Natural Invariant Measures

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Outline of talk

I. SRB measures -- from Axiom A to general diffeomorphisms

II. Conditions for existence and statistical properties of SRB measures

III. A class of “strange attractors” and some concrete examples

IV. Extending the scope of previous work, to infinite dim, random etc.
SRB measures for Axiom A attractors (1970s)

\[ M = \text{cpt Riem manifold}, \quad f = \text{map} \quad \text{or} \quad f_t = \text{flow} \]

Assume uniformly hyperbolic or Axiom A attractor

A very important discovery of Sinai, Ruelle and Bowen is that these attractors have a special invariant prob meas \( \mu \) with the following properties:

1. (time avg = space avg)
   \[
   \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) \to \int \varphi \, d\mu \quad \text{Leb-a.e. } x
   \]
   for all cts observables

2. (characteristic \( W^u \) geometry) \( \mu \) has conditional densities on unstable manifolds

3. (entropy formula)
   \[
   h_\mu(f) = \int \log |\det(Df|E^u)| \, d\mu
   \]

Moreover, \( (1) \iff (2) \iff (3) \)

Proofs involves Markov partitions & connection to stat mech
Next drop Axiom A assumption.

**How general is the idea of SRB measures?** 

\[ M = \text{cpct Riem manifold}, \quad f = \text{arbitrary diffeomorphism or flow} \]

Recall: properties of SRB measures in Axiom A setting:

1. time avg = space avg,
2. characteristic \( W^u \) geometry,
3. entropy formula

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**Theorem** [Ledrappier-Strelcyn, L, L-Young 1980s]

Let \((f, \mu)\) be given where \(\mu\) is an arbitrary invariant Borel prob. Then (2) \(\iff\) (3); more precisely:

\((f, \mu)\) has pos Lyap exp a.e. and \(\mu\) has densities on \(W^u\)

\[ h_\mu(f) = \int \sum_i \lambda_i^+ m_i \ d\mu \quad \text{where} \quad \lambda_i \text{ are Lyap exp with multiplicities } m_i \]

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We defined SRB measures for general \((f, \mu)\) by (2).

Note: Entropy formula proved for volume-preserving diffeos (Pesin, 1970)

Entropy inequality \((\leq)\) proved for all \((f, \mu)\) (Ruelle, 1970s)
What is the meaning of all this?

For finite dim dynamical systems, an often adopted point of view is

**observable events = positive Leb measure sets**

For Hamiltonian systems,

Liouville measure = *the* important invariant measure
Same for volume preserving dynamical systems

But what about “dissipative” systems, e.g., one with an attractor?

Suppose \( f : U \to U, \ f(\bar{U}) \subset U, \) and \( \Lambda = \bigcap_{n=0}^{\infty} f^n(U) \)

Assume \( f \) is volume decreasing.

Then \( \text{Leb}(\Lambda) = 0 \), and all inv meas are supported on \( \Lambda \)

i.e., no inv meas has a density wrt Leb

This does not necessary imply no inv meas can be *physically relevant*

= reflecting the properties of Lebesgue
Here is how it works:

μ has densities on $W^u$ together with

if

$$\bar{\varphi}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \varphi(f^i x)$$

then

$$\bar{\varphi}(x) = \bar{\varphi}(y) \quad \forall y \in W^s(x)$$

integrating out along $W^s$, properties on $W^u$ passed to basin

Crucial to this argument is the absolute continuity of the $W^s$ foliation proved in nonuniform setting [Pugh-Shub 1990]

To summarize:

- one way to define \textit{SRB meas for general} $(f, \mu)$ is pos Lyap exp + conditional densities on $W^u$ i.e. property (2)

- conditional densities on $W^u$ implies \textit{physical relevance}

under assumptions of ergodicity and no 0 Lyap exp i.e. property (1)
And how is the entropy formula related to all this?

entropy comes from expansion but not all expansion goes into making entropy

Ruelle’s entropy inequality

But whether entropy = sum of pos Lyap exp, what does that have to do with backward-time dynamics?

