Bessel potentials and optimal Hardy and Hardy-Rellich inequalities

Nassif Ghoussoub
University of British Columbia

Abstract

We give necessary and sufficient conditions on a pair of positive radial functions $V$ and $W$ on a ball $\Omega$ of radius $R$ in $\mathbb{R}^n$, $n \geq 2$, so that the following inequalities hold for all $u \in C_0^\infty(\Omega)$:

$$\int_\Omega V(x)|\nabla u|^2\,dx \geq \int_\Omega W(x)u^2\,dx$$  \hspace{1cm} (1)

and

$$\int_\Omega V(x)\Delta u|^2\,dx \geq \int_\Omega W(x)|\nabla u|^2\,dx + (n - 1) \int_\Omega \left(\frac{V(x)}{|x|^2} - \frac{v(|x|)}{|x|}\right)|\nabla u|^2\,dx.$$  \hspace{1cm} (2)

We then identify a large number of such couples $(V, W)$ – that we call Bessel pairs – and the best constants in the corresponding inequalities. This will allow us to complete, improve, extend, and unify most related results –old and new– about Hardy and Hardy-Rellich type inequalities which were obtained by Caffarelli-Kohn-Nirenberg, Brezis-Vázquez, Adimurthi-Chaudhuri-Ramaswamy, Filippas-Tertikas, Adimurthi-Grossi-Santra, as well as some very recent work by Tertikas-Zographopoulos, Liskevich-Lyachova-Moroz, and Blanchet-Bonforte-Dolbeault-Grillo-Vasquez, among others.

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