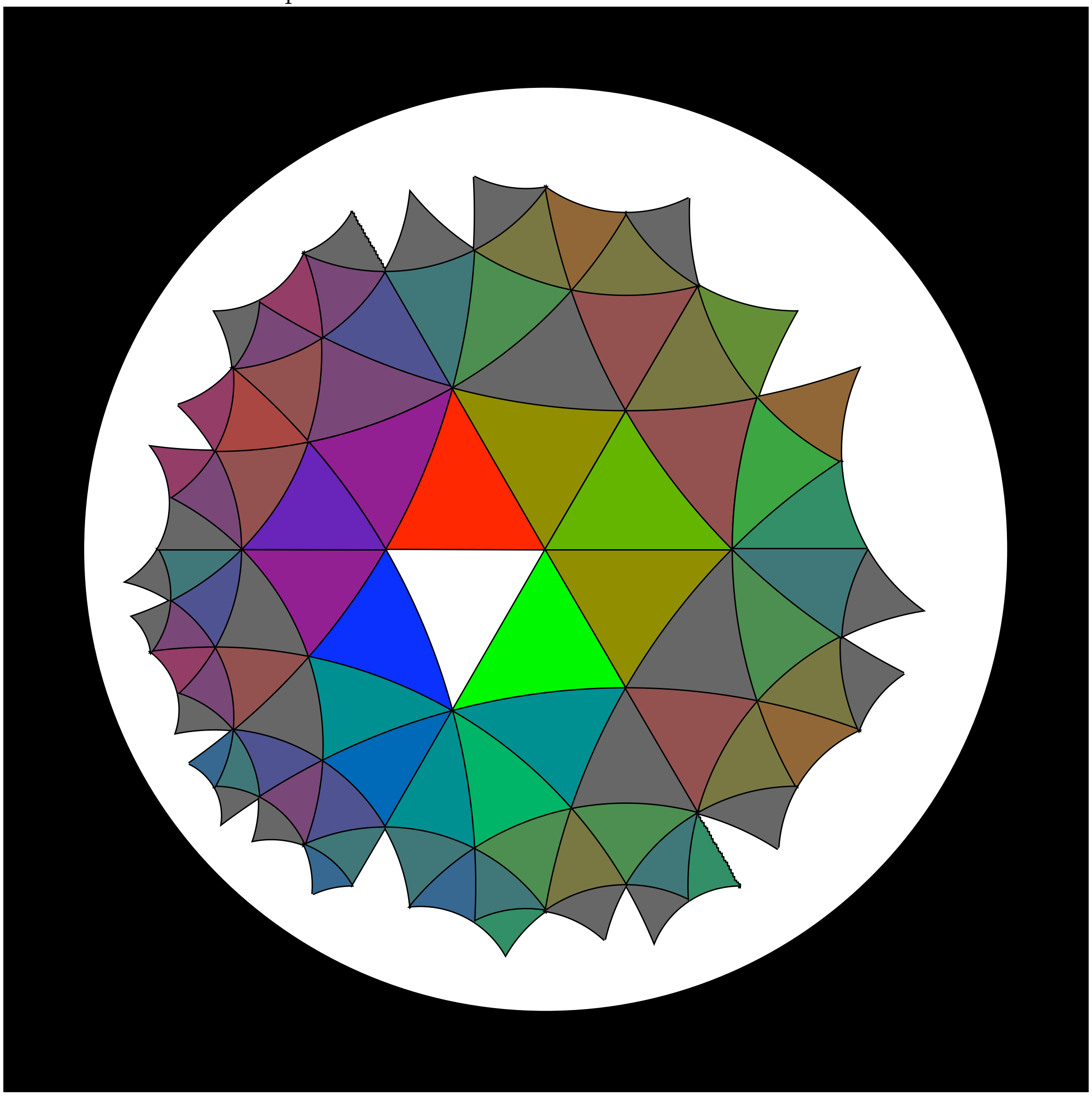
VÉRONIQUE SANGIN

The most interesting thing about triangle groups is that they can be graphically represented. At Sage Days 9 I learned how to code such graphics. First of all, I drew the hyperbolic circle with Sage, and then to fill the triangle with color, I used Piscript. Here are the results:

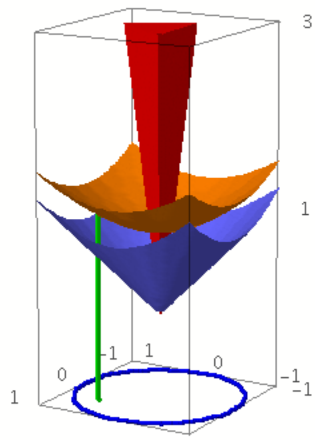
The version with Sage:



The version with Piscript:



Of course I don't show all the possible triangles - because there is an infinit number of them; I show only about 70 of them.



The mathematical part of this subject is interesting as well. We have an abelian group W with the generators $e1, e2, e3$. Which have the following property :

$$e_1 * e_1 = e_2 * e_2 = e_3 * e_3 = (e_1 * e_2)^{m_{12}} = (e_1 * e_3)^{m_{13}} = (e_2 * e_3)^{m_{23}} = e_W$$

We have also a bilinear form: $B(ei, ej) = -\cos(\frac{\pi}{m_{ij}})$

And the Tits Cone is $\{w|B(e_1, w) > 0, B(e_2, w) > 0, B(e3, w) > 0\}$. It is the red cone in the graph.

Which means that the Tits Cone is cone delimited by 3 hyperplanes, say H_1, H_2, H_3.

A reflexion of x by Hi is a matrix of the form : $s_i(x) = x - 2B(x, e_i)e_i$

We are interresseted to see the impact of the reflexion of each hyperplane on the Tits Cone. Then the problem of finding what triangle represents what product of reflexion of the Tits Cone is reduced to a product of matrix $s_1, s_2, s_3$and then to apply the Minkowski -Poincarré transformation. This application takes a point on the hyperboloid (in yellow) and maps it to the unit disk (it is showed with the green line on the previous graphic) This transformation have the great property that it keeps the angle the same.

I considered the case where $m_{12} = m_{13} = 3, m_{23} = 4$, but we could make a similar image - hyperboloic circle - as long as $1/m_{12} + 1/m_{13} + 1/m_{23} < 1$

Finaly, the color in the piscript version isn't only esthetic, I wanted to show the proportion of generator that each has. The generators s_1, s_2, s_3 are filled with the colors red,green,blue and the identity element in white.