Entropy formula holds iff system is conservative in forward time = an interpretation of SRB measure

Meaning of gap in Ruelle’s Inequality:

**Theorem** [Ledrappier-Young, 1980s] \((f, \mu)\) as above; assume ergodic for simplicity.

Then

\[
h_\mu(f) = \sum_i \lambda_i^+ \delta_i \quad \text{where} \quad \delta_i \in [0, \dim E_i]
\]

“in the direction of \(E_i\)”

Interpretation: \(\dim(\mu|W^u) = \sum \delta_i\) is a measure of dissipativeness
A difference between results for Axiom A and general diffeos is that no existence is claimed.

A natural condition that guarantees existence:

Let $R = \text{return time}$
$m = \text{Leb meas in } E^u$

**Prop** [Young 80s] If $\int Rdm < \infty$, the SRB meas exists.

Most SRB meas (outside of Axiom A) were constructed this way. First time I used it: piecewise unif hyperbolic maps of $\mathbb{R}^2$ [Young, 1980s]
In the same spirit that (finite) Markov partitions facilitated the study of statistical properties of Axiom A systems, I proposed (1990s) that

(1) stats of systems that admit countable Markov extensions can be expressed in terms of their renewal times, and

(2) this may provide a unified view of a class of nonuniformly hyperbolic systems that have “controlled hyperbolicity”

By (1), I mean given \( f \), seek

\[
\begin{align*}
\Delta & \xrightarrow{F} \Delta \\
\downarrow & \\
M & \xrightarrow{f} M
\end{align*}
\]

s.t. \((F, \Delta)\) has a countable Markov partition

In practice, fix a reference set \( \Lambda_0 \) with hyperbolic (product) structure. Build skyscraper until “good return“

“dynamical renewal”
Theorem [Young, 90s]
Suppose $f$ admits a Markov extension with return time $R$, $m=\text{Leb},$

(a) If $\int Rdm < \infty$, then $f$ has an SRB meas $\mu$
(b) If $m\{R > n\} < C\theta^n, \theta < 1$, then $(f, \mu)$ has exp decay of correl
(c) If $m\{R > n\} = O(n^{-\alpha}), \alpha > 1$, then decay $\sim n^{-\alpha+1}$
(d) If $m\{R > n\} = O(n^{-\alpha}), \alpha > 2$, then CLT holds.

Idea is to **swap messy dynamics for a nice space w/ Markov structures**

Construction of Markov extension was carried out for several known examples
e.g.

Theorem [Young, 1990s] Exponential decay of time correlations for collision map of 2D periodic Lorentz gas

Remarks 1. Important progress in hyperbolic theory is the understanding that **deterministic chaotic systems** produce stats very similar to those from (random) **stochastic processes**

2. Above are conditions for natural inv meas & their statistical properties. To check these conditions, **need some degree of hyperbolicity** for the dyn sys
Part III. Proving positivity of Lyap exp in systems w/out inv cones

Major challenge even when there is a lot of expansion
Reason: where there is expansion, there is also contraction ....

\[ v_0 = \text{tangent vector at } x, \quad v_n = Df^n_x(v_0) \]

\[ \|v_n\| \text{ sometimes grows, sometimes shrinks} \]

cancellation can be delicate

A breakthrough, and an important paradigm:

**Theorems**

\[ f_a(x) = 1 - ax^2, \quad a \in [0, 2] \]

1. [Jakobson 1981] There is a positive meas set of \( a \) for which
   \( f_a \) has an invariant density and a pos Lyap exp.

