Invariant measures for non-minimal stationary Bratteli diagrams and aperiodic substitutions

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Bratteli diagrams

DEFINITION. A *Bratteli diagram* is an infinite graph $B = (V, E)$ with vertex set $V = \bigcup_{i \geq 0} V_i$ and edge set $E = \bigcup_{i \geq 1} E_i$:

1. $V_0 = \{v_0\}$ is a single point;
2. V_i and E_i are finite sets;
3. edges E_i connect V_i to V_{i+1}:
   there exist a range map r and a source map s from E to V such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \emptyset$, $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.
4. $F_n =$ incidence matrix of size $|V_{n+1}| \times |V_n|$.
5. B is *stationary* if $F_n = F_1$ for $n \geq 2$.

Ordered Bratteli diagrams, Vershik maps

$X_B =$ the space of infinite paths in B from the root v_0; topology defined by cylinder sets.

$\mathcal{R} =$ cofinal equivalence relation on B: two paths are cofinal if they have the same tail.

DEFINITION. A Bratteli diagram $B = (V, E)$ is *ordered* if every set $r^{-1}(v)$ is linearly ordered.

- This defines a lexicographic order on X_B.
- The *Vershik map*, or *adic transformation*, ϕ_B, is the immediate successor transformation, defined on $X_B \setminus X_B^{\max}$. The inverse ϕ_B^{-1} is defined on $X_B \setminus X_B^{\min}$.

History (very incomplete...)

- Bratteli (1972): AF-algebras
- Elliott (1976), Effros, Handelman, Shen (1979-1981): dimension groups
- Vershik (1981-1982): 'adic transformation model theorem' in measurable dynamics
- Herman, Putnam, Skau (1992): 'Bratteli-Vershik model theorem' in topological dynamics
- Karl Petersen, students, and co-authors (X. Méla, S. Bailey Frick, I. Salama, M. Keane,...): Pascal adic, Euler adic, and generalizations

Invariant measures for Bratteli diagrams

- $M_1(\mathcal{R}) = \{$invariant probability measures for the cofinal equivalence relation $\mathcal{R}$ on $X_B\}$

- $M_1(\mathcal{R})$ are called *central measures* by Kerov and Vershik

- $M_1(\mathcal{R}) =$ $\{$normalized states on the dimension group associated with $B\}$

Invariant measures for Bratteli diagrams (cont.)

- B is a *simple* diagram if it has only nonzero entries in the incidence matrices at each level, after 'telescoping'
- In the stationary case, 'simple' is equivalent to F being primitive.
- As far as we aware, systematic study of stationary non-simple diagrams has not been done till now. Although...
- Handelman (1982) considered the case of 2 irreducible components
- What is the connection with the work of Cuntz, Krieger, D. Huang, and others on reducible shifts of finite type?

Invariant measures for stationary diagrams

- B — stationary Bratteli diagram, N vertices on each level
- F — incidence matrix, $A = F^T$
- $E(v_0, w)$ — the set of paths from v_0 to w, $h_w^{(n)} = |E(v_0, w)|$
- $X_w^{(n)}(\overline{e})$ — cylinder set, where $w \in V_n$, $\overline{e} \in E(v_0, w)$

Invariant measures for stationary diagrams (cont.)

Theorem. *Let* μ *be a Borel probability* $\mathcal{R}$*-invariant measure on* X_B*. Set*

$$\overline{p}^{(n)} = (\mu(X_w^{(n)}(\overline{e})))_{w \in V_n} \quad \textit{for any } \overline{e} \in E(v_0, w)$$

Then

(1) $\overline{p}^{(n)} = A\overline{p}^{(n+1)}, \ \ n \geq 1;$

(2) $\overline{p}^{(n)} \in core(A) := \bigcap_{k \geq 1} A^k(\mathbb{R}_+^N), \ n \geq 1;$

(3) $\sum_{w \in V_n} h_w^{(n)} p_w^{(n)} = 1, \ n \geq 1.$

Conversely, if $\{\overline{p}^{(n)}\} \in \mathbb{R}_+^N$ *satisfy (1)–(3), then there exists an invariant probability* μ *on* X_B *with* $p_w^{(n)} = \mu(X_w^{(n)}(\overline{e}))$*.*

Core of a non-negative matrix

$$core(A) := \bigcap_{k\geq 1} A^k(\mathbb{R}_+^N)$$

Assume that the irreducible components of A are primitive (mixing, aperiodic); this can be achieved by raising A to a power (telescoping the diagram).

Theorem. [Frobenius 1912, Victory 1985, Schneider 1986] *Suppose A is non-negative, with a positive spectral radius, with primitive irreducible components. Then*

$core(A)$ is a simplicial cone with extreme rays corresponding to non-negative eigenvectors of A;

Core of a non-negative matrix (cont.)

- For an irreducible component A_α let $\lambda_\alpha = \rho(A_\alpha)$
- Write $\beta \succ \alpha$ if component β "has access to" α (communicates with α), but $\alpha \neq \beta$
- irreducible component α is *distinguished* if

$$\lambda_\alpha > \lambda_\beta, \;\; \forall \, \beta \succ \alpha$$

Frobenius Theorem continued: *non-negative eigenvectors of A are in 1-1 correspondence with distinguished irreducible components.*

Invariant measures for stationary diagrams (cont.)

Corollary.

1. If F has aperiodic irreducible components and $\rho(F) > 0$, then the ergodic invariant probabilities measures for $(X_B, \mathcal{R})$ are in 1-1 correspondence with the distinguished irreducible components.
2. Moreover, if α is such a component, then the corresponding measure μ_α is supported on the set of paths which are cofinal to paths in that component.

