Stochastic integration

- Let's go back to the definition of an integral:

$$\int_0^t f(t)\,dt = \lim_{n\to\infty}\sum_{j=1}^{n} f(\tau_j)(t_{j+1}-t_j)$$

where τ_j is in the interval $[t_j, t_{j+1}]$

- More generally have Riemann-Stieltjes integral

$$\int_0^t f(t)dg(t) = \lim_{n\to\infty}\sum_{j=1}^{n} f(\tau_j)(g(t_{j+1})-g(t_j))$$

- For a smooth measure $g(t)$, limit converges to a unique value regardless of where τ_j taken in interval $[t_j, t_{j+1}]$

Stochastic integration

- **HOWEVER:** $W(t)$ is not smooth. In fact, $\dot{W}(t)$ is delta-autocorrelated: in any interval of the real line, white noise fluctuates an infinite number of times with infinite variance
- The limit that defines the integral depends on where τ_j is taken to lie in interval $[t_j, t_{j+1}]$
- Different choices lead to different stochastic calculi:
  - $\tau_j = t_j \Rightarrow$ **Ito calculus**
  - $\tau_j = (t_j + t_{j+1})/2 \Rightarrow$ **Stratonovich calculus**
- This may look arbitrary, but luckily we know when to use what calculus (and how to translate between them)

Ito & Stratonovich calculi

- We start right off the bat with a strange result:

$$(I)\int_0^t W(t')dW(t') = \frac{1}{2}[W(t)^2 - W(0)^2 - t]$$

while

$$(S)\int_0^t W(t')dW(t') = \frac{1}{2}[W(t)^2 - W(0)^2]$$

- "Normal rules of calculus" don't apply to Ito integral, but do apply to Stratonovich
- Ultimately, weird behaviour comes from fact that (loosely) $(\delta W)^2 \sim \delta t$; in Taylor series expansion terms in $(\delta W)^2$ enter at same order as δt. Formally, for Ito integral, $dW^2 = dt$ in the sense that

$$\int_{t_1}^{t_2} G(t')\,[dW(t')]^2 = \int_{t_1}^{t_2} G(t')\,dt$$

Ito and Stratonovich SDEs

- Will use different notation to distinguish between SDEs interpreted in Ito and Stratonovich senses:

$$(I)\ :\ \frac{dx}{dt} = a(x,t) + b(x,t)\dot{W}(t)$$

$$(S)\ :\ \frac{dx}{dt} = a(x,t) + b(x,t)\circ\dot{W}(t)$$

- **Ito's formula** gives us a "chain rule" for solution of Ito SDE:

$$\begin{aligned} df(x(t)) &= f(x(t)+dx(t)) - f(x(t)) \\ &= f'[x(t)]dx(t) + \frac{1}{2}f''[x(t)]dx(t)^2 + ... \\ &= \left(a(x,t)f'(x) + \frac{1}{2}b(x,t)^2 f''(x)\right)dt + b(x,t)f'(x)dW \end{aligned}$$

to leading order in dt

Ito and Stratonovich SDEs

- We can use Ito's formula to find a "translation" between Ito and Stratonovich SDEs:

  Ito SDE
  $$\frac{dx}{dt} = a(x,t) + b(x,t)\dot{W}(t)$$
  is the same as the Stratonovich SDE
  $$\frac{dx}{dt} = \left(a(x,t) - \frac{1}{2}b(x,t)\partial_x b(x,t)\right) + b(x,t) \circ \dot{W}(t)$$
- Can go between Ito and Stratonovich by appropriately modifying drift; correction term often called **"noise-induced drift"**
- Note dimensions of $b(x,t)$: $[b] = [x][t]^{-1/2}$ because $[\dot{W}] = [t]^{-1/2}$. $b(x,t)$ is **not** standard deviation of white noise - it's a scaling factor.

Solving an Ito SDE: An example

- To solve Ito SDE
  $$\frac{dx}{dt} = x\dot{W}(t)$$
  can transform to Stratonovich SDE
  $$\frac{dx}{dt} = -\frac{1}{2}x + x \circ \dot{W}(t)$$
  for which "normal rules of calculus" apply:
  $$\int_0^t \frac{1}{x}\frac{dx}{dt}\, dt = \int_0^t \left(-\frac{1}{2} + \dot{W}(t)\right)$$
  so
  $$x(t) = x(0)\exp\left(-\frac{t}{2} + W(t) - W(0)\right)$$