2. [Lyubich; Graczyk-Swiatek 1990s] Parameter space \([0, 2] = \mathcal{A} \cup \mathcal{B} \mod \text{Leb 0}\)
   s.t. \( \mathcal{A} \) is open and dense and \( a \in \mathcal{A} \implies f_a \) has sinks
   \( \mathcal{B} \) has positive meas and \( a \in \mathcal{B} \implies f_a \) has acim & pos exp

Intermingling of opposite dynamical types makes it impossible
to determine pos Lyap exp from finite precision or finite # iterates
Next breakthrough: The Henon maps [Benedicks-Carleson 1990]

\[ T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_{a,b}(x, y) = (1 - ax^2 + y, bx) \]

[BC] devised (i) an inductive algorithm to identify a “critical set”, and (ii) a scheme to keep track of derivative growth for points that do not approach the “critical set” faster than exponentially.

Borrowing [BC]’s techniques:

**Theorem** [Wang-Young 2000s] [technical details omitted]

**Setting:** \( F_{a,\varepsilon} : M \cap \epsilon \quad \text{where} \quad M = S_1 \times D_m \) (m-dim disk)

\( a = \) parameter, \( \varepsilon^m = \) “determinant” (dissipation)

**Assume**

1. singular limit defined, i.e. \( F_{a,\varepsilon} \to F_{a,0} \) as \( \varepsilon \to 0 \)
2. \( f_a = F_{a,0}|(S_1 \times \{0\}) : S_1 \cap \text{ has “enough expansion”} \)
3. nondegeneracy + transversality conditions

**Then** for all suff small \( \varepsilon > 0 \), \( \exists \Delta(\varepsilon) = \) pos meas set of \( a \)

s.t. (a) \( F_{a,\varepsilon} \) has an ergodic SRB measure
(b) \( \lambda_{\text{max}} > 0 \) Leb-a.e. in \( M \)
We called the resulting attractors “rank-one attractors”

= 1-D instability, strong codim 1 contraction

• “fattening up expanding circle maps e.g. \( z \mapsto z^2 \) gives solenoid maps;
slight “fattening up” of 1D maps (w/ singularities) gives rank-one maps

• passage to singular limit = lower dim’l object makes problem tractable

• rank-one attractors (generalization of Henon attractors) are currently
  the only class of nonuniformly hyperbolic attractors amenable to analysis

• proof in [BC] is computational, using formula of Henon maps;
  [WY]’s formulation + proof are geometric, independent of [BC]

• Motivation: rank-one attractors likely occur naturally, shortly after
  a system’s loss of stability

• [WY] gives checkable conditions so results can be applied
  without going thru 100+ page proof each time
Application of rank-one attractor ideas

Shear-induced chaos in periodically kicked oscillators

Simplest version: linear shear flow

\[
\begin{align*}
\frac{d\theta}{dt} &= 1 + \sigma y \\
\frac{dy}{dt} &= -\lambda y + A \sin(2\pi \theta) \sum_{n=0}^{\infty} \delta(t - nT)
\end{align*}
\]

kicks delivered with period \(T\)

\(\theta \in S^1, \ y \in \mathbb{R}, \ \sigma = \text{shear}, \ \lambda = \text{damping}, \ T \gg 1\)

Unforced equation: \(\{y = 0\} = \text{attractive limit cycle}\)

key:

\[
\frac{\sigma}{\lambda} \cdot A = \frac{\text{shear}}{\text{damping}} \cdot \text{deformation}
\]

assuming \(e^{-\lambda T} \ll 1\)
\[
\dot{\theta} = 1 + \sigma y \\
\dot{y} = -\lambda y + \text{kick} \\
T = \text{kick period} \\
\sigma \frac{A}{\lambda} = \text{shear damping} \cdot \text{deformation}
\]

**Theorem** [Wang-Young 2000s]

(a) small \( \sigma \frac{A}{\lambda} \) : invariant closed curve

(b) as \( \sigma \frac{A}{\lambda} \) increases : invariant curve breaks, horseshoes develop

(c) large \( \sigma \frac{A}{\lambda} \) : “dichotomy”

\[ \lambda_{\text{max}} > 0 \quad \text{pos meas set parameters} \]

\[ \lambda_{\text{max}} < 0 \quad \text{horseshoes + sinks} \]

Proof obtained by checking conditions in [WY]; general limit cycles OK.

