Complete Statistical Mechanics and Emergence of Large Scale Coherent Structure

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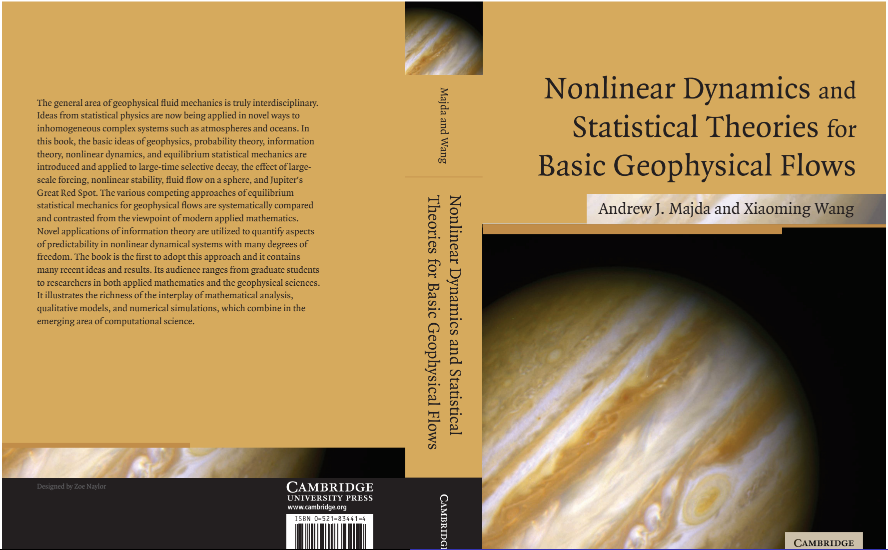
Stochastic and Probabilistic Methods in Ocean-Atmosphere Dynamics, Victoria, July 2008

Main reference



Liouville property

- Approximate PDE by ODE system.
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$$\frac{d\vec{X}}{dt} = \vec{F}(\vec{X}),\ \vec{X} \in \mathcal{R}^N,\ \vec{F} = (F_1, \cdots, F_N), \quad N \gg 1, \quad \vec{X}|_{t=0} = \vec{X}_0.$$

  $S(t)$: solution semi-group.
- Liouville property

$$\nabla_{\vec{X}} \vec{F} = \sum_{j=1}^{N} \frac{\partial F_j}{\partial X_j} = 0.$$

  Example: Hamiltonian system.
- Liouville equation and its solution

$$\frac{\partial p}{\partial t} + \vec{F} \cdot \nabla_{\vec{X}} p = 0$$

$$p(\vec{X}, t) = p_0(S^{-1}(t)(\vec{X}))$$

- Conservation

$$\frac{d}{dt} \int_{\mathcal{R}^N} G(p(\vec{X}, t)) = 0$$

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Maximum entropy principle applied to ODE

- Conserved quantities $E_l(\vec{X}(t)) = E_l(\vec{X}_0), \quad 1 \le l \le L$
- Statistical version of the conserved quantities $\overline{E}_l = \langle E_l \rangle_p \equiv \int_{\mathcal{R}^N} E_l(\vec{X}) p(\vec{X})\, d\vec{X}, \quad 1 \le l \le L.$
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$$\begin{aligned} \mathcal{S}(p^*) &= \max_{p \in \mathcal{C}} \mathcal{S}(p), \quad \mathcal{S}(p) = -\int_{\mathcal{R}^N} p(\vec{X}) \ln p(\vec{X})\, d\vec{X}, \\ \mathcal{C} &= \left\{ p(\vec{X}) \ge 0, \int_{\mathcal{R}^N} p(\vec{X})\, d\vec{X} = 1, \langle E_l \rangle_p = \overline{E}_l, 1 \le l \le L \right\} \end{aligned}$$

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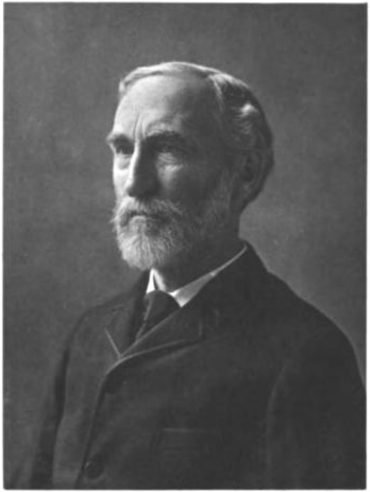
Josiah Willard Gibbs