SDEs and Fokker-Planck Equations

- Can use Ito's formula to show that pdf $p(x(t))$ of Ito SDE satisfies FPE
  $$\partial_t p = -\partial_x[a(x,t)p] + \frac{1}{2}\partial^2_{xx}[b^2(x,t)p]$$
  and pdf of Stratonovich SDE satisfies
  $$\begin{aligned}\partial_t p &= -\partial_x[a(x,t)p] + \frac{1}{2}\partial_x\left[b(x,t)\partial_x[b(x,t)p]\right] \\ &= -\partial_x\left(\left[a(x,t) + \frac{1}{2}b(x,t)\partial_x b(x,t)\right]p\right) + \frac{1}{2}\partial^2_{xx}[b^2(x,t)p]\end{aligned}$$
- Again we see the connection between solutions of SDEs and diffusion of probability
- For every SDE there is a unique FPE, but every FPE has a set of associated SDEs (because b^2 appears in FPE)

Ito vs. Stratonovich: How to choose?

- White noise is an idealisation; real fluctuating forcing has finite amplitude and timescale
- If white noise is approximation to continuously fluctuating noise with finite memory (much shorter than dynamical timescales), appropriate representation is *Stratonovich* (Wong-Zakai Theorem)
- If white noise approximates set of discrete pulses with finite separation to which system responds, or SDE continuous approximation to discrete system, then *Ito* representation appropriate
- Because in an atmosphere/ocean/climate context "driving noise" a representation of "fast" part of continuous fluid dynamical system, Stratonovich SDEs usually most natural

Ito vs. Stratonovich: How to choose?

- For example, consider 2D SDE

$$\begin{aligned}\frac{dx}{dt} &= a(x,t) + b(x,t)\eta \\ \frac{d\eta}{dt} &= -\frac{1}{\tau}\eta + \frac{\sigma}{\tau}\dot{W}\end{aligned}$$

- As $\tau \to 0$, $\eta \to \dot{W}$ and x satisfies the Stratonovich SDE

$$\frac{dx}{dt} = a(x,t) + b(x,t) \circ \dot{W}$$

- Operationally: Stratonovich SDEs easier to solve analytically, but Ito SDEs more natural starting point for numerical schemes

Nonlinearity or multiplicative noise?

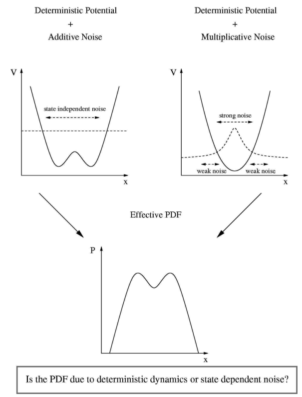
- The pdf of a linear SDE with additive noise is Gaussian
- More general SDE

$$\dot{x} = a(x,t) + b(x,t) \circ \dot{W}(t)$$

has stationary pdf (from FPE)

$$p(x) = \mathcal{N} \exp\left(\int_0^x \left[\frac{a(x)}{b(x)} - \frac{1}{2}\partial_x b(x)\right] dx\right)$$

- Non-Gaussianity can arise from nonlinearity of $a(x)$, or from multiplicative noise: $b(x) \neq$ const.



Sura et al. *JAS* 2005

Convergence

- Have been cavalier up to this point about what is meant by "convergence" in stochastic processes
- Different levels of convergence of random sequence $\{x_n\}$ as $n \to \infty$
  a. almost sure

$$P(x_n \to x) = 1$$

  b. mean square

$$\mathsf{E}\{(x_n - x)^2\} \to 0$$

  c. in distribution

$$P(x_n) \to P(x)$$

- a $\Rightarrow$ c and b $\Rightarrow$ c, but not vice-versa (convergence in distribution is relatively weak)

SDEs from chaotic dynamics: Rigorous results

- Coupled slow/fast system with "chaotic" fast dynamics

$$\begin{aligned}\frac{dx}{dt} &= f(x,y) \quad \text{(slow climate mode)} \\ \frac{dy}{dt} &= \frac{1}{\epsilon}g(x,y) \quad \text{(fast weather mode)}\end{aligned}$$

- As $\epsilon \to 0$, $x \to X$ in distribution, where X satisfies:

$$\frac{dX}{dt} = \overline{f}(X) + \epsilon D(X) + \sqrt{\epsilon}\sigma(X)\frac{dW}{dt}$$

- Have explicit formulae for $\overline{f}(X)$, $D(X)$, $\sigma(X)$ in terms of the "stationary distribution" of the fast dynamics
- Another related (but distinct) approach to "stochastic mode reduction" is **MTV (Majda, Timofeyev, & Vanden-Eijnden) Theory**