**Other applications of this body of ideas**

- homoclinic bifurcations [Mora-Viana 1990s]
- periodically forced Hopf bifurcations [Wang-Young 2000s]
- forced relaxation oscillators [Guckenheimer-Weschelberger-Young 2000s]
- Shilnikov homoclinic loops [Ott and Wang 2010s]
- forced Hopf bif in parabolic PDEs, appl to chemical networks [Lu-Wang-Young 2010s]
Part IV. Extending the scope of existing theory

A. Infinite dimensional systems

Dynamical setting for certain classes of PDEs

Consider

\[
\frac{du}{dt} + Au = f(u)
\]

where \( u \in X \) = function space, \( A \) = linear operator, \( f \) = nonlinear term

To define a \( C^r \) dynamical system, need \( (X, \| \cdot \|) \) s.t.

1. \( u_0 \in X \implies u(t) \) exists and is unique in \( X \) for all \( t \geq 0 \),
   so semiflow \( f^t : X \to X \) is well defined

2. \( t \mapsto u(t) \) is continuous for \( t \geq 0 \)

3. \( f^t \in C^r \) for each \( t \)

Remark. Dissipative PDEs (e.g. reaction diffusion eqtns) have attractors with a very finite dimensional character -- natural place to start

E.g. Multiplicative Ergodic Theorem proved only for Hilbert/Banach space operators that are quasi-compact \([\text{Mane, Ruelle, Thieullen, Lian-Lu ....}]\)
In infinite dimensions: what plays the role of Leb measure? More concretely, what is a "typical" solution for a PDE?

Sample results

Theorem  Under global invariant cones conditions:

(a) Existence of center manifold \( W^c \)

(b) Existence of stable \( W^s \) foliation [known]

(c) Absolute continuity of \( W^s \)-foliation in the case \( \dim(W^c) < \infty \)

i.e. if \( \Sigma_1, \Sigma_2 = \) disks transversal to \( W^s \), and \( \theta : \Sigma_1 \to \Sigma_2 \) is holonomy along \( W^s \)-leaves, then \( \text{Leb}(\theta(A)) \leq c \text{Leb}(A) \) for all Borel \( A \subset \Sigma_1 \).

Interpretation

Notion of "almost everywhere" in Banach space inherited from Leb measure class on \( W^c \)
e.g. a.e. in the sense of k-parameters of initial conditions

General idea: use of finite dim’l probes in infinite dim sp
More general setting $X = \text{Banach or Hilbert space}$

\[ F : [0, \infty) \times X \to X \quad \text{cts semiflow}, \quad f^t(x) = F(t, x) \]

Assume

1. \( F|_{(0, \infty) \times X} \) is \( C^2 \)
2. \( f^t, \ Df^t_x \) injective \quad \text{[backward uniqueness]} \n3. existence of compact \( A \subset X, \ f^t(A) = A \) \quad \text{[attractor]}

**Theorem**

Assume no 0 Lyap exponents.

(a) \([\text{Li-Shu, Blumenthal-Young 2010s}]\) \( \mu \) is an SRB measure if and only if

\[ h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \, d\mu \]

(b) \([\text{Blumenthal-Young 2010s}]\) \text{Absolute continuity of } W^s

i.e. statistics of SRB \text{ visible}

**B. Random dynamical systems \ (RDS)\**

\[ \cdots \circ f_{\omega_3} \circ f_{\omega_2} \circ f_{\omega_1}, \quad \text{i.i.d. with law } \nu \]

where \( \nu \) is a Borel probability on \( C^r(M) = \text{space of self-maps of} \)

Motivation: small random perturbations of deterministic maps, \ SDEs
Two notions of invariant measures