Figure: Josiah Willard Gibbs, 1839-1903



Remarks on the maximum entropy principle

- Entropy is conserved in the deterministic case.
- $$\frac{d\vec{X}}{dt} = \vec{F}(\vec{X}) + \epsilon\frac{d\vec{W}}{dt}$$
- Fokker-Planck equation (Kolmogorov, Smoluchowski)
$$\frac{\partial p}{\partial t} + \vec{F}\cdot\nabla_{\vec{X}} p - \frac{\epsilon^2}{2}\Delta_{\vec{X}} p = 0$$
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Adriaan Fokker, Max Planck, Andrey Kolmogorov, Marian Smoluchowski

Figure: Fokker, 1887-1972

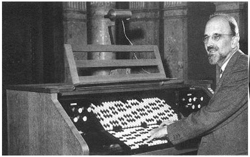

Figure: Planck, 1858-1947

Figure: Kolmogorov, 1903-1987

Figure: Smoluchowski, 1872-1917

Barotropic quasi-geostrophic equation with topography

- Barotropic quasi-geostrophic equation

$$\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q = 0, \quad q = \Delta\psi + h$$

q: potential vorticity, ψ: stream-function, h: bottom topography, $\Omega = (0, 2\pi) \times (0, 2\pi)$, per. b.c.

- Conserved quantities: kinetic energy E and total enstrophy $\mathcal{E}$

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Application to barotropic QG: general strategy

1. Finite dimensional (Galerkin) truncation (approximate PDE by ODE)
2. Verify (i) the conservation of energy enstrophy and (ii) the Liouville property
3. Apply equilibrium statistical mechanics theory to the truncated ODE.
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2. Verify (i) the conservation of energy enstrophy and (ii) the Liouville property
3. Apply equilibrium statistical mechanics theory to the truncated ODE.
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Jule Charney, Robert Kraichnan

Figure: Jule Charney, 1917-1981

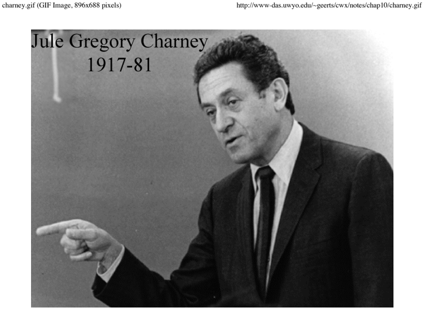
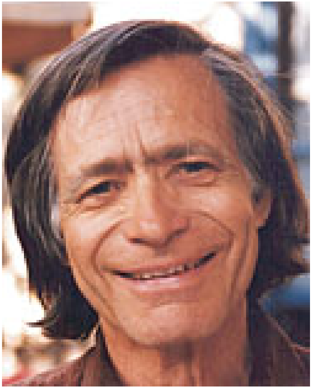


Figure: Robert Kraichnan, 1928-2008



Truncated QG

- Fourier basis: $B_\Lambda = \left\{ \exp\left(i\vec{k}\cdot\vec{x} \right) \mid 1 \le |\vec{k}|^2 \le \Lambda \right\}$.
-

$$
\begin{aligned}
\psi_\Lambda &\equiv \sum_{1\le|\vec{k}|^2\le\Lambda} \hat{\psi}_{\vec{k}}(t)e^{i\vec{x}\cdot\vec{k}} = -\sum_{1\le|\vec{k}|^2\le\Lambda} \frac{1}{|\vec{k}|^2}\hat{\omega}_{\vec{k}}(t)e^{i\vec{x}\cdot\vec{k}}, \\
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\end{aligned}
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-

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\frac{\partial\omega_\Lambda}{\partial t} + P_\Lambda\left(\nabla^\perp\psi_\Lambda\cdot\nabla(\omega_\Lambda + h_\Lambda)\right)
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\frac{d\hat{\omega}_{\vec{k}}}{dt} - \sum_{\substack{\vec{l}+\vec{m}=\vec{k},\\ |\vec{l}|^2\le\Lambda, |\vec{m}|^2\le\Lambda}} \frac{\vec{l}^\perp\cdot\vec{m}}{|\vec{l}|^2}\hat{\omega}_{\vec{l}}(\hat{\omega}_{\vec{m}} + \hat{h}_{\vec{m}}) = 0.
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Conservation of truncated energy and enstrophy and Liouville's property