Examples

$$\textbf{(1)} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \quad \text{with all } a_{ij} > 0.$$

If $a_{22} > a_{11}$, then there are two ergodic measures, otherwise, there is a unique invariant measure, supported on the minimal component.

$$\textbf{(2)} \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \quad \text{with all } a_{ij} > 0.$$

Case 1: $a_{11} < a_{22} < a_{33}$, three ergodic measures;
Case 2: $a_{11} < a_{22} \geq a_{33}$ or $a_{22} \leq a_{11} < a_{33}$, two ergodic measures;
Case 3: $a_{11} \geq a_{22} \geq a_{33}$, one ergodic measure.

Another viewpoint

- Suppose F has aperiodic irreducible components and $\rho(F) > 0$.
- Let α be an irreducible component, with a positive spectral radius, and B_α is the corresponding simple sub-diagram, $X_\alpha = X_{B_\alpha}$.
- There is a unique invariant probability ν_α on X_α.
- Extend the measure to the set of paths cofinal to those in X_α (in a unique way, if it is invariant).
- The resulting measure is finite iff α is a distinguished component for $A = F^T$; otherwise, it is σ-finite. It is non-atomic if $\lambda_\alpha > 1$.

σ-finite invariant measures

Theorem. *Suppose F has aperiodic irreducible components, each with a spectral radius greater than one. Then every ergodic σ-finite invariant measure for X_B, which is positive and finite on some cylinder set, arises from a non-distinguished component of $A = F^T$.*

σ-finite invariant measures (cont.)

<u>*Proof Sketch.*</u> We can equip the diagram with *any* stationary order. Then the *Vershik map* is defined on the complement of a countable set (the half-orbits of the maximal and minimal paths).

The Vershik map on X_α induces a primitive (integral) transformation on a subset of X_B. The induced measure is finite iff the time of first return is integrable, which happens iff $\rho(A_\alpha)$ is greater than the spectral radii of the components which have access to it.

Program: from minimal to aperiodic

Cantor minimal systems

- Herman, Putnam, Skau (1992): topological Bratteli-Vershik model
- Giordano, Putnam, Skau (1995): orbit equivalence
- Forrest (1997), Durand, Host, Skau (1999): stationary **Cantor minimal** Bratteli-Vershik maps are either odometers or primitive substitutions

Aperiodic Cantor systems

- Bezuglyi, Dooley, Medynets (2005): Rokhlin Lemma for homeomorphisms of a Cantor set
- Medynets (2006): topological Bratteli-Vershik model
- Bezuglyi, Medynets, Kwiatkowksi (2008): aperiodic substitution systems and their Bratteli-Vershik models

Substitutions

- **Alphabet:** $\mathcal{A} = \{1, \ldots, m\}, \quad \mathcal{A}^+ = \cup_{n \geq 1} \mathcal{A}^n$
- **Substitution** (=non-erasing morphism) $\zeta : \mathcal{A} \to \mathcal{A}^+$
- **Standing assumption** $(*)$**:** $|\zeta^n(\alpha)| \to \infty$, as $n \to \infty$, for all $\alpha \in \mathcal{A}$.
- **Substitution space:**

$$X_\zeta = \{x \in \mathcal{A}^{\mathbb{Z}} : \text{ every subword of } x \text{ occurs in some } \zeta^n(\alpha)\}.$$

- **Substitution Dynamical System:** (X_ζ, σ), where σ is the left shift.

Substitution Matrix, Primitive Substitutions

- **Substitution matrix:** $M_\zeta(i,j) =$ the number of times the letter i occurs in $\zeta(j)$.
- **Substitution ζ is primitive** if M_ζ is primitive, i.e. $M_\zeta^k > 0$ for some $k \geq 1$.

Theorem. *Assuming* $(*)$, (X_ζ, σ) *is minimal iff* ζ *is primitive.*

Theorem. [Michel 1974] *Primitive substitution dynamical systems are uniquely ergodic.*

Invariant measures for non-minimal aperiodic substitutions

Theorem. *Let ζ be a substitution satisfying $(*)$, without periodic points, and such that the substitution matrix has primitive irreducible components with spectral radius greater than 1. Then the ergodic invariant probability measures for (X_ζ, σ) are in 1-1 correspondence with the distinguished components of M_ζ.*

Proof uses Bratteli-Vershik realization from [BKM 2008].

Examples

1 $a \longrightarrow abb,\ b \longrightarrow ab,\ c \longrightarrow accb,\ \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Uniquely ergodic (measure supported on the minimal component).

2 $a \longrightarrow abb,\ b \longrightarrow ab,\ c \longrightarrow acccb,\ \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

Two ergodic measure (one supported on the minimal component and one supported on the complement).

Other non-minimal substitutions

- A. Fisher (1992): $0 \to 000$, $1 \to 101$ generates the integer Cantor set $101000101000000000101000101\ldots$, there is a unique non-atomic invariant measure normalized by $\mu([1]) = 1$, which is σ-finite. Proved an order-two ergodic theorem for this system.

- H. Yuasa (2007): "almost primitive" substitutions, generalizes the above (has two irreducible components, communicating, with a single minimal component 1×1; $\exists\, a$, with $\zeta(a) = a^p$).

Open questions and further directions

1. Orbit equivalence, K-theory, C^*-algebras
2. Ergodic theory, spectral properties, diffraction spectrum.
3. $\mathbb{Z}^d$ actions, tilings.
4. Order-two ergodic theorems.