Stationary measure

\[ \mu(A) = \int P(A|x) \, d\mu(x) \]

Equivalently, \( \mu = \int (f_\omega)_* \mu \, \mathbb{P}(d\omega) \) in the random maps representation

Sample measures = \( \mu \) conditioned on the past

\[ \omega = (\omega_n)_{n=-\infty}^{\infty} \quad \mu_\omega = \lim_{n \to \infty} (f_{\omega_{-1}} \circ \cdots \circ f_{\omega_{-n+1}} \circ f_{\omega_{-n}})_* \mu \]

Interpretation: \( \mu_\omega \) describes what we see at time 0 given that the transformations \( f_{\omega_n}, n \leq 0 \), have occurred.

**Theorem.** Given RDS with stationary \( \mu \), \( \lambda_{\text{max}} = \text{largest Lyap exp} \)

(a) [Le Jan, 1980s] If \( \lambda_{\text{max}} < 0 \), then \( \mu_\omega \) is supported on a finite set of points for \( \nu^\mathbb{Z} - \text{a.e.}\omega \) called random sinks.

(b) [Ledrappier-Young, 1980s] If \( \mu \) has a density and \( \lambda_{\text{max}} > 0 \), then entropy formula holds and \( \mu_\omega \) are random SRB measures for \( \nu^\mathbb{Z} - \text{a.e.}\omega \)

(c) [L-Y 1980s] Additional Hormander condition on derivative process partial dimensions satisfy

\[ \delta_i = 1 \quad \text{for} \quad i < i_0, \quad \delta_i = 0 \quad \text{for} \quad i > i_0 \]
Application: **reliability** of biological and engineered systems

$$I_\omega(t) \xrightarrow{\text{fluctuating input}} (\text{large}) \text{ dynam sys} \xrightarrow{R(t)} \text{response}$$

$$x_0 = \text{is internal state of system at time of presentation}$$

Say the dyn sys is **reliable** wrt a class of signals if the dependence of $$R(t)$$ on $$x_0$$ tends to 0 with as $$t$$ increases.

If $$\lambda_{\max} < 0$$ and a.e. $$\mu_\omega$$ is, e.g. supported on a single point, then $$x_t = \text{state of system at time } t \text{ is largely indep of } x_0 : \text{reliable}$$

If $$\lambda_{\max} > 0$$ then $$\mu_\omega$$ is supported on stacks of lower dim’l surfaces, $$x_t$$ depends on $$x_0$$ no matter how long we wait: **unreliable**

Example: coupled oscillators

at $$t = 50, 500, 2000$$
Another application of RDS: **climate** e.g. Ghil group

stationary meas: theoretical avg vs sample meas: now given history

C. Open dynamical systems

= systems in contact with external world (rel to nonequilibrium stat mech)

A simple situation is leaky systems, i.e., systems with holes

Questions include escape rate $\rho$, surviving distributions etc

Sample result:

**THEOREM** [Demers-Wright-Young 2000s] Billiard tables with holes

1. escape rate $\rho$ is well defined
2. limiting surviving distribution $\mu_\infty$ well defined & conditionally invariant

$$f_\star(\mu_\infty)|_{M \setminus H} = e^{-\rho} \mu_\infty$$

3. characterized by SRB geometry and entropy formula

$$h - \sum \lambda_i^+ m_i = -\rho$$

4. tends to SRB measure as hole size goes to 0

Result extendable to systems admitting Markov extension
D. Farther afield

In biological systems, I’ve encountered the following challenges:

(1) **Inverse problems**: Given basic structure + outputs of system, deduce dynamics and (nonequilibrium) steady states

(2) Continuous adaptation to (changing) stimulus, & partial convergence to *time-dependent steady states*

Concluding remarks

- Importance of idea measured by impact and how it shapes future development, SRB ideas truly lasting

- **Dynamical systems** has evolved since the 1970s, will remain *fun, vibrant*, and *relevant* as long as it continues to evolve ....