- Truncated energy and enstrophy

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- Liouville property (detailed)

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F_j(\vec{X}) = F_j(X_1, \cdots, X_{j-1}, X_{j+1}, \cdots, X_{2M}),
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Prediction of the truncated system

- Gibbs measure

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\mathcal{G}_{\alpha,\mu,\Lambda} &= \mathcal{Z}^{-1}_{\alpha,\mu,\Lambda}\exp\Big(-\frac{\alpha}{2}\sum_{1\le|\vec{k}|^2\le\Lambda}|-|\vec{k}|^2\hat{\psi}_{\vec{k}}+\hat{h}_{\vec{k}}|^2\Big)-\frac{\theta}{2}\sum_{1\le|\vec{k}|^2\le\Lambda}|\vec{k}|^2|\hat{\psi}_{\vec{k}}\\
&= \mathcal{Z}_{\alpha,\mu,\Lambda}\exp\Big(-\frac{\alpha}{2}\sum_{1\le|\vec{k}|^2\le\Lambda}|\vec{k}|^2(|\vec{k}|^2+\mu)(\hat{\psi}_{\vec{k}}-\overline{\hat{\psi}}_{\vec{k}})^2\Big)\Big)\\
&= \Pi_{j=1}^{2M}\mathcal{G}^{j}_{\alpha,\mu,\Lambda}(X_j)
\end{aligned}$$

α, θ Lagrange multipliers for enstrophy and energy, $\mu = \frac{\theta}{\alpha}$.

- Mean state and equation

$$\overline{\hat{\psi}}_{\vec{k}} = \frac{\hat{h}_{\vec{k}}}{\mu+|\vec{k}|^2}, \quad \overline{\psi}_\Lambda(\vec{x},t) = \sum_{1\le|\vec{k}|^2\le\Lambda}\overline{\hat{\psi}}_{\vec{k}}e^{i\vec{x}\cdot\vec{k}}$$

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Continuum limit

- Asymptotic constraints

$$\lim_{\Lambda\to\infty}\langle E_\Lambda\rangle_{\mathcal{G}} = E_0, \lim_{\Lambda\to\infty}\langle \mathcal{E}_\Lambda\rangle_{\mathcal{G}} = \mathcal{E}_0.$$

-

$$\begin{aligned}\langle E_\Lambda\rangle_{\mathcal{G}} &= \overline{E}_\Lambda + E'_\Lambda = \frac{1}{2}\sum_{1\le|\vec{k}|^2\le\Lambda}\frac{|\vec{k}|^2|\hat{h}_{\vec{k}}|^2}{(\mu+|\vec{k}|^2)^2} + \frac{\alpha^{-1}}{2}\sum_{1\le|\vec{k}|^2\le\Lambda}\frac{1}{\mu+|\vec{k}|^2}\\ \langle \mathcal{E}_\Lambda\rangle_{\mathcal{G}} &= \overline{\mathcal{E}}_\Lambda + \mathcal{E}'_\Lambda = \frac{1}{2}\sum_{1\le|\vec{k}|^2\le\Lambda}\frac{\mu^2|\hat{h}_{\vec{k}}|^2}{(\mu+|\vec{k}|^2)^2} + \frac{\alpha^{-1}}{2}\sum_{1\le|\vec{k}|^2\le\Lambda}\frac{|\vec{k}|^2}{\mu+|\vec{k}|^2}\end{aligned}$$

-

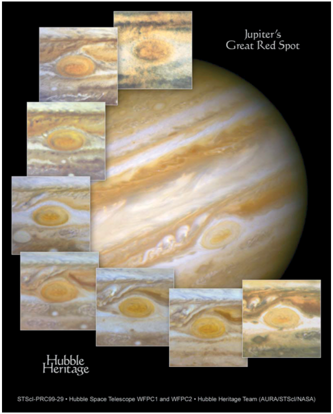
$$\mu_\Lambda \to \mu_\infty, \alpha_\Lambda \to \infty.$$

-

$$\Delta\overline{\psi}_{\mu_\infty} + h = \mu_\infty\overline{\psi}_{\mu_\infty}$$

$\mu_\infty \in (-1,\infty)$: $E(\overline{\psi}_{\mu_\infty}) = E_0$.

Large coherent structure



- Understand the emergence and persistence of such large scale coherent structure
- Prediction

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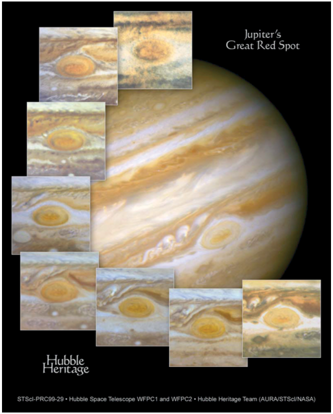

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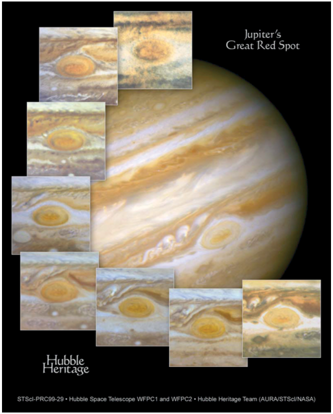

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Mathematical model

One layer model (Two dimensional fluid system for potential vorticity)

$$\begin{aligned}
\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q &= \mathcal{D}(-\Delta)\psi + \mathcal{F}, \\
q &= \Delta\psi + \beta y - F\psi + h \\
\mathcal{D}(-\Delta)\psi &= \sum_{j\geq 1} d_j(-\Delta)^j\psi
\end{aligned}$$

d_1: Ekman damping, d_2:Newtonian viscosity, $d_j, j \geq 3$: hyper-viscosity

Equilibrium/empirical statistical mechanics

- Undamped/unforced setting customary
- Information theoretical approach: maximize Shannon entropy with given information
- Conserved quantity becomes constraints on ρ
- **Mean field equation**

$$\bar{q} = \mathcal{G}(\bar{\psi})$$

- Most of them are stable under appropriate assumptions
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Dynamical approach / non-equilibrium

- Forcing on the largest scale (Yudovitch, Marchioro, Constantin-Foias-Temam)
- Selective decay (decaying flow) (Foias-Saut, Majda-Shim-W., Montgomery, McWilliam etc)
- Large scale structure: ground energy shell

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- unresolved small scale in forcing (small scale convection on Jupiter weather layer, storms for the oceans' mixing layer)
- random small scale forcing (in Jupiter's case: predominantly positive)
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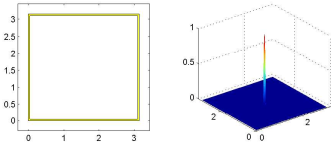
Simple model

Two dimensional Navier-Stokes equation (vorticity-stream function)

$$
\begin{aligned}
\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q &= \nu \Delta q + \mathcal{F}, \\
\Delta \psi &= q, \\
q|_{t=0} &= q_0 (\geq 0) \\
\psi = q &= 0, \text{on } \partial Q
\end{aligned}
$$

$Q = [0, \pi] \times [0, \pi]$

Impulse(kick) random small scale forcing



- $$\mathcal{F} = \sum_{j=1}^{\infty} \delta(t - j\Delta t) A \omega_r(\mathbf{x} - \mathbf{x}_j)$$
- $$\omega_r(\mathbf{x}) = \begin{cases} \left(1 - |\mathbf{x} - \mathbf{x}_j|^2 / r^2\right)^2, \\ 0, \end{cases}$$
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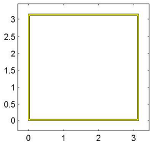

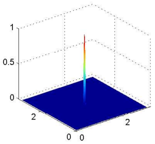

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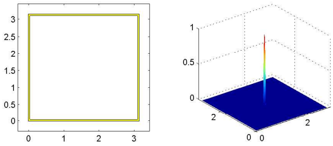

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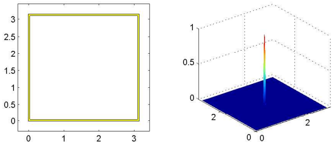

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Numerical results

Figure: Contour

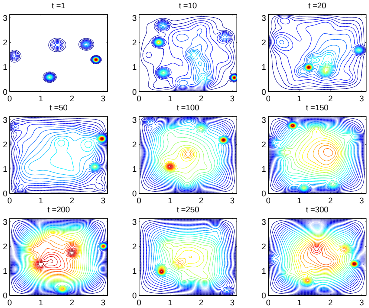
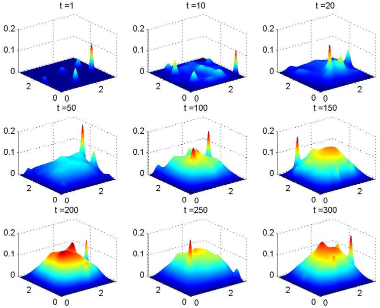


Figure: Vorticity



Numerical results (correlation, Dirichlet quotient, energy)

Figure: Correlation and D quotient

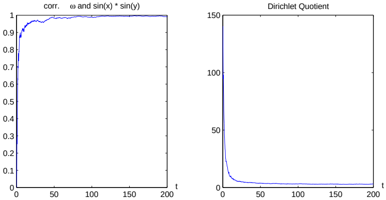
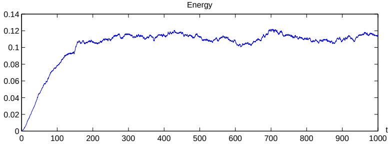


Figure: Energy



Stochastic approach

- Decomposition of the kick as mean plus fluctuation

$$\omega_r = \bar{\omega}_r + \omega_r', \quad \bar{\omega}_r = \mathbb{E}\omega_r$$

- cumulative forcing effect (deterministic part)

$$\lfloor \frac{t}{\Delta t} \rfloor A \bar{\omega}_r$$

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- The continuous equation

$$\begin{aligned} \frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q &= \nu \Delta q + c_r \bar{\omega}_r + c_r \epsilon \frac{dG}{dt}, \\ q &= \Delta \psi \end{aligned}$$

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Heuristic limit

- heuristic limit as $\epsilon \to 0$

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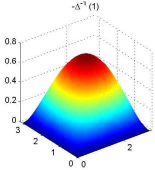

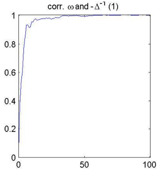

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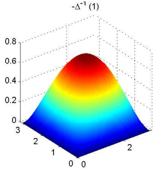

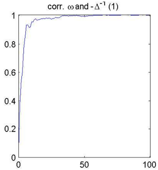

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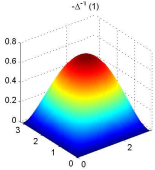

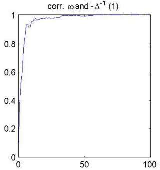

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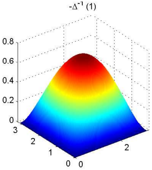

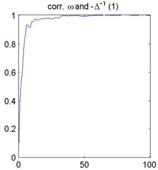

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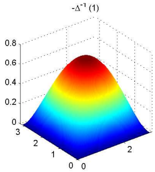

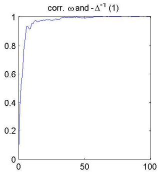

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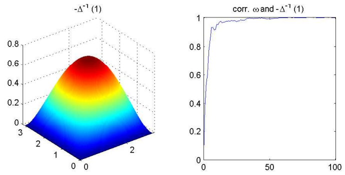

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Rate of convergence (Majda-W., 2006)

- **Theorem**

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$$\lim_{\epsilon \longrightarrow 0} dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) = 0, \ a.s.$$

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Commutative diagram (Majda-W., 2006)

Theorem

$$
\begin{array}{ccc}
q(t,\omega) & \longrightarrow & q_\infty(\omega) \\
\Downarrow & & \Downarrow \\
q^0(t) & \longrightarrow & q^0_\infty
\end{array}
$$

Uniqueness of invariant measure

- Invariant measure $\mu_0(du)$

$$\int_H F(u)\mu_0(du) = \int_H \mathbb{E}F(\varphi(t,\omega,u))\mu_0(du)$$

- Invariant measure is unique for small data
- $q' = q^2 - q^1$

$$\frac{d}{dt}\|q'\|^2 \leq (-2\nu - c\frac{\|\nabla q^1\|^2}{\nu})\|q'\|^2$$

- Main ingredient: contraction, Ito+Burkholder (with mean forcing and dependent Brownian motion)
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Summary

- Random small scale bombardments could induce large coherent structure
- Large structures well predicted by equilibrium statistical theory
- Random bombardment could alter sign as long as the mean is not